Prof. M. Krstic

Solutions, HW 3

- **2.42**

$$\begin{aligned}\dot{x}_1 &= \tan^{-1}(ax_1) - x_1 x_2 \\ \dot{x}_2 &= bx_1^2 - cx_2\end{aligned}$$

Let $\lambda = [a, b, c]^T$. The nominal values are $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$. The Jacobian matrices $[\partial f/\partial x]$ and $[\partial f/\partial \lambda]$, are given by

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{a}{1+a^2x_1^2} - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix}, \quad \frac{\partial f}{\partial \lambda} = \begin{bmatrix} \frac{x_1}{1+a^2x_1^2} & 0 & 0 \\ 0 & x_1^2 & -x_2 \end{bmatrix}$$

Let

$$S = \left.\frac{\partial x}{\partial \lambda}\right|_{\text{nominal}} = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix}$$

Then

$$\dot{S} = \left.\frac{\partial f}{\partial x}\right|_{\text{nominal}} S + \left.\frac{\partial f}{\partial \lambda}\right|_{\text{nominal}}, \quad S(0) = 0$$

The augmented equation (2.11) is given by

$$\begin{aligned}\dot{x}_1 &= \tan^{-1}(x_1) - x_1 x_2 \\ \dot{x}_2 &= -x_2 \\ \dot{x}_3 &= \left(\frac{1}{1+x_1^2} - x_2\right) x_3 - x_1 x_4 + \frac{x_1}{1+x_1^2} \\ \dot{x}_4 &= -x_4 \\ \dot{x}_5 &= \left(\frac{1}{1+x_1^2} - x_2\right) x_5 - x_1 x_6 \\ \dot{x}_6 &= -x_6 + x_1^2 \\ \dot{x}_7 &= \left(\frac{1}{1+x_1^2} - x_2\right) x_7 - x_1 x_8 \\ \dot{x}_8 &= -x_8 - x_2\end{aligned}$$

with the initial conditions

$$x_1(0) = x_{10}, \; x_2(0) = x_{20}, \; x_3(0) = x_4(0) = x_5(0) = x_6(0) = x_7(0) = x_8(0) = 0$$

• **2.46**

$$x(t,a,\eta) = \eta + \int_a^t f(s, x(s,a,\eta))\, ds$$

$$\begin{aligned} x_a(t) &= \frac{\partial}{\partial a} x(t,a,\eta) &= -f(a,\eta) + \int_a^t \frac{\partial f}{\partial x}(s, x(s,a,\eta)) \frac{\partial}{\partial a} x(s,a,\eta)\, ds \\ x_\eta(t) &= \frac{\partial}{\partial \eta} x(t,a,\eta) &= I + \int_a^t \frac{\partial f}{\partial x}(s, x(s,a,\eta)) \frac{\partial}{\partial \eta} x(s,a,\eta)\, ds \end{aligned}$$

Therefore

$$x_a(t) + x_\eta(t) f(a,\eta) = \int_a^t \left\{ \frac{\partial f}{\partial x}(s, x(s,a,\eta))[x_a(s) + x_\eta(s) f(a,\eta)] \right\} ds$$

Differentiating with respect to t, we see that $x_a(t) + x_\eta(t) f(a,\eta)$ satisfies the differential equation

$$\frac{\partial}{\partial t}[x_a(t) + x_\eta(t) f(a,\eta)] = \frac{\partial f}{\partial x}(t, x(t,a,\eta))[x_a(t) + x_\eta(t) f(a,\eta)]$$

with initial condition

$$x_a(a) + x_\eta(a) f(a,\eta) = -f(a,\eta) + f(a,\eta) = 0$$

Thus

$$x_a(t) + x_\eta(t) f(a,\eta) \equiv 0, \quad \forall\, t \in [a, t_1]$$

Lemma B.6 *Let v, l_1, and l_2 be real-valued functions defined on $\mathbb{R}_+$, and let c be a positive constant. If l_1 and l_2 are nonnegative and in $\mathcal{L}_1$ and satisfy the differential inequality*

$$\dot{v} \leq -cv + l_1(t)v + l_2(t), \qquad v(0) \geq 0 \tag{B.17}$$

then $v \in \mathcal{L}_\infty \cap \mathcal{L}_1$ and

$$\begin{aligned} v(t) &\leq \left(v(0)e^{-ct} + \|l_2\|_1\right) e^{\|l_1\|_1} && \text{(B.18)} \\ \|v\|_1 &\leq \frac{1}{c}\left(v(0) + \|l_2\|_1\right) e^{\|l_1\|_1}. && \text{(B.19)} \end{aligned}$$

Proof. Using the fact that fact that $w(t) \leq v(t)$, $\dot{w} = -cw + l_1(t)w + l_2(t)$, $w(0) = v(0)$ (the comparison principle; see, for example, [108, 132]), and

applying the variation of constants formula, the differential inequality (B.17) is rewritten as

$$\begin{aligned} v(t) &\leq v(0)e^{\int_0^t [-c+l_1(s)]ds} + \int_0^t e^{\int_\tau^t [-c+l_1(s)]ds} l_2(\tau)d\tau \\ &\leq v(0)e^{-ct}e^{\int_0^\infty l_1(s)ds} + \int_0^t e^{-c(t-\tau)} l_2(\tau)d\tau e^{\int_0^\infty l_1(s)ds} \\ &\leq \left[v(0)e^{-ct} + \int_0^t e^{-c(t-\tau)} l_2(\tau)d\tau\right] e^{\|l_1\|_1}. && \text{(B.20)} \end{aligned}$$

By taking a supremum of $e^{-c(t-\tau)}$ over $[0, \infty]$, we obtain (B.18). Integrating (B.20) over $[0, \infty]$, we get

$$\int_0^t v(\tau)d\tau \leq \left(\frac{1}{c}v(0) + \int_0^t \left[\int_0^\tau e^{-c(\tau-s)} l_2(s)ds\right] d\tau\right) e^{\|l_1\|_1}. \tag{B.21}$$

Applying Theorem B.4 to the double integral, we arrive at (B.19). □

Remark B.7 An alternative proof that $v \in \mathcal{L}_\infty \cap \mathcal{L}_1$ in Lemma B.6 is using the Gronwall lemma (Lemma B.11). However, with the Gronwall lemma, the estimates of the bounds (B.18) and (B.19) are more conservative:

$$\begin{aligned} v(t) &\leq \left(v(0)e^{-ct} + \|l_2\|_1\right)\left(1 + \|l_1\|_1 e^{\|l_1\|_1}\right) && \text{(B.22)} \\ \|v\|_1 &\leq \frac{1}{c}\left(v(0) + \|l_2\|_1\right)\left(1 + \|l_1\|_1 e^{\|l_1\|_1}\right), && \text{(B.23)} \end{aligned}$$

because $e^x < (1 + xe^x)$, $\forall x > 0$. Note that the ratio between the bounds (B.22) and (B.18), and the ratio between the bounds (B.23) and (B.19), are of order $\|l_1\|_1$ when $\|l_1\|_1 \to \infty$. ◇

Introduce $z = v \cdot e^{ct}$

Then

$\dot{v} \le -cv + e_1 \cdot v + e_2 \quad / \cdot e^{ct}$

becomes

$\dot{z} \le e_1 z + e_2 \cdot e^{ct} \le e_1 z + |e_2| \cdot e^{ct}$

Integrating from 0 to t we get

$$z(t) - z(0) \le \int_0^t |e_2(s)| e^{cs} ds + \int_0^t e_1(s) \cdot z(s) ds$$

$$\overset{c>0}{\le} e^{ct} \cdot \int_0^t |e_2(s)| ds + \int_0^t e_1(s) \cdot z(s) ds$$

$$\overset{t<\infty}{\le} e^{ct} \cdot \|e_2\|_1 + \int_0^t e_1(s) \cdot z(s) ds$$

i.e

$$z(t) \le \left(z(0) + e^{ct} \cdot \|e_2\|_1\right) + \int_0^t e_1(s) \cdot z(s) ds \qquad (10)$$

Assuming that $e_1 > 0$ we apply Gronwall lemma & get

$$z(t) \le \left(z(0) + e^{ct} \|e_2\|_1\right) + \int_0^t \left(z(0) + e^{cs} \|e_2\|_1\right) \cdot e_1(s) \cdot e^{\int_s^t e_1(\tau) d\tau} \quad !$$

Since $\int_s^t e_1(\tau) d\tau \le \int_0^t e_1(\tau) d\tau \le \|e_1\|_1$ we get

$$z(t) \le \left(z(0) + e^{ct} \|e_2\|_1\right) + \int_0^t \left(z(0) + e^{cs} \|e_2\|_1\right) \cdot e_1(s) \cdot e^{\|e_1\|_1} \quad !$$

$$\le \left(z(0) + e^{ct} \|e_2\|_1\right) \left(1 + \int_0^t e_1(s) ds \cdot e^{\|e_1\|_1}\right)$$

$$\le \left(z(0) + e^{ct} \|e_2\|_1\right) \left(1 + \|e_1\|_1 \cdot e^{\|e_1\|_1}\right) \qquad (20)$$

which is same as

$$v(t) \le \left(v(0) \cdot e^{-ct} + \|e_2\|_1\right) \left(1 + \|e_1\|_1 \cdot e^{\|e_1\|_1}\right)$$