ENME 606
NONLINEAR CONTROL DESIGN

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Solutions HW 2

- 2.7 Let $z(t) = y(t)e^{\alpha(t-t_0)}$. Then

$$
\begin{aligned}
z(t) &\le k_1 + \int_{t_0}^{t} e^{\alpha(\tau-t_0)}[k_2 y(\tau) + k_3]\, d\tau = k_1 + \int_{t_0}^{t} \left[k_2 z(\tau) + k_3 e^{\alpha(\tau-t_0)}\right] d\tau \\
&= k_1 + k_2 \int_{t_0}^{t} z(\tau)\, d\tau + (k_3/\alpha)\left[\exp^{\alpha(t-t_0)} - 1\right]
\end{aligned}
$$

From Gronwall-Bellman inequality,

$$
z(t) \le k_1 + (k_3/\alpha)\left[\exp^{\alpha(t-t_0)} - 1\right] + \int_{t_0}^{t} \left\{k_1 + (k_3/\alpha)\left[\exp^{\alpha(s-t_0)} - 1\right]\right\} k_2 e^{k_2(t-s)}\, ds
$$

By evaluating the integral, it can be shown that

$$
z(t) \le \frac{k_3}{\alpha} e^{\alpha(t-t_0)} + \left(k_1 - \frac{k_3}{\alpha}\right) e^{k_2(t-t_0)} + \frac{k_2 k_3}{\alpha(\alpha - k_2)}\left[e^{\alpha(t-t_0)} - e^{k_2(t-t_0)}\right]
$$

Hence

$$
\begin{aligned}
y(t) &= z(t)e^{-\alpha(t-t_0)} \le \frac{k_3}{\alpha}\left[1 + \frac{k_2}{(\alpha - k_2)}\right] + \left[k_1 - \frac{k_3}{\alpha} - \frac{k_2 k_3}{\alpha(\alpha - k_2)}\right] e^{(k_2-\alpha)(t-t_0)} \\
&= \frac{k_3}{(\alpha - k_2)} + \left[k_1 - \frac{k_3}{(\alpha - k_2)}\right] e^{(k_2-\alpha)(t-t_0)} \\
&= k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{(\alpha - k_2)}\left[1 - e^{-(\alpha-k_2)(t-t_0)}\right]
\end{aligned}
$$

- 2.5 Set $g(\sigma) = f(\sigma x)$ for $0 \le \sigma \le 1$. Since D is convex, $\sigma x \in D$ for $0 \le \sigma \le 1$.

$$
g'(\sigma) = \frac{\partial f}{\partial x}(\sigma x)\frac{\partial \sigma x}{\partial \sigma} = \frac{\partial f}{\partial x}(\sigma x)\, x
$$

$$
f(x) = f(x) - f(0) = g(1) - g(0) = \int_0^1 g'(\sigma)\, d\sigma = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)\, d\sigma\, x
$$

• 2.27

(1) $\text{sgn}(x_2)$ is a discontinuous function. Hence, the answer to all parts (a) to (e) is NO.

(2) $\text{sat}(x_2)$ is continuous, globally Lipschitz, but not continuously differentiable. The linear functions x_1 and x_2 as well as $\sin(x_2)$ are continuously differentiable and globally Lipschitz. Therefore, the answer to part (a) is NO, and to parts (b), (c), (d), and (e) is YES.

(3) $x_3\text{sat}(x_1+x_2)$ is continuous, locally Lipschitz, but not continuously differentiable nor globally Lipschitz. The function x_2^2 is continuously differentiable but not globally Lipschitz. Therefore, the answer to parts (a), (d), and (e) is NO, and to parts (b) and (c) is YES.

• 2.34 (a)

$$\begin{aligned}
x(t) &= x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau \\
\|x(t)\| &\le \|x_0\| + \int_{t_0}^{t} \|f(\tau, x(\tau))\|\, d\tau \\
&\le \|x_0\| + \int_{t_0}^{t} [k_1 + k_2\|x(\tau)\|]\, d\tau \\
&= \|x_0\| + k_1(t-t_0) + k_2\int_{t_0}^{t} \|x(\tau)\|\, d\tau
\end{aligned}$$

By Gronwall-Bellman inequality

$$\|x(t)\| \le \|x_0\| + k_1(t-t_0) + \int_{t_0}^{t} [\|x_0\| + k_1(s-t_0)]k_2 e^{k_2(t-s)}\, ds$$

Integrating by parts, we obtain

$$\|x(t)\| \le \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)] - 1\}, \quad \forall\, t \ge t_0$$

(b) The upper bound on $\|x(t)\|$ is finite for every finite t. It tends to ∞ as $t \to \infty$. Hence the solution of the system cannot have a finite escape time.