MAE 281A Prof. M. Krstic

Nonlinear Systems Winter 2003

FINAL EXAM

Take home. Open books and notes.

Total points: 65

Due Friday, March 14, 2003, at 4:00 pm in Professor Krstic's office. (4pm is a hard deadline, I am leaving the office at 4pm.)

Late submissions will not be accepted. Collaboration not allowed.

Problem 1. Consider the system

$$\begin{aligned}
\dot p &= -\mu\epsilon\gamma \sin\phi \left(r^2 \sin^2 t - q\right) \\
\dot q &= \mu\epsilon\gamma \left(r^2 \sin^2 t - q\right) \\
\dot\phi &= \mu\epsilon \\
\dot r &= \mu r \cos^2 t \left(1 + (p + \sin\phi)^2 - r^2 \sin^2 t\right) ,
\end{aligned}$$

where $r(0) \geq 0$ (note that this implies that $r(t) \geq 0$ for all time because $r = 0$ sets $\dot r = 0$). Denote

$$x = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \text{and} \qquad x_{\mathrm{r}}(t) = \begin{bmatrix} 0 \\ 3 \\ 2\sqrt{1 + \sin(\mu\epsilon t)^2} \end{bmatrix}$$

Show that, for sufficiently small μ, ϵ, and γ, the solution $x(t, \mu, \epsilon, \gamma)$ locally exponentially converges to $x_{\mathrm{r}}(t) + \mathrm{O}(\mu + \epsilon + \gamma)$, at least on a finite time interval.

Hint. This is a complicated problem that involves *four time scales.* (I hope it will not take you as much time to solve it as it took me to construct it and double check its solvability.) The four time scales are (going from fastest to slowest):

- $\sin t$
- r
- $\sin(\mu\epsilon t)$
- p, q.

Your analysis should apply the following steps:

- One step of averaging for the complete system, treating μ as small.
- One step of singular perturbation, treating r as fast and ϵ as small, and introducing the new time $\tau = \epsilon t$ to put the system (after averaging) into the standard singular perturbation form. To derive the boundary layer model, you will need to introduce "another" time variable $\hat t = \tau/\epsilon = t$.

- A second step of averaging on the (p, q) system with $\phi = \mu\tau = \mu\epsilon t$ as time, treating γ as small (again).

Note that the hardest part of the problem is not to mechanically perform the approximations but to connect them all, through appropriate theorems, to draw the final conclusion. Make sure you do quote the theorems as you go from the last step of simplification backwards towards the original system. I will be quite unimpressed to see that you only know how to calculte an average system or how to find a quasi steady state. Note that, in order to draw the final conclusion, exponential stability needs to be satisfied every step of the way.

Problem 2. Consider the system

$$\begin{aligned}\dot{x} &= -x + xz + y(1-y) \\ \dot{y} &= -x(1-y) \\ \dot{z} &= -x^2 .\end{aligned}$$

Give the most precise statement you can on stability and convergence (global and local) of solutions of this system. Note that this is an open ended problem. Since the system has two entire lines of equilibria, analyzing *all* of them might take many days of work, leading you to use not only the Lyapunov, LaSalle invariance, and linearization theorems, but even the center manifold and Chetaev theorems. Go as far as you can with your ideas and these tools.

Hints. If you try to study individual equilibria, the first thing to note is that, since they all belong to continuous sets of equilibria, none of them can be *asymptotically* stable. So, the equilibria fall into one of the two categories: stable or unstable. By taking linearizations around equilibria, you will note that all of them have at least one eigenvalue at zero in their Jacobians. So, unless you find them unstable by linearization, you may need the center manifold theorem. Note that, since none of the equilibria are asymptotically stable, the center manifold theorem should work only for the equilibria that happen to be unstable. Don't immediately look for complicated center manifolds—trivial ones ($h(\cdot) = 0$) will carry you a long way. For those equilibria that happen to be stable you can use Lyapunov functions parametrized by the equilibria. For one of the equilibria, (0,1,1), even center manifold is not enough and the stability question needs to be resolved by direct Lyapunov or Chetaev. I personally have not figured out this one as of this writing.