MAE 281A

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NONLINEAR SYSTEMS

Solutions, HW 1

- 1.13 (1)

$$\begin{aligned} 0 &= x_2 \\ 0 &= -x_1 + \frac{1}{6}x_1^3 - x_2 \Rightarrow x_1 = 0, \sqrt{6}, -\sqrt{6} \end{aligned}$$

There are three equilibrium points at $(0,0)$, $(\sqrt{6},0)$, $(-\sqrt{6},0)$.

$$f(x) = \begin{bmatrix} x_2 \\ -x_1 + \frac{1}{6}x_1^3 - x_2 \end{bmatrix}, \quad \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{1}{2}x_1^2 & -1 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}; \quad \text{Eigenvalues}: -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$(0,0)$ is a stable focus.

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=\sqrt{6}} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}; \quad \text{Eigenvalues}: 1, -2$$

$(\sqrt{6},0)$ is a saddle. Similarly, $(-\sqrt{6},0)$ is a saddle.

(2)

$$\begin{aligned} 0 &= -x_1 + x_2 \\ 0 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3 \end{aligned}$$

Hence $x_2 = x_1$ and

$$0 = x_1(1.9 + x_1 + 0.1x_1^2) \Rightarrow x_1 = 0, -2.5505, \text{ or } -7.4495$$

There are three equilibrium points at $(0,0)$, $(-2.5505,-2.5505)$, and $(-7.4495,-7.4495)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ 0.1 - 2x_1 - 0.3x_1^2 & -2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0} = \begin{bmatrix} -1 & 1 \\ 0.1 & -2 \end{bmatrix}; \quad \text{Eigenvalues} : -0.9084, -2.0916$$

(0, 0) is a stable node.

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=-2.5505} = \begin{bmatrix} -1 & 1 \\ 3.2495 & -2 \end{bmatrix}; \quad \text{Eigenvalues} : 0.3707, -3.3707$$

(−2.5505, −2.5505) is a saddle point.

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=-7.4495} = \begin{bmatrix} -1 & 1 \\ -1.8695 & -2 \end{bmatrix}; \quad \text{Eigenvalues} : -1.5 \pm j1.183$$

(−7.4495, −7.4495) is a stable focus.

(3)

$$\begin{aligned} 0 &= (1-x_1)x_1 - \frac{2x_1x_2}{1+x_1} \\ 0 &= \left(2 - \frac{x_2}{1+x_1}\right)x_2 \end{aligned}$$

From the second equation, $x_2 = 0$ or $x_2 = 2(1+x_1)$.

$$x_2 = 0 \;\Rightarrow\; x_1 = 0 \text{ or } x_1 = 1$$

$$x_2 = 2(1+x_1) \;\Rightarrow 0 = (x_1+3)x_1 \;\Rightarrow\; x_1 = 0 \text{ or } x_1 = -3$$

There are four equilibrium points at (0, 0), (1, 0), (0, 2), and (−3, −4). Notice that we have assumed $1 + x_1 \neq 0$; otherwise the equation would not be well defined.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 - 2x_1 - \frac{2x_2}{(1+x_1)^2} & -2\frac{x_1}{(1+x_1)} \\ \frac{x_2^2}{(1+x_1)^2} & 2 - \frac{2x_2}{(1+x_1)} \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \text{ Eigenvalues} : 1, 2 \Rightarrow (0,0) \text{ is unstable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(1,0)} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}; \text{ Eigenvalues} : -1, 2 \Rightarrow (1,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,2)} = \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix}; \text{ Eigenvalues} : -3, -2 \Rightarrow (0,2) \text{ is a stable node}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(-3,-4)} = \begin{bmatrix} 9 & -3 \\ 4 & -2 \end{bmatrix}; \text{ Eigenvalues} : 7.722, -0.772 \Rightarrow (-3,-4) \text{ is a saddle}$$

(4)

$$\begin{aligned} 0 &= x_2 \\ 0 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) \end{aligned}$$

There is a unique equilibrium point at $(0,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -1-6x_1x_2 & 1-3x_1^2-6x_2^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Eigenvalues are $1/2 \pm j\sqrt{3}/2$; hence, $(0,0)$ is unstable focus.

(5)

$$\begin{aligned} 0 &= -x_1 + x_2(1+x_1) \\ 0 &= -x_1(1+x_1) \Rightarrow x_1 = 0 \text{ or } x_1 = -1 \end{aligned}$$

At $x_1 = 0$, $x_2 = 0$. At $x_1 = -1$, $\dot{x}_1 = 1 \neq 0$. Hence, there is a unique equilibrium point at $(0,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} -1+x_2 & 1+x_1 \\ -1+2x_1 & 0 \end{bmatrix}_{(0,0)} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

Eigenvalues are $-1/2 \pm j\sqrt{3}/2$; hence, $(0,0)$ is stable focus.

(6)

$$\begin{aligned} 0 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ 0 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

$x_1^2 + x_2^2 = 1$ is an equilibrium set and $(0,0)$ is an isolated equilibrium point.

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 3x_1^2+x_2^2-1-2x_1x_2 & 2x_1x_2-x_1^2-3x_2^2+1 \\ 3x_1^2+x_2^2-1+2x_1x_2 & x_1^2+3x_2^2-1+2x_1x_2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

Eigenvalues are $-1 \pm j$; hence, $(0,0)$ is stable focus.

(7)

$$\begin{aligned} 0 &= -x_1^3 + x_2 \\ 0 &= x_1 - x_2^3 \end{aligned}$$

$$x_2 = x_1^3 \Rightarrow x_1(1 - x_1^8) = 0 \Rightarrow x_1 = 0 \text{ or } x_1^8 = 1$$

The equation $x_1^8 = 1$ has two real roots at $x_1 = \pm 1$. Thus, there are three equilibrium points at $(0,0)$, $(1,1)$, $(-1,-1)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \text{Eigenvalues} : 1, -1 \Rightarrow (0,0) \text{ is a saddle}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(1,1)} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}; \quad \text{Eigenvalues} : -2, -4 \Rightarrow (1,1) \text{ is a stable node}$$

Similarly, $(-1,-1)$ is a stable node.

• 1.15 (a)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 - x_1x_2\alpha(x) & -\beta(x) + x_2^2\alpha(x) \\ \beta(x) - x_1^2\alpha(x) & -1 + x_1x_2\alpha(x) \end{bmatrix}$$

where

$$\alpha(x) = \frac{1}{(x_1^2 + x_2^2)(\ln\sqrt{x_1^2 + x_2^2})^2}, \quad \beta(x) = \frac{1}{\ln\sqrt{x_1^2 + x_2^2}}$$

Noting that $\lim_{x\to 0} x_i x_i \beta(x) = 0$ for $i, j = 1, 2$ and $\lim_{x\to 0} \beta(x) = 0$, it can be seen that

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{the origin is a stable node}$$

(b) Transform the state equation into the polar coordinates

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

to obtain

$$\dot{r} = -r \Rightarrow r(t) = r_0 e^{-t}$$

and

$$\dot{\theta} = \frac{1}{\ln r} = \frac{1}{\ln r_0 - t} \Rightarrow \theta(t) = \theta_0 - \ln|\ln r_0 - t| + \ln|\ln r_0|$$

Hence, for $0 < r_0 < 1$, $r(t)$ and $\theta(t)$ are strictly decreasing and $\lim_{t\to\infty} r(t) = 0$, $\lim_{t\to\infty} \theta(t) = -\infty$. Thus, the trajectory spirals clockwise toward the origin.
(c) $f(x)$ is continuously differentiable, but not analytic, in the neighborhood of $x = 0$. See the discussion on page 40.

• 1.22 (1) The system

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2(1 - x_1^2 + 0.1x_1^4) \end{aligned}$$

has a unique equilibrium point at the origin. The Jacobian at the origin is given by

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{Eigenvalues } = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$$

Hence the origin is a stable focus. The phase portrait is shown in Figure 1.18(a) with the arrow heads. The direction of the arrow heads can be determined by inspection of the vector field. In particular, since $f_1(x) = -x_2$, we see that f_1 is negative in the upper half of the plane, and positive in the lower half. The system has two limit cycles. The inner limit cycle is unstable, while the outer one is stable. All trajectories starting inside the inner limit cycle spiral toward the origin. All trajectories starting outside the inner limit cycle approach the stable (outer) limit cycle.

(2) The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + x_2 - 3\tan^{-1}(x_1 + x_2) \end{aligned}$$

has three equilibrium points at $(0, 0)$, $(a, 0)$, and $(-a, 0)$, where a is the root of

$$a - \tan\left(\frac{a}{3}\right) = 0$$

which is approximated by $a = 3.9726$. The Jacobian matrix is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - \frac{3}{1+(x_1+x_2)^2} & 1 - \frac{3}{1+(x_1+x_2)^2} \end{bmatrix}$$

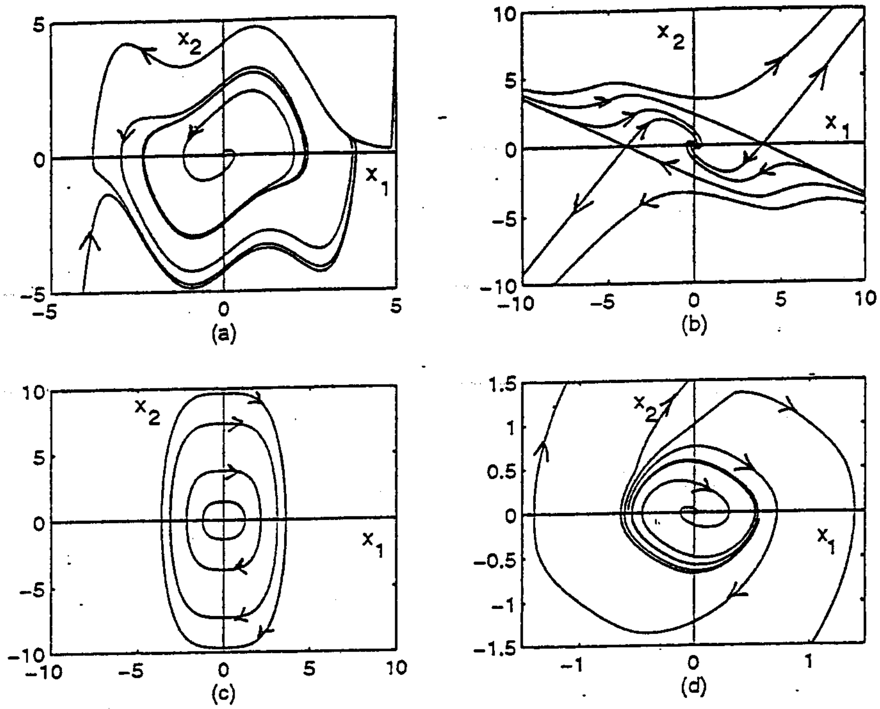

Figure 1.18: Exercise 1.22

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = -1 \pm j \Rightarrow \ (0,0) \text{ is a stable focus}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=\pm 3.9762;x_2=0} = \begin{bmatrix} 0 & 1 \\ 0.821 & 0.821 \end{bmatrix} \Rightarrow \text{ Eigenvalues } = 1.4\& - 0.58 \Rightarrow (\pm 3.9762, 0) \text{ is a saddle}$$

The phase portrait is shown in Figure 1.18(b) with the arrow heads. The direction of the arrow heads can be determined by inspection of the vector field. In particular, since $f_1(x) = x_2$, we see that f_1 is positive in the upper half of the plane, and negative in the lower half. The stable trajectories of the saddle points form two separatrices which divide the plane into three regions. Trajectories starting in the middle region spiral toward the origin, while those starting in the outer regions approach infinity.

(3) The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(0.5x_1 + x_1^3) \end{aligned}$$

has one equilibrium point at the origin.

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} 0 & 1 \\ -0.5 & 0 \end{bmatrix}$$

Linearization fails to determine the type of the equilibrium point. The phase portrait is shown in Figure 1.18(c) with the arrow heads. The direction of the arrow heads can be determined by inspection of the vector field. In particular, since $f_1(x) = x_2$, we see that f_1 is positive in the upper half of the plane, and negative in the lower half. All trajectories are closed orbits centered at the origin. From this we see that the origin is a center.

(4) The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \psi(x_1 - x_2) \end{aligned}$$

has one equilibrium point at the origin. Noting that $\psi'(y) = 3y^2 = 0.5$ for small $|y|$ and $\psi'(0) = 0$, we have

$$\left.\frac{\partial f}{\partial x}\right|_{x_1=0;x_2=0} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.5 \end{bmatrix} \Rightarrow \text{Eigenvalues } = -0.25 \pm j0.6641 \Rightarrow \ (\pm 3.9762, 0) \text{ is a stable focus}$$

The phase portrait is shown in Figure 1.18(d) with the arrow heads. The direction of the arrow heads can be determined by inspection of the vector field. In particular, since $f_1(x) = x_2$, we see that f_1 is positive in the upper half of the plane, and negative in the lower half. The systems has an unstable limit cycle. All trajectories inside the limit cycle spiral toward the origin, while all trajectories outside it diverge to infinity.