MAE 281A
Nonlinear Systems

Prof. M. Krstic

Homework 1

Due: Thursday, 1/17/02

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1. Khalil, Exercise 1.13, Parts 1, 2, 5, and 7.
Use Matlab to compute eigenvalues.

2. Khalil, Exercise 1.15. [illegible]

3. Khalil, Exercise 1.22.

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Homework 2

Due: August 29, 2002

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1. Khalil, Exercise 2.5.
This is a special case of the mean value theorem. We will use it in several control designs.

2. Khalil, Exercise 2.7.
This exercise shows how to apply the Gronwal lemma to the case:

$$y(t) \leq \lambda(t) + \rho(t)\int \mu(\tau)y(\tau)d\tau.$$

3. Khalil, Exercise 2.27.

4. Khalil, Exercise 2.34.
Hint: Apply Gronwall lemma. (1.1) means $\dot{x} = f(t,x)$

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Homework 3

Due: February 5, 2002

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1. Khalil, Exercise 2.42.

2. Khalil, Exercise 2.46.

3. Using the comparison principle, show that if v, l_1 and l_2 are functions that satisfy

$$\dot{v} \le -cv + l_1(t)v + l_2(t), \quad v(0) \ge 0$$

and if c>0, then

$$v(t) \le \left(v(0)e^{-ct} + \|l_2\|_1\right)e^{\|l_1\|_1},$$

where $\|\bullet\|_1$ denotes the L_1 norm defined as

$$\|f\|_1 = \int_0^\infty |f(t)|dt$$

Using Gronwall's lemma show that

$$v(t) \le \left(v(0)e^{-ct} + \|l_2\|_1\right)\left(1 + \|l_1\|_1 e^{\|l_1\|_1}\right)$$

Which of the two bounds is less conservative?

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Homework 4

Due: February 19, 2002

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1. Khalil, Exercise 3.3, Parts (1) and (2).

2. Khalil, Exercise 3.12.
Hint: Use Chetaev's theorem with $V(x) = -\frac{1}{6}x_1^6 + \frac{1}{4}x_2^4$.

3. Khalil, Exercise 3.17, Parts (a) and (b).
Hint: In Part (b), use LaSalle's theorem.

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Homework 5

Due: March 12, 2002

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1. Khalil, Exercise 3.41.

Hint: In part (b), study (local) u.a.s. of the solution $x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$.

2. Khalil, Exercise 5.5.

Hint: Use completion of squares in $\dot{V}$.

3. Khalil, Exercise 5.18, Parts (a)-(c).
Hint: In Part (c), show that stability is not *global* by showing that there exist multiple equilibria.

4. Khalil, Exercise 5.21, Parts (2) and (3).

5. Khalil, Exercise 5.22, Parts (1) and (4).
Hints:

- Part (1): use Lemma C.4 in [KKK].
- Part (4): use Young's inequality which gives $x_1u_1 \le \frac{1}{2}x_1^2 + \frac{1}{2}u_1^2$ and $x_2u_2 \le \frac{1}{4}x_2^4 + \frac{3}{4}u_2^{4/3}$; then apply Theorem C.3 in [KKK].

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Homework 6

Due: March 18, 2002

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1. Khalil, Exercise 4.6, Parts (1) and (2).

2. Khalil, Exercise 4.7, Part (4).
Hint: take $z_1 = x_1 + x_3, z_2 = x_2 - x_3$.

3. Khalil, Exercise 4.8, Parts (1), (2), (3), (4).

4. Khalil, Exercise 8.5, Parts (a), (b), (c).

5. Khalil, Exercise 8.8, Parts (a), (b), (c).

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Homework 7

Due: April 16, 2002

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1. Khalil, Exercise 8.16, Parts (a) and (b).

2. Khalil, Exercise 8.17.

3. Khalil, Exercise 9.7, Part (a).

4. Khalil, Exercise 9.11.
Hint: try substituting the quasi-steady state into the (exact) manifold condition.

5. Khalil, Exercise 9.12.

6. Khalil, Exercise 9.29. (Thm 9.4 in 2n edition means Thm 11.2 in 3rd edition)

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Homework 8

Due: ??

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1. Khalil, Exercise 10.33, Parts (1), (2), (3).

2. Khalil, Exercise 10.35. (Fig. 10.17 in 2nd ed. = Fig. 7.1 in 3rd ed)

3. Khalil, Exercise 10.37.

third equation is a torque equation for the shaft, with J as the rotor inertia and c_3 as damping coefficient. The term $c_1 i_f \omega$ is the back e.m.f. induced in the armature circuit, and $c_2 i_f i_a$ is the torque produced by the interaction of the armature current with the field circuit flux. The voltage v_a is held constant, and the voltage v_f is the control input. Choose appropriate state variables and write down the state equation.

Exercise 1.13 For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium.

$$(1) \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \frac{x_1^3}{6} - x_2 \end{aligned}$$

$$(2) \qquad \begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3 \end{aligned}$$

$$(3) \qquad \begin{aligned} \dot{x}_1 &= (1 - x_1)x_1 - \frac{2x_1x_2}{1 + x_1} \\ \dot{x}_2 &= \left(2 - \frac{x_2}{1 + x_1}\right) x_2 \end{aligned}$$

$$(4) \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) \end{aligned}$$

$$(5) \qquad \begin{aligned} \dot{x}_1 &= -x_1 + x_2(1 + x_1) \\ \dot{x}_2 &= -x_1(1 + x_1) \end{aligned}$$

$$(6) \qquad \begin{aligned} \dot{x}_1 &= (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right) \\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right) \end{aligned}$$

$$(7) \qquad \begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{aligned}$$

Exercise 1.14 Find all equilibrium points of the system

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_1x_2 \\ \dot{x}_2 &= bx_1^2 - cx_2 \end{aligned}$$

for all positive real values of a, b, and c, and determine the type of each equilibrium.

Exercise 1.15 The system

$$\begin{aligned}\dot{x}_1 &= -x_1 - \frac{x_2}{\ln\sqrt{x_1^2+x_2^2}}\\ \dot{x}_2 &= -x_2 + \frac{x_1}{\ln\sqrt{x_1^2+x_2^2}}\end{aligned}$$

has an equilibrium point at the origin.

(a) Linearize the system about the origin and find the type of the origin as an equilibrium point of the linear system.

(b) Find the phase portrait of the nonlinear system near the origin, and show that the portrait resembles a stable focus.
Hint: Transform the equations into polar coordinates.

(c) Explain the discrepancy between the results of parts (a) and (b).

Exercise 1.16 For each of the following systems, construct the phase portrait using the isocline method and discuss the qualitative behavior of the system. You can use information about the equilibrium points or the vector field, but do not use a computer program to generate the phase portrait.

(1)
$$\begin{aligned}\dot{x}_1 &= x_2 \cos x_1\\ \dot{x}_2 &= \sin x_1\end{aligned}$$

(2)
$$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)\left(x_1^2 + x_2^2 - 1\right)\\ \dot{x}_2 &= (x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right)\end{aligned}$$

(3)
$$\begin{aligned}\dot{x}_1 &= \left(1 - \frac{x_1}{x_2}\right)\\ \dot{x}_2 &= -\frac{x_1}{x_2}\left(1 - \frac{x_1}{x_2}\right)\end{aligned}$$

(4)
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - \tfrac{1}{3}x_1^3 - x_2\end{aligned}$$

(5)
$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2\\ \dot{x}_2 &= -x_1^3 - 4x_2\end{aligned}$$

(6)
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - \tfrac{1}{6}x_1^3\end{aligned}$$

(b) Take $\lambda = 5$. Construct the phase portrait, preferably using a computer program, and discuss the qualitative behavior of the system.

Exercise 1.22 The phase portraits of the following four systems are shown in Figures 1.37: parts (a), (b), (c), and (d), respectively. Mark the arrowheads and discuss the qualitative behavior of each system.

(1)
$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 - x_2\left(1 - x_1^2 + 0.1x_1^4\right) \end{aligned}$$

(2)
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + x_2 - 3\tan^{-1}(x_1 + x_2) \end{aligned}$$

(3)
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(0.5x_1 + x_1^3) \end{aligned}$$

(4)
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \psi(x_1 - x_2) \end{aligned}$$

where $\psi(y) = y^3 + 0.5y$ if $|y| \leq 1$ and $\psi(y) = 2y - 0.5$ if $|y| > 1$.

Exercise 1.23 An equivalent circuit of the Wien-Bridge oscillator is shown in Figure 1.38 [34], where $g(v)$ is a nonlinear voltage-controlled voltage source.

(a) With $x_1 = v_{C1}$ and $x_2 = v_{C2} = v$ as the state variables, show that the state equation is

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C_1R_1}[-x_1 + x_2 - g(x_2)] \\ \dot{x}_2 &= -\frac{1}{C_2R_1}[-x_1 + x_2 - g(x_2)] - \frac{1}{C_2R_2}x_2 \end{aligned}$$

(b) Let $C_1 = C_2 = R_1 = R_2 = 1$ and $g(v) = 3.234v - 2.195v^3 + 0.666v^5$. Construct the phase portrait, preferably using a computer program, and discuss the qualitative behavior of the system.

Exercise 1.24 Consider the mass-spring system with dry friction

$$\ddot{y} + ky + c\dot{y} + \eta(y, \dot{y}) = 0$$

where η is defined in Section 1.1.3. Use piecewise linear analysis to construct the phase portrait qualitatively (without numerical data), and discuss the qualitative behavior of the system.

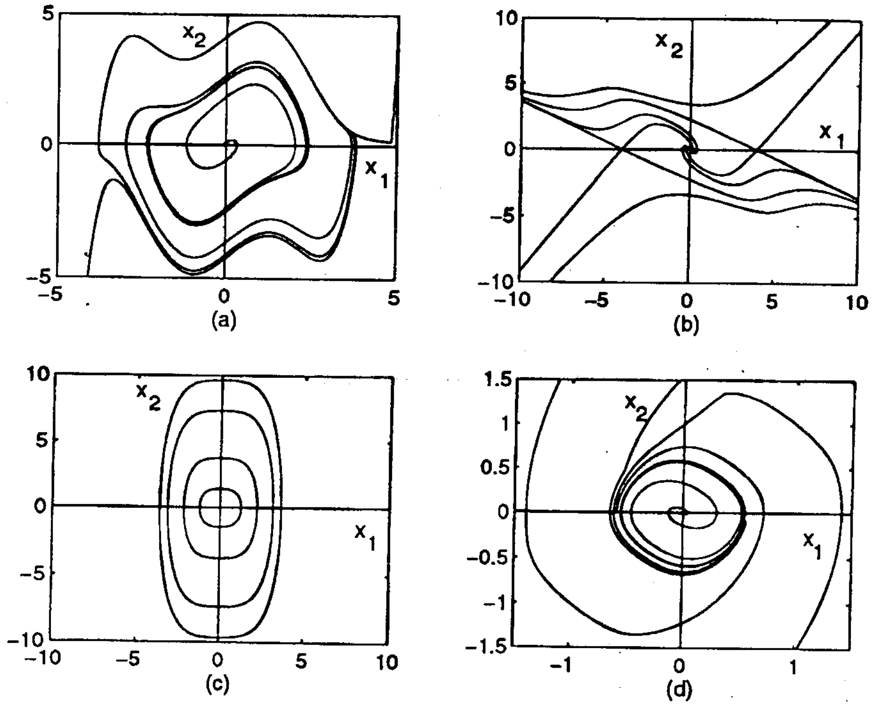

Figure 1.37: Exercise 1.22.

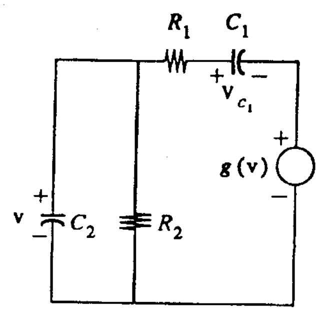

Figure 1.38: Exercise 1.23.

2.6 Exercises

Exercise 2.1 Show that, for any $x \in R^n$, we have

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\ \|x\|_2$$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\ \|x\|_\infty$$

$$\|x\|_\infty \leq \|x\|_1 \leq n\ \|x\|_\infty$$

Exercise 2.2 Show that, for any $m \times n$ real matrix A and any $n \times q$ real matrix B, we have

$$\|A\|_2 \leq \sqrt{\|A\|_1\ \|A\|_\infty}$$

$$\frac{1}{\sqrt{n}}\ \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\ \|A\|_\infty$$

$$\frac{1}{\sqrt{m}}\ \|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\ \|A\|_1$$

$$\|AB\|_p \leq \|A\|_p\ \|B\|_p$$

Exercise 2.3 Consider the set $S = \{x \in R^2 \mid -1 < x_i \leq 1,\ i = 1, 2\}$. Is S open? Is it closed? Find the closure, interior, and boundary of S.

Exercise 2.4 Let $u_T(t)$ be the unit step function, defined by $u_T(t) = 0$ for $t < T$ and $u_T(t) = 1$ for $t \geq T$.

(a) Show that $u_T(t)$ is piecewise continuous.

(b) Show that $f(t) = g(t)u_T(t)$, for any continuous function $g(t)$, is piecewise continuous.

(c) Show that the periodic square waveform is piecewise continuous.

Exercise 2.5 Let $f(x)$ be a continuously differentiable function that maps a convex domain $D \subset R^n$ into R^n. Suppose D contains the origin $x = 0$ and $f(0) = 0$. Show that

$$f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)\ d\sigma\ x, \quad \forall\ x \in D$$

Hint: Set $g(\sigma) = f(\sigma x)$ for $0 \leq \sigma \leq 1$ and use the fact that $g(1) - g(0) = \int_0^1 g'(\sigma)\ d\sigma$.

Exercise 2.6 Let $f(x)$ be continuously differentiable. Show that an equilibrium point x^* of $\dot{x} = f(x)$ is isolated if the Jacobian matrix $[\partial f/\partial x](x^*)$ is nonsingular.
Hint: Use the implicit function theorem.

Exercise 2.7 Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \le k_1 e^{-\alpha(t-t_0)} + \int_{t_0}^{t} e^{-\alpha(t-\tau)}[k_2 y(\tau) + k_3]\ d\tau$$

where k_1, k_2, k_3 are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall-Bellman inequality, show that

$$y(t) \le k_1 e^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\left[1 - e^{-(\alpha-k_2)(t-t_0)}\right]$$

Hint: Take $z(t) = y(t)e^{\alpha(t-t_0)}$ and find the inequality satisfied by z.

Exercise 2.8 Let $\mathcal{L}_2$ be the set of all piecewise continuous functions $u : [0,\infty) \to R^n$ with the property that each component is square integrable on $[0,\infty)$; that is, $\int_0^\infty |u_i(t)|^2\ dt < \infty$. Define $\|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\ dt}$. Show that $\|u\|_{\mathcal{L}_2}$ is a well-defined norm.
Hint: Use the Cauchy-Schwartz inequality

$$\int_a^b v(t)u(t)\ dt \le \sqrt{\int_a^b v^2(t)\ dt \int_a^b u^2(t)\ dt}$$

for all nonnegative scalar functions $u(t)$ and $v(t)$.

Exercise 2.9 Let l_2 be the set of all sequences of scalars $\{\eta_1, \eta_2, \ldots\}$ for which $\sum_{i=1}^{\infty} |\eta_i|^2 < \infty$ and define

$$\|x\|_l = \left[\sum_{i=1}^{\infty} |\eta_i|^2\right]^{1/2}$$

(a) Show that $\|x\|_l$ is a well-defined norm.

(b) Show that l_2 with the norm $\|x\|_l$ is a Banach space.

Exercise 2.10 Let $\mathcal{S}$ be the set of all half-wave symmetric periodic signals of fundamental frequency ω, which have finite energy on any finite interval. A signal $y \in \mathcal{S}$ can be represented by its Fourier series

$$y(t) = \sum_{k\ \text{odd}} a_k \exp(jk\omega t), \qquad \sum_{k\ \text{odd}} |a_k|^2 < \infty$$

Define

$$\|y\|_{\mathcal{S}} = \left[\frac{\omega}{\pi}\int_0^{2\pi/\omega} y^2(t)\ dt\right]^{1/2}$$

(b) Show that

$$\|x_0\|_2 \exp[-L(t-t_0)] \le \|x(t)\|_2 \le \|x_0\|_2 \exp[L(t-t_0)]$$

Exercise 2.23 Let $x : R \to R^n$ be a differentiable function that satisfies

$$\|\dot{x}(t)\| \le g(t), \quad \forall\ t \ge t_0$$

Show that

$$\|x(t)\| \le \|x(t_0)\| + \int_{t_0}^{t} g(s)\ ds$$

Exercise 2.24 ([36]) Give another proof of Theorem 2.2 by direct application of the contraction principle in the space of continuous functions with the norm

$$\|x\|_{\mathcal{X}} = \max_{t_0 \le t \le t_1} \left\{ e^{-L(t-t_0)/\rho} \|x(t)\| \right\}, \quad 0 < \rho < 1, \quad L > 0$$

Exercise 2.25 ([7]) Let $f : R \to R$. Suppose f satisfies the inequality

$$|f(x) - f(x_0)| \le M|x - x_0|^{\alpha}, \quad M \ge 0,\ \alpha \ge 0$$

in some neighborhood of x_0. Show that f is continuous at x_0 if $\alpha > 0$, and differentiable at x_0 if $\alpha > 1$.

Exercise 2.26 For each of the following functions $f : R \to R$, find whether (a) f is continuously differentiable at $x = 0$; (b) f is locally Lipschitz at $x = 0$; (c) f is continuous at $x = 0$; (d) f is globally Lipschitz; (e) f is uniformly continuous on R; (f) f is Lipschitz on $(-1, 1)$.

(1) $f(x) = \begin{cases} x^2 \sin(1/x), & \text{for } x \ne 0 \\ 0, & \text{for } x = 0 \end{cases}$

(2) $f(x) = \begin{cases} x^3 \sin(1/x), & \text{for } x \ne 0 \\ 0, & \text{for } x = 0 \end{cases}$

(3) $f(x) = \tan(\pi x/2)$

Exercise 2.27 For each of the following functions $f : R^n \to R^n$, find whether (a) f is continuously differentiable; (b) f is locally Lipschitz; (c) f is continuous; (d) f is globally Lipschitz; (e) f is uniformly continuous on R^n.

(1) $f(x) = \begin{bmatrix} x_1 + \text{sgn}\ (x_2) \\ x_2 \end{bmatrix}$ **(2)** $f(x) = \begin{bmatrix} x_1 + \text{sat}\ (x_2) \\ x_1 + \sin x_2 \end{bmatrix}$

(3) $f(x) = \begin{bmatrix} x_3 \text{sat}\ (x_1 + x_2) \\ x_2^2 \\ x_1 \end{bmatrix}$

Exercise 2.28 Let $D_r = \{x \in R^n \mid \|x\| < r\}$. For each of the following systems, represented as $\dot{x} = f(t, x)$, find whether (a) f is locally Lipschitz in x on D_r, for sufficiently small r; (b) f is locally Lipschitz in x on D_r, for any finite $r > 0$; (c) f is globally Lipschitz in x.

(1) The pendulum equation with friction and constant input torque (Section 1.1.1).

(2) The tunnel diode circuit (Example 1.2).

(3) The mass-spring equation with linear spring, linear viscous damping, dry friction, and zero external force (Section 1.1.3).

(4) The Van der Pol oscillator (Example 1.7).

(5) The closed-loop equation of a third-order adaptive control system (Section 1.1.6).

(6) The system $\dot{x} = Ax - B\psi(Cx)$, where A, B, and C are $n \times n$, $n \times 1$, and $1 \times n$ matrices, respectively, and $\psi(\cdot)$ is a dead-zone nonlinearity, defined by

$$\psi(y) = \begin{cases} y + d, & \text{for } y < -d \\ 0, & \text{for } -d \le y \le d \\ y - d, & \text{for } y > d \end{cases}$$

Exercise 2.29 Show that if $f_1 : R \to R$ and $f_2 : R \to R$ are locally Lipschitz, then $f_1 + f_2$, $f_1 f_2$ and $f_2 \circ f_1$ are locally Lipschitz.

Exercise 2.30 Let $f : R^n \to R^n$ be defined by

$$f(x) = \begin{cases} \frac{1}{\|Kx\|} Kx, & \text{if } g(x)\|Kx\| \ge \mu > 0 \\ \frac{g(x)}{\mu} Kx, & \text{if } g(x)\|Kx\| < \mu \end{cases}$$

where $g : R^n \to R$ is locally Lipschitz and nonnegative, and K is a constant matrix. Show that $f(x)$ is Lipschitz on any compact subset of R^n.

Exercise 2.31 Let $V : R \times R^n \to R$ be continuously differentiable. Suppose $V(t, 0) = 0$ for all $t \ge 0$ and

$$V(t,x) \ge c_1\|x\|^2; \quad \left\| \frac{\partial V}{\partial x}(t,x) \right\| \le c_4\|x\|, \quad \forall\ (t,x) \in [0,\infty) \times D$$

where c_1 and c_4 are positive constants and $D \subset R^n$ is a convex domain that contains the origin $x = 0$.

(a) Show that $V(t,x) \le \frac{1}{2}c_4\|x\|^2$ for all $x \in D$.
Hint: Use the representation $V(t,x) = \int_0^1 \frac{\partial V}{\partial x}(t,\sigma x)\, d\sigma\; x$.

(b) Show that the constants c_1 and c_4 must satisfy $2c_1 \le c_4$.

(c) Show that $W(t,x) = \sqrt{V(t,x)}$ satisfies the Lipschitz condition

$$|W(t,x_2) - W(t,x_1)| \le \frac{c_4}{2\sqrt{c_1}}\|x_2 - x_1\|, \quad \forall\, t \ge 0,\ \forall\, x_1, x_2 \in D$$

Exercise 2.32 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0,t_1] \times D$, for some domain $D \subset R^n$. Let W be a compact subset of D. Let $x(t)$ be the solution of $\dot{x} = f(t,x)$ starting at $x(t_0) = x_0 \in W$. Suppose $x(t)$ is defined and $x(t) \in W$ for all $t \in [t_0,T)$, $T < t_1$.

(a) Show that $x(t)$ is uniformly continuous on $[t_0,T)$.

(b) Show that $x(T)$ is defined and belongs to W, and $x(t)$ is a solution on $[t_0,T]$.

(c) Show that there is $\delta > 0$ such that the solution can be extended to $[t_0, T+\delta]$.

Exercise 2.33 Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x on $[t_0,t_1] \times D$, for some domain $D \subset R^n$. Let $y(t)$ be a solution of (2.1) on a maximal open interval $[t_0,T) \subset [t_0,t_1]$ with $T < \infty$. Let W be any compact subset of D. Show that there is some $t \in [t_0,T)$ with $y(t) \notin W$.
Hint: Use the previous exercise.

Exercise 2.34 Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, and

$$\|f(t,x)\| \le k_1 + k_2\|x\|, \quad \forall\, (t,x) \in [t_0,\infty) \times R^n$$

(a) Show that the solution of (2.1) satisfies

$$\|x(t)\| \le \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)] - 1\}$$

for all $t \ge t_0$ for which the solution exists.

(b) Can the solution have a finite escape time?

Exercise 2.35 Let $g : R^n \to R^n$ be continuously differentiable for all $x \in R^n$, and define $f(x)$ by

$$f(x) = \frac{1}{1 + g^T(x)g(x)}\, g(x)$$

Show that $\dot{x} = f(x)$, $x(0) = x_0$, has a unique solution defined for all $t \ge 0$.

Exercise 2.36 Show that the state equation

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{2x_2}{1+x_2^2}, \quad x_1(0) = a\\ \dot{x}_2 &= -x_2 + \frac{2x_1}{1+x_1^2}, \quad x_2(0) = b\end{aligned}$$

has a unique solution defined for all $t \geq 0$.

Exercise 2.37 Suppose the second-order system $\dot{x} = f(x)$, with a locally Lipschitz $f(x)$, has a limit cycle. Show that any solution that starts in the region enclosed by the limit cycle cannot have a finite escape time.

Exercise 2.38 ([36]) Let $x_1 : R \to R^n$ and $x_2 : R \to R^n$ be differentiable functions such that

$$\|x_1(a) - x_2(a)\| \leq \gamma, \quad \|\dot{x}_i(t) - f((t, x_i(t))\| \leq \mu_i, \text{ for } i = 1, 2$$

for $a \leq t \leq b$. If f satisfies the Lipschitz condition (2.2), show that

$$\|x_1(t) - x_2(t)\| \leq \gamma e^{L(t-a)} + (\mu_1 + \mu_2)\left[\frac{e^{L(t-a)} - 1}{L}\right], \quad \text{for } a \leq t \leq b$$

Exercise 2.39 Derive the sensitivity equations for the tunnel diode circuit of Example 1.2 as L and C vary from their nominal values.

Exercise 2.40 Derive the sensitivity equations for the Van der Pol oscillator of Example 1.7 as ϵ varies from its nominal value. Use the state equation in the x-coordinates.

Exercise 2.41 Repeat the previous exercise using the state equation in the z-coordinates.

Exercise 2.42 Derive the sensitivity equations for the system

$$\begin{aligned}\dot{x}_1 &= \tan^{-1}(ax_1) - x_1x_2\\ \dot{x}_2 &= bx_1^2 - cx_2\end{aligned}$$

as the parameters a, b, c vary from their nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$.

Exercise 2.43 Let $f(x)$ be continuously differentiable, $f(0) = 0$, and

$$\left\|\frac{\partial f_i}{\partial x}(x) - \frac{\partial f_i}{\partial x}(0)\right\|_2 \leq L_i\|x\|_2, \quad \text{for } 1 \leq i \leq n$$

Show that

$$\left\|f(x) - \frac{\partial f}{\partial x}(0)x\right\|_2 \leq L\|x\|_2^2, \quad \text{where } L = \sqrt{\sum_{i=1}^{n} L_i^2}$$

Exercise 2.44 Show, under the assumptions of Theorem 2.6, that the solution of (2.1) depends continuously on the initial time t_0.

Exercise 2.45 Let $f(t,x)$ and its partial derivative with respect to x be continuous in (t,x) for all $(t,x) \in [t_0,t_1] \times R^n$. Let $x(t,\eta)$ be the solution of (2.1) that starts at $x(t_0) = \eta$ and suppose $x(t,\eta)$ is defined on $[t_0,t_1]$. Show that $x(t,\eta)$ is continuously differentiable with respect to η and find the variational equation satisfied by $[\partial x/\partial \eta]$.
Hint: Put $y = x - \eta$ to transform (2.1) into

$$\dot{y} = f(t, y+\eta), \quad y(t_0) = 0$$

with η as a parameter.

Exercise 2.46 Let $f(t,x)$ and its partial derivative with respect to x be continuous in (t,x) for all $(t,x) \in R \times R^n$. Let $x(t,a,\eta)$ be the solution of (2.1) that starts at $x(a) = \eta$ and suppose that $x(t,a,\eta)$ is defined on $[a,t_1]$. Show that $x(t,a,\eta)$ is continuously differentiable with respect to a and η and let $x_a(t)$ and $x_\eta(t)$ denote $[\partial x/\partial a]$ and $[\partial x/\partial \eta]$, respectively. Show that $x_a(t)$ and $x_\eta(t)$ satisfy the identity

$$x_a(t) + x_\eta(t) f(a,\eta) \equiv 0, \quad \forall\, t \in [a, t_1]$$

Exercise 2.47 ([36]) Let $f : R \times R \to R$ be a continuous function. Suppose $f(t,x)$ is locally Lipschitz and nondecreasing in x for each fixed value of t. Let $x(t)$ be a solution of $\dot{x} = f(t,x)$ on an interval $[a,b]$. If the continuous function $y(t)$ satisfies the integral inequality

$$y(t) \le x(a) + \int_a^t f(s, y(s))\, ds$$

for $a \le t \le b$, show that $y(t) \le x(t)$ throughout this interval.

a class $\mathcal{KL}$ function and r_0 be a positive constant such that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in R^n \mid \|x\| < r_0\}$. Assume that the trajectory of the system satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ x(t_0) \in D_0,\ \forall\ t \geq t_0 \geq 0$$

Then, there is a continuously differentiable function $V : [0, \infty) \times D_0 \to R$ that satisfies the inequalities

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|)$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$, and $\alpha_4(\cdot)$ are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independent of t. ◇

Proof: Appendix A.6.

3.7 Exercises

Exercise 3.1 Consider a second-order autonomous system $\dot{x} = f(x)$. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable. Justify your answer using phase portraits.

(1) stable node	(2) unstable node	(3) stable focus
(4) unstable focus	(5) center	(6) saddle

Exercise 3.2 Consider the scalar system $\dot{x} = ax^p + g(x)$, where p is a positive integer and $g(x)$ satisfies $|g(x)| \leq k|x|^{p+1}$ in some neighborhood of the origin $x = 0$. Show that the origin is asymptotically stable if p is odd and $a < 0$. Show that it is unstable if p is odd and $a > 0$ or p is even and $a \neq 0$.

Exercise 3.3 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

(1) $$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 \end{aligned}$$

(2) $$\begin{aligned} \dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned}$$

(3) $$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 + x_1 \end{aligned}$$

(4) $$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 \\ \dot{x}_2 &= x_1 - x_2^3 \end{aligned}$$

(b) Show that all trajectories starting in the first quadrant to the right of the curve $x_1 x_2 = c$ (with sufficiently large $c > 0$) cannot reach the origin.

(c) Show that the origin is not globally asymptotically stable.

Hint: In part (b), consider $V(x) = x_1 x_2$; calculate $\dot{V}(x)$ and show that on the curve $V(x) = c$ the derivative $\dot{V}(x) > 0$ when c is large enough.

Exercise 3.9 (Krasovskii's Method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f/\partial x]$ satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall\, x \in R^n, \quad \text{where } P = P^T > 0$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x)x\, d\sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall\, x \in R^n$$

(b) Show that $V(x) = f^T(x) P f(x)$ is positive definite for all $x \in R^n$.

(c) Show that $V(x)$ is radially unbounded.

(d) Using $V(x)$ as a Lyapunov function candidate, show that the origin is globally asymptotically stable.

Exercise 3.10 Using Theorem 3.3, prove Lyapunov's first instability theorem: For the system (3.1), if a continuous function $V_1(x)$ with continuous first partial derivatives can be found in a neighborhood of the origin such that $V_1(0) = 0$, and $\dot{V}_1$ along the trajectories of the system is positive definite, but V_1 itself is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Exercise 3.11 Using Theorem 3.3, prove Lyapunov's second instability theorem: For the system (3.1), if in a neighborhood D of the origin, a continuously differentiable function $V_1(x)$ exists such that $V_1(0) = 0$ and $\dot{V}_1$ along the trajectories of the system is of the form $\dot{V}_1 = \lambda V_1 + W(x)$ where $\lambda > 0$ and $W(x) \geq 0$ in D, and if $V_1(x)$ is not negative definite or negative semidefinite arbitrarily near the origin, then the origin is unstable.

Exercise 3.12 Show that the origin of the following system is unstable.

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^6 \\ \dot{x}_2 &= x_2^3 + x_1^6 \end{aligned}$$

Exercise 3.17 Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + g(x_3) \\ \dot{x}_2 &= -g(x_3) \\ \dot{x}_3 &= -ax_1 + bx_2 - cg(x_3) \end{aligned}$$

where a, b, and c are positive constants and $g(\cdot)$ satisfies

$$g(0) = 0 \;\text{ and }\; yg(y) > 0, \;\; \forall\; 0 < |y| < k, \;\; k > 0$$

(a) Show that the origin is an isolated equilibrium point.

(b) With $V(x) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2 + \int_0^{x_3} g(y)\; dy$ as a Lyapunov function candidate, show that the origin is asymptotically stable.

(c) Suppose $yg(y) > 0\; \forall\; y \in R - \{0\}$. Is the origin globally asymptotically stable?

Exercise 3.18 ([67]) Consider Liénard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\; dy + \frac{1}{2}x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat (b) using $V(x) = \frac{1}{2}\left[x_2 + \int_0^{x_1} h(y)\; dy\right]^2 + \int_0^{x_1} g(y)\; dy$.

Exercise 3.19 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin x_1 - kx_1 - dx_2 - cx_3 \\ \dot{x}_3 &= -x_3 + x_2 \end{aligned}$$

where all coefficients are positive and $k > a$. Using

$$V(x) = 2a\int_0^{x_1} \sin y\; dy + kx_1^2 + x_2^2 + px_3^2$$

with some $p > 0$, show that the origin is globally asymptotically stable.

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \le -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(\|x\|^3)$, where $O(\|x\|^3)$ is a term bounded by $k\|x\|^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

Exercise 3.40 (Floquet theory) [29] Consider the linear system $\dot{x} = A(t)x$, where $A(t) = A(t+T)$. Let $\Phi(\cdot,\cdot)$ be the state transition matrix. Define a constant matrix B via the equation $\exp(BT) = \Phi(T,0)$, and let $P(t) = \exp(Bt)\Phi(0,t)$. Show that

(a) $P(t+T) = P(t)$.

(b) $\Phi(t,\tau) = P^{-1}(t)\exp[(t-\tau)B]P(\tau)$.

(c) the origin of $\dot{x} = A(t)x$ is exponentially stable if and only if B is Hurwitz.

Exercise 3.41 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 2x_1x_2 + 3t + 2 - 3x_1 - 2(t+1)x_2\end{aligned}$$

(a) Verify that $x_1(t) = t$, $x_2(t) = 1$ is a solution.

(b) Show that if $x(0)$ is sufficiently close to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $x(t)$ approaches $\begin{bmatrix} t \\ 1 \end{bmatrix}$ as $t \to \infty$.

Exercise 3.42 Consider the system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + g(t)x_2\\ \dot{x}_2 &= g(t)x_1 - 2x_2\end{aligned}$$

where $g(t)$ is continuously differentiable and $|g(t)| \le 1$ for all $t \ge 0$. Show that the origin is uniformly asymptotically stable.

Exercise 3.43 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - (1 + b\cos t)x_2\end{aligned}$$

Find $b^* > 0$ such that the origin is exponentially stable for all $|b| < b^*$.

[29] See [142] for a comprehensive treatment of Floquet theory.

Exercise 5.5 Consider the perturbed system $\dot{x} = Ax + B[u + g(t,x)]$, where $g(t,x)$ is continuously differentiable and satisfies $\|g(t,x)\|_2 \le k\|x\|_2$, $\forall\, t \ge 0$, $\forall\, x \in B_r$ for some $r > 0$. Let $P = P^T > 0$ be the solution of the Riccati equation

$$PA + A^TP + Q - PBB^TP + 2\alpha P = 0$$

where $Q \ge k^2 I$ and $\alpha > 0$. Show that $u = -B^TPx$ stabilizes the origin of the perturbed system.

Exercise 5.6 ([92]) Consider the perturbed system $\dot{x} = Ax + Bu + Dg(t,y)$, $y = Cx$, where $g(t,y)$ is continuously differentiable and satisfies $\|g(t,y)\|_2 \le k\|y\|_2$, $\forall\, t \ge 0$, $\forall\, \|y\|_2 \le r$ for some $r > 0$. Suppose the equation

$$PA + A^TP + \epsilon Q - \frac{1}{\epsilon}PBB^TP + \frac{1}{\gamma}PDD^TP + \frac{1}{\gamma}C^TC = 0$$

where $Q = Q^T > 0$, $\epsilon > 0$, and $0 < \gamma < 1/k$ has a positive definite solution $P = P^T > 0$. Show that $u = -(1/2\epsilon)B^TPx$ stabilizes the origin of the perturbed system.

Exercise 5.7 Consider the system

$$\begin{aligned}\dot{x}_1 &= -\alpha x_1 - \omega x_2 + (\beta x_1 - \gamma x_2)(x_1^2 + x_2^2)\\ \dot{x}_2 &= \omega x_1 - \alpha x_2 + (\gamma x_1 + \beta x_2)(x_1^2 + x_2^2)\end{aligned}$$

where $\alpha > 0$, β, γ, and $\omega > 0$ are constants.

(a) By viewing this system as a perturbation of the linear system

$$\begin{aligned}\dot{x}_1 &= -\alpha x_1 - \omega x_2\\ \dot{x}_2 &= \omega x_1 - \alpha x_2\end{aligned}$$

show that the origin of the perturbed system is exponentially stable with $\{\|x\|_2 \le r\}$ included in the region of attraction, provided $\beta|$ and $|\gamma|$ are sufficiently small. Find upper bounds on $|\beta|$ and $|\gamma|$ in terms of r.

(b) Using $V(x) = x_1^2 + x_2^2$ as a Lyapunov function candidate for the perturbed system, show that the origin is globally exponentially stable when $\beta < 0$ and exponentially stable with $\{\|x\|_2 < \sqrt{\alpha/\beta}\}$ included in the region of attraction when $\beta \ge 0$.

(c) Compare the results of (a) and (b) and comment on the conservative nature of the result of (a).

(c) Discuss the results of parts (a) and (b) in view of the robustness results of Section 5.1, and show that when $b = 0$ the origin is not globally exponentially stable.

Exercise 5.18 ([6]) Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + (x_1 + a)x_2, \quad a \neq 0 \\ \dot{x}_2 &= -x_1(x_1 + a) + bx_2 \end{aligned}$$

(a) Let $b = 0$. Show that the origin is globally asymptotically stable. Is it exponentially stable?

(b) Let $b > 0$. Show that the origin is exponentially stable for $b < \min\{1, a^2\}$.

(c) Show that the origin is not globally asymptotically stable for any $b > 0$.

(d) Discuss the results of parts (a)–(c) in view of the robustness results of Section 5.1, and show that when $b = 0$ the origin is not globally exponentially stable.

Hint: In part (d), note that the Jacobian matrix of the nominal system is not globally bounded.

Exercise 5.19 Consider the scalar system $\dot{x} = -x/(1 + x^2)$ and $V(x) = x^4$.

(a) Show that inequalities (5.20)–(5.22) are satisfied globally with

$$\alpha_1(r) = \alpha_2(r) = r^4; \ \alpha_3(r) = \frac{4r^4}{1 + r^2}; \ \alpha_4(r) = 4r^3$$

(b) Verify that these functions belong to class $\mathcal{K}_\infty$.

(c) Show that the right-hand side of (5.23) approaches zero as $r \to \infty$.

(d) Consider the perturbed system $\dot{x} = -x/(1 + x^2) + \delta$, where δ is a positive constant. Show that whenever $\delta > \frac{1}{2}$, the solution $x(t)$ escapes to ∞ for any initial state $x(0)$.

Exercise 5.20 Consider the scalar system $\dot{x} = -x^3 + e^{-t}$. Show that $x(t) \to 0$ as $t \to \infty$.

Exercise 5.21 For each of the following scalar systems, investigate the input-to-state stability.

(1) $\dot{x} = -(1 + u)x^3$ (2) $\dot{x} = -(1 + u)x^3 - x^5$

(3) $\dot{x} = -x + x^2u$ (4) $\dot{x} = x - x^3 + u$

Exercise 5.22 For each of the following systems, investigate the input-to-state stability.

(1) $\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + u \end{aligned}$ (2) $\begin{aligned} \dot{x}_1 &= (x_1 - x_2 + u)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (x_1 + x_2 + u)(x_1^2 + x_2^2 - 1) \end{aligned}$

(3) $\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 + x_1 + u \end{aligned}$ (4) $\begin{aligned} \dot{x}_1 &= -x_1 - x_2 + u_1 \\ \dot{x}_2 &= x_1 - x_2^3 + u_2 \end{aligned}$

Exercise 5.23 *Consider the system*

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + Bu \end{aligned}$$

where $\eta \in R^{n-r}$, $\xi \in R^r$ *for some* $1 \le r < n$, (A, B) *is controllable, and the system* $\dot{\eta} = f_0(\eta, \xi)$, *with* ξ *viewed as the input, is locally input-to-state stable. Find a state feedback control* $u = \gamma(\eta, \xi)$ *that stabilizes the origin of the full system. If* $\dot{\eta} = f_0(\eta, \xi)$ *is input-to-state stable, design the feedback control such that the origin of the closed-loop system is globally asymptotically stable.*

Exercise 5.24 Using Lemma 5.6, show that the origin of the system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -x_2^3 \end{aligned}$$

is asymptotically stable.

Exercise 5.25 Prove another version of Theorem 5.2, where all the assumptions are the same except that inequality (5.29) is replaced by

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \le -\alpha_3(\|x\|) + \psi(u)$$

where $\alpha_3(\cdot)$ is a class $\mathcal{K}_\infty$ function and $\psi(u)$ is a continuous function of u with $\psi(0) = 0$.

Exercise 5.26 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 \left[\left(\sin \frac{\pi x_2}{2} \right)^2 - 1 \right] \\ \dot{x}_2 &= -x_2 + u \end{aligned}$$

(a) With $u = 0$, show that the origin is globally asymptotically stable.

(b) Show that for any bounded input $u(t)$, the state $x(t)$ is bounded.

Exercise 4.5 ([80]) Consider the system

$$\begin{aligned}\dot{x}_a &= f_a(x_a, x_b)\\ \dot{x}_b &= A_b x_b + f_b(x_a, x_b)\end{aligned}$$

where $\dim(x_a) = n_1$, $\dim(x_b) = n_2$, A_b is a Hurwitz matrix, f_a and f_b are continuously differentiable, $[\partial f_b/\partial x_b](0,0) = 0$, and $f_b(x_a, 0) = 0$ in a neighborhood of $x_a = 0$.

(a) Show that if the origin $x_a = 0$ is an exponentially stable equilibrium point of $\dot{x}_a = f_a(x_a, 0)$, then the origin $(x_a, x_b) = (0,0)$ is an exponentially stable equilibrium point of the full system.

(b) Using the center manifold theorem, show that if the origin $x_a = 0$ is an asymptotically (but not exponentially) stable equilibrium point of $\dot{x}_a = f_a(x_a, 0)$, then the origin $(x_a, x_b) = (0,0)$ is an asymptotically stable equilibrium point of the full system.

Exercise 4.6 ([59]) For each of the following systems, investigate the stability of the origin using the center manifold theorem.

(1) $$\begin{aligned}\dot{x}_1 &= -x_2^2\\ \dot{x}_2 &= -x_2 + x_1^2 + x_1 x_2\end{aligned}$$

(2) $$\begin{aligned}\dot{x}_1 &= a x_1^2 - x_2^2, \quad a \neq 0\\ \dot{x}_2 &= -x_2 + x_1^2 + x_1 x_2\end{aligned}$$

(3) $$\begin{aligned}\dot{x}_1 &= -x_2 + x_1 x_3\\ \dot{x}_2 &= x_1 + x_2 x_3\\ \dot{x}_3 &= -x_3 - (x_1^2 + x_2^2) + x_3^2\end{aligned}$$

Exercise 4.7 For each of the following systems, investigate the stability of the origin using the center manifold theorem.

(1) $$\begin{aligned}\dot{x}_1 &= x_1 x_2^3\\ \dot{x}_2 &= -x_2 - x_1^2 + 2x_1^8\end{aligned}$$

(2) $$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^3(x_1 + x_2 - 1)\\ \dot{x}_2 &= x_2^3(x_1 + x_2 - 1)\end{aligned}$$

(3) $$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_2 + a x_1^3/(1 + x_1^2)\\ & \quad a \neq 0\end{aligned}$$

(4) $$\begin{aligned}\dot{x}_1 &= -2x_1 - 3x_2 + x_3 + x_3^2\\ \dot{x}_2 &= x_1 + x_1^2 + x_2\\ \dot{x}_3 &= x_1^2\end{aligned}$$

Exercise 4.8 ([28]) Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1 x_2 + a x_1^3 + b x_1 x_2^2\\ \dot{x}_2 &= -x_2 + c x_1^2 + d x_1^2 x_2\end{aligned}$$

Investigate the stability of the origin using the center manifold theorem for each of the following cases.

(1) $a + c > 0$ (2) $a + c < 0$
(3) $a + c = 0$ and $cd + bc^2 < 0$ (4) $a + c = 0$ and $cd + bc^2 > 0$
(5) $a + c = cd + bc^2 = 0$

Exercise 4.9 ([28]) Consider the system

$$\begin{aligned}\dot{x}_1 &= ax_1^3 + x_1^2 x_2 \\ \dot{x}_2 &= -x_2 + x_2^2 + x_1 x_2 - x_1^3\end{aligned}$$

Investigate the stability of the origin using the center manifold theorem for all possible values of the real parameter a.

Exercise 4.10 ([80]) Consider the system

$$\begin{aligned}\dot{x}_1 &= ax_1 x_2 - x_1^3 \\ \dot{x}_2 &= -x_2 + bx_1 x_2 + cx_1^2\end{aligned}$$

Investigate the stability of the origin using the center manifold theorem for all possible values of the real constants a, b, c.

Exercise 4.11 (Zubov's Theorem) Consider the system (3.1) and let $G \subset R^n$ be a domain containing the origin. Suppose there exist two functions $V : G \rightarrow R$ and $h : R^n \rightarrow R$ with the following properties:

- V is continuously differentiable and positive definite in G and satisfies the inequality
$$0 < V(x) < 1, \quad \forall\, x \in G - \{0\}$$
- As x approaches the boundary of G, or in case of unbounded G as $\|x\| \rightarrow \infty$, $\lim V(x) = 1$.
- h is continuous and positive definite on R^n.
- For $x \in G$, $V(x)$ satisfies the partial differential equation
$$\frac{\partial V}{\partial x} f(x) = -h(x)[1 - V(x)] \tag{4.15}$$

Show that $x = 0$ is asymptotically stable and G is the region of attraction.

Exercise 4.12 ([62]) Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= -h(x_1) + g_2(x_2) \\ \dot{x}_2 &= -g_1(x_1)\end{aligned}$$

Recalling the first case of Example 8.12, we see that $A(t)$ has an average with convergence function $\sigma(T) = 1/(T+1)$. Hence, the class $\mathcal{K}$ function of Theorem 8.4 is $\alpha(\eta) = k\eta$. Let $x(t,\epsilon)$ and $x_{\rm av}(t,\epsilon)$ denote solutions of the original and averaged systems which start from the same initial state. By Theorem 8.4,

$$\|x(t,\epsilon) - x_{\rm av}(t,\epsilon)\| = O(\epsilon), \quad \forall\, t \ge 0$$

$\triangle$

8.6 Exercises

Exercise 8.1 Using Theorem 2.5, verify inequality (8.4).

Exercise 8.2 If $\delta(\epsilon) = O(\epsilon)$, is it $O(\epsilon^{1/2})$? Is it $O(\epsilon^{3/2})$?

Exercise 8.3 If $\delta(\epsilon) = \epsilon^{1/n}$, where $n > 1$ is a positive integer, is there a positive integer N such that $\delta(\epsilon) = O(\epsilon^N)$?

Exercise 8.4 Consider the initial value problem

$$\begin{aligned}
\dot{x}_1 &= -(0.2+\epsilon)x_1 + \frac{\pi}{4} - \tan^{-1} x_1 + \epsilon \tan^{-1} x_2, \quad x_1(0) = \eta_1 \\
\dot{x}_2 &= -(0.2+\epsilon)x_2 + \frac{\pi}{4} - \tan^{-1} x_2 + \epsilon \tan^{-1} x_1, \quad x_2(0) = \eta_2
\end{aligned}$$

(a) Find an $O(\epsilon)$ approximation.

(b) Find an $O(\epsilon^2)$ approximation.

(c) Investigate the validity of the approximation on the infinite interval.

(d) Calculate, using a computer program, the exact solution, the $O(\epsilon)$ approximation, and the $O(\epsilon^2)$ approximation for $\epsilon = 0.1$, $\eta_1 = 0.5$, and $\eta_2 = 1.5$ on the time interval $[0, 3]$. Comment on the accuracy of the approximation.

Hint: In parts (a) and (b), it is sufficient to give the equations defining the approximation. You are not required to find an analytic closed-form expression for the approximation.

Exercise 8.5 Repeat Exercise 8.4 for the system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_2 + \epsilon x_1^3
\end{aligned}$$

In part (d), let $\epsilon = 0.1$, $\eta_1 = 1.0$, $\eta_2 = 0.0$, and the time interval be $[0, 5]$.

Exercise 8.6 Repeat Exercise 8.4 for the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= \epsilon x_1 - x_2 - \tfrac{1}{3}x_2^3\end{aligned}$$

In part (d), let $\epsilon = 0.2$, $\eta_1 = 1.0$, $\eta_2 = 0.0$, and the time interval be $[0, 4]$.

Exercise 8.7 ([150]) Repeat Exercise 8.4 for the system

$$\begin{aligned}\dot{x}_1 &= x_1 - x_1^2 + \epsilon x_1 x_2\\ \dot{x}_2 &= 2x_2 - x_2^2 - \epsilon x_1 x_2\end{aligned}$$

In part (d), let $\epsilon = 0.2$, $\eta_1 = 0.5$, $\eta_2 = 1.0$, and the time interval be $[0, 4]$.

Exercise 8.8 Repeat Exercise 8.4 for the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2(1 + x_1) + \epsilon(1 + x_1)^2\\ \dot{x}_2 &= -x_1(x_1 + 1)\end{aligned}$$

In part (d), let $\epsilon = -0.1$, $\eta_1 = -1$, and $\eta_2 = 2$. Repeat the calculation for $\epsilon = -0.05$ and $\epsilon = -0.2$ and comment on the accuracy of the approximation.

Exercise 8.9 Consider the initial value problem

$$\begin{aligned}\dot{x}_1 &= -x_1 + \epsilon x_2, \quad x_1(0) = \eta\\ \dot{x}_2 &= -x_2 - \epsilon x_1, \quad x_2(0) = \eta\end{aligned}$$

Find an $O(\epsilon)$ approximation. Calculate the exact and approximate solutions at $\epsilon = 0.1$ for two different sets of initial conditions: (1) $\eta = 1$, (2) $\eta = 10$. Comment on the approximation accuracy. Explain any discrepancy with Theorem 8.1.

Exercise 8.10 ([59]) Study, using the averaging method, each of the following scalar systems.

(1) $\dot{x} = \epsilon(x - x^2)\sin^2 t$ (2) $\dot{x} = \epsilon(x\cos^2 t - \frac{1}{2}x^2)$

(3) $\dot{x} = \epsilon(-x + \cos^2 t)$ (4) $\dot{x} = -\epsilon x \cos t$

Exercise 8.11 For each of the following systems, show that for sufficiently small $\epsilon > 0$, the origin is exponentially stable.

(1)
$$\begin{aligned}\dot{x}_1 &= \epsilon x_2\\ \dot{x}_2 &= -\epsilon(1 + 2\sin t)x_2 - \epsilon(1 + \cos t)\sin x_1\end{aligned}$$

(2)
$$\begin{aligned}\dot{x}_1 &= \epsilon[(-1 + 1.5\cos^2 t)x_1 + (1 - 1.5\sin t\cos t)x_2]\\ \dot{x}_2 &= \epsilon[(-1 - 1.5\sin t\cos t)x_1 + (-1 + 1.5\sin^2 t)x_2]\end{aligned}$$

(3)
$$\dot{x} = \epsilon\,(-x\sin^2 t + x^2\sin t + xe^{-t}), \quad \epsilon > 0$$

Exercise 8.12 Consider the system $\dot{y} = Ay + \epsilon g(t, y, \epsilon)$, $\epsilon > 0$, where the $n \times n$ matrix A has only simple eigenvalues on the imaginary axis.

(a) Show that $\exp(At)$ and $\exp(-At)$ are bounded for all $t \geq 0$.

(b) Show that the change of variables $y = \exp(At)x$ transforms the system into the form $\dot{x} = \epsilon f(t, x, \epsilon)$ and give an expression for f in terms of g and $\exp(At)$.

Exercise 8.13 ([150]) Study Mathieu's equation $\ddot{y} + (1 + 2\epsilon \cos 2t)y = 0$, $\epsilon > 0$, using the averaging method.
Hint: Use Exercise 8.12.

Exercise 8.14 ([150]) Study the equation $\ddot{y} + y = 8\epsilon(\dot{y})^2 \cos t$ using the averaging method.
Hint: Use Exercise 8.12.

Exercise 8.15 Apply the averaging method to study the existence of limit cycles for each of the following second-order systems. If there is a limit cycle, estimate its location in the state plane and the period of oscillation, and determine whether it is stable or unstable.

(1) $\ddot{y} + y = -\epsilon \dot{y}(1 - y^2)$ (2) $\ddot{y} + y = \epsilon \dot{y}(1 - y^2) - \epsilon y^3$

(3) $\ddot{y} + y = -\epsilon\left(1 - \frac{3\pi}{4}|y|\right)\dot{y}$ (4) $\ddot{y} + y = -\epsilon\left(1 - \frac{3\pi}{4}|\dot{y}|\right)\dot{y}$

(5) $\ddot{y} + y = -\epsilon(\dot{y} - y^3)$ (6) $\ddot{y} + y = \epsilon \dot{y}(1 - y^2 - \dot{y}^2)$

Exercise 8.16 Consider Rayleigh's equation

$$m\frac{d^2u}{dt^2} + ku = \lambda\left[1 - \alpha\left(\frac{du}{dt}\right)^2\right]\frac{du}{dt}$$

where m, k, λ, and α are positive constants.

(a) Using the dimensionless variables $y = u/u^*$, $\tau = t/t^*$, and $\epsilon = \lambda/\lambda^*$, where $(u^*)^2\alpha k = m/3$, $t^* = \sqrt{m/k}$, and $\lambda^* = \sqrt{km}$, show that the equation can be normalized to

$$\ddot{y} + y = \epsilon\left(\dot{y} - \tfrac{1}{3}\dot{y}^3\right)$$

where $\dot{y}$ denotes the derivative of y with respect to τ.

(b) Apply the averaging method to show that the normalized Rayleigh equation has a stable limit cycle. Estimate the location of the limit cycle in the plane $(y, \dot{y})$.

(c) Using a numerical algorithm, obtain the phase portrait of the normalized Rayleigh equation in the plane $(y, \dot{y})$ for

$$\text{(i) } \epsilon = 1, \quad \text{(ii) } \epsilon = 0.1, \quad \text{and} \quad \text{(iii) } \epsilon = 0.01,$$

Compare with the results of part (b).

Exercise 8.17 Consider Duffing's equation

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = A\cos\omega t$$

where A, a, c, k, m and ω are positive constants.

(a) Taking $x_1 = y$, $x_2 = \dot{y}$, $\tau = \omega t$, and $\epsilon = 1/\omega$, show that the equation can be represented as $\frac{dx}{d\tau} = \epsilon f(\tau, x, \epsilon)$.

(b) Show that the system has an exponentially stable periodic solution for sufficiently large ω. Estimate the frequency of oscillation and the location of the periodic orbit in the phase plane.

Exercise 8.18 Verify (8.37).
Hint: Start from (8.36). In majorizing $\sigma(t)$, use the fact that $\sigma(t)$ is bounded for $t \leq 1/\sqrt{\eta}$, while for $t \geq 1/\sqrt{\eta}$ use the inequality $\sigma(t) \leq \sigma(1/\sqrt{\eta})$.

Exercise 8.19 Study, using general averaging, the scalar system

$$\dot{x} = \epsilon\left(\sin^2 t + \sin 1.5t + e^{-t}\right) x$$

Exercise 8.20 ([151]) The output of an nth-order linear time-invariant single-input–single-output system can be represented by $y(t) = \theta^T w(t)$, where θ is a $(2n+1)$-dimensional vector of constant parameters and $w(t)$ is an auxiliary signal which can be synthesized from the system's input and output, without knowing θ. Suppose that the vector θ is unknown and denote its value by θ^*. In identification experiments, the parameter $\theta(t)$ is updated using an adaptation law of the form $\dot{\theta} = -\epsilon e(t)w(t)$, where $e(t) = [\theta(t) - \theta^*]^T w(t)$ is the error between the actual system's output and the estimated output using $\theta(t)$. Let $\phi(t) = \theta(t) - \theta^*$ denote the parameter error.

(a) Show that $\dot{\phi} = \epsilon A(t)\phi$, where $A(t) = -w(t)w^T(t)$.

(b) Using (general) averaging, derive a condition on $w(t)$ which ensures that, for sufficiently small ϵ, $\theta(t) \to \theta^*$ as $t \to \infty$.

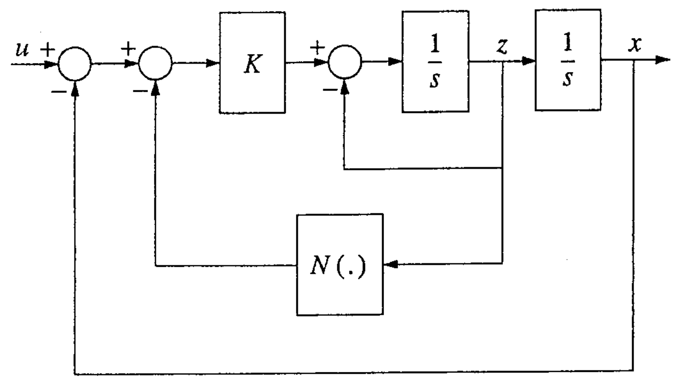

Figure 9.14: Exercise 9.4.

Exercise 9.2 Consider the RC circuit of Figure 9.13 and suppose that the resistor R_1 is small relative to R_2, while $C_1 = C_2 = C$. Represent the system in the standard singularly perturbed form.

Exercise 9.3 Consider the tunnel diode circuit of Section 1.1.2 and suppose that the inductance L is relatively small so that the time constant L/R is much smaller than the time constant CR. Represent the system as a standard singularly perturbed model with $\epsilon = L/CR^2$.

Exercise 9.4 ([95]) The feedback system of Figure 9.14 has a high-gain amplifier K and a nonlinear element $N(\cdot)$. Represent the system as a standard singularly perturbed model with $\epsilon = 1/K$.

Exercise 9.5 Show that if the Jacobian $[\partial g/\partial y]$ satisfies the eigenvalue condition (9.16), then there exist constants k, γ, and ρ_0 for which inequality (9.15) is satisfied.

Exercise 9.6 Show that if there is a Lyapunov function satisfying (9.17)–(9.18), then inequality (9.15) is satisfied with the estimates (9.19).

Exercise 9.7 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x^2 + z, & x(0) = \xi \\ \epsilon \dot{z} &= x^2 - z + 1, & z(0) = \eta \end{aligned}$$

(a) Find an $O(\epsilon)$ approximation of x and z on the time interval $[0, 1]$.

(b) Let $\xi = \eta = 0$. Simulate x and z for

$$(1)\ \epsilon = 0.1 \quad \text{and} \quad (2)\ \epsilon = 0.05$$

and compare with the approximation derived in part (a). In carrying out the computer simulation note that the system has a finite escape time shortly after $t = 1$.

Exercise 9.8 Consider the singular perturbation problem

$$\begin{aligned} \dot{x} &= x + z, \quad x(0) = \xi \\ \epsilon\dot{z} &= -\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{2}(2x + z)\right), \quad z(0) = \eta \end{aligned}$$

(a) Find an $O(\epsilon)$ approximation of x and z on the time interval $[0, 1]$.

(b) Let $\xi = \eta = 1$. Simulate x and z for

$$(1)\ \epsilon = 0.2 \quad \text{and} \quad (2)\ \epsilon = 0.1$$

and compare with the approximation derived in part (a).

Exercise 9.9 Consider the singularly perturbed system

$$\begin{aligned} \dot{x} &= z \\ \epsilon\dot{z} &= -x - \epsilon z - \exp(z) + 1 + u(t) \end{aligned}$$

Find the reduced and boundary-layer models and analyze the stability properties of the boundary-layer model.

Exercise 9.10 ([95]) Consider the singularly perturbed system

$$\begin{aligned} \dot{x} &= \frac{x^2 t}{z} \\ \epsilon\dot{z} &= -(z + xt)(z - 2)(z - 4) \end{aligned}$$

(a) How many reduced models can this system have?

(b) Investigate boundary-layer stability for each reduced model.

(c) Let $x(0) = 1$ and $z(0) = a$. Find an $O(\epsilon)$ approximation of x and z on the time interval $[0, 1]$ for all values of a in the interval $[-2, 6]$.

Exercise 9.11 ([95]) Find the exact slow manifold of the singularly perturbed system

$$\begin{aligned} \dot{x} &= xz^3 \\ \epsilon\dot{z} &= -z - x^{4/3} + \tfrac{4}{3}\epsilon x^{16/3} \end{aligned}$$

Exercise 9.12 ([95]) How many slow manifolds does the following system have? Which of these manifolds will attract trajectories of the system?

$$\begin{aligned} \dot{x} &= -xz \\ \epsilon\dot{z} &= -(z - \sin^2 x)(z - e^{ax})(z - 2e^{2ax}), \quad a > 0 \end{aligned}$$

Exercise 9.13 ([95]) Consider the linear autonomous singularly perturbed system

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z \\ \epsilon\dot{z} &= A_{21}x + A_{22}z \end{aligned}$$

where $x \in R^n$, $z \in R^m$, and A_{22} is a Hurwitz matrix.

(a) Show that for sufficiently small ϵ, the system has an exact slow manifold $z = -L(\epsilon)x$, where L satisfies the algebraic equation

$$-\epsilon L(A_{11} - A_{12}L) = A_{21} - A_{22}L$$

(b) Show that the change of variables

$$\eta = z + L(\epsilon)x$$

transforms the system into a block triangular form.

(c) Show that the eigenvalues of the system cluster into a group of n slow eigenvalues of order $O(1)$ and m fast eigenvalues of order $O(1/\epsilon)$.

(d) Let $H(\epsilon)$ be the solution of the linear equation

$$\epsilon(A_{11} - A_{12}L)H - H(A_{22} + \epsilon L A_{12}) + A_{12} = 0$$

Show that the similarity transformation

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

transforms the system into the block modal form

$$\dot{\xi} = A_s(\epsilon)\xi; \quad \epsilon\dot{\eta} = A_f(\epsilon)\eta$$

where the eigenvalues of A_s and A_f/ϵ are, respectively, the slow and fast eigenvalues of the full singularly perturbed system.

(e) Show that the component of the fast mode in x is $O(\epsilon)$.

(f) Give an independent proof of Tikhonov's theorem in the current case.

(c) Let $\nu(w, y, u) = V(w, u) + \frac{1}{2}y^T y$. Verify that there exist $\epsilon^* > 0$ and $\mu^* > 0$ such that for all $0 < \epsilon < \epsilon^*$ and $0 \leq \mu < \mu^*$, the inequality

$$\dot{\nu} \leq -\alpha\nu + \beta\sqrt{\nu}\,\|\dot{u}\|$$

is satisfied in a neighborhood of $(w, y) = (0, 0)$ for some $\alpha > 0$ and $\beta \geq 0$.

(d) Show that there exist positive constants ρ_1, ρ_2, ρ_3, ρ_4 such that if $0 < \epsilon < \rho_1$, $\mu < \rho_2$, $\|x(0) - h(u(0))\| < \rho_3$, and $\|z(0) - g(x(0), u(0))\| < \rho_4$, then $x(t)$ and $z(t)$ are uniformly bounded for all $t \geq 0$, and $x(t) - h(u(t))$ is uniformly ultimately bounded by $k\mu$ for some $k > 0$.

(e) Show that if (in addition to the previous assumptions) $u(t) \to u_\infty$ and $\dot{u}(t) \to 0$ as $t \to \infty$, then $x(t) \to h(u_\infty)$ as $t \to \infty$.

Hint: In part (b), use Lemma 5.12 and in parts (d) and (e), apply the comparison lemma

Exercise 9.29 Apply Theorem 9.4 to study the asymptotic behavior of the system

$$\begin{aligned} \dot{x} &= -x + z - \sin t \\ \epsilon\dot{z} &= -z + \sin t \end{aligned}$$

as $t \to \infty$.

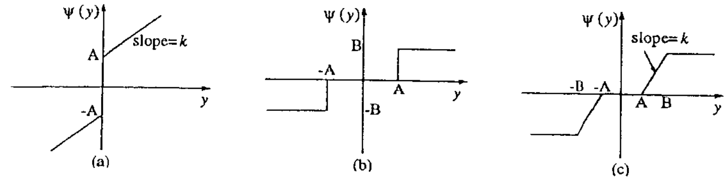

Figure 10.27: Exercise 10.33.

Exercise 10.33 For each odd nonlinearity $\psi(y)$ on the following list, verify the given expression of the describing function $\Psi(a)$.

(1) $\psi(y) = y^5; \quad \Psi(a) = 5a^4/8$

(2) $\psi(y) = y^3|y|; \quad \Psi(a) = 32a^3/15\pi$

(3) $\psi(y)$: Figure 10.27(a); $\quad \Psi(a) = K + \dfrac{4A}{\pi a}$

(4) $\psi(y)$: Figure 10.27(b)

$$\Psi(a) = \begin{cases} 0, & \text{for } a \le A \\ (4B/\pi a)[1-(A/a)^2]^{1/2}, & \text{for } a \ge A \end{cases}$$

(5) $\psi(y)$: Figure 10.27(c)

$$\Psi(a) = \begin{cases} 0, & \text{for } a \le A \\ K[1-N(a/A)], & \text{for } A \le a \le B \\ K[N(a/B)-N(a/A)], & \text{for } a \ge B \end{cases}$$

where

$$N(x) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{1}{x}\right) + \frac{1}{x}\sqrt{1-\left(\frac{1}{x}\right)^2}\right]$$

Exercise 10.34 Consider the feedback connection of Figure 10.17 with

$$G(s) = \frac{1-s}{s(s+1)}$$

Using the describing function method, investigate the possibility of existence of periodic solutions and the possible frequency and amplitude of oscillation for each of the following nonlinearities.

(1) $\psi(y) = y^5$.

(2) The nonlinearity of Exercise 10.33, part (5), with $A = 1$, $B = \frac{3}{2}$, and $K = 2$.

Exercise 10.35 Consider the feedback connection of Figure 10.17 with

$$G(s) = \frac{1}{(s+1)^6} \quad \text{and} \quad \psi(y) = \text{sgn}(y)$$

Using the describing function method, investigate the possibility of existence of periodic solutions and the possible frequency and amplitude of oscillation.

Exercise 10.36 Repeat Exercise 10.35 with

$$G(s) = \frac{s+6}{s(s+2)(s+3)} \quad \text{and} \quad \psi(y) = \text{sgn}(y)$$

Exercise 10.37 Repeat Exercise 10.35 with

$$G(s) = \frac{s}{s^2-s+1} \quad \text{and} \quad \psi(y) = y^5$$

Exercise 10.38 Consider the feedback connection of Figure 10.17 with

$$G(s) = \frac{5(s+0.25)}{s^2(s+2)^2}$$

Using the describing function method, investigate the possibility of existence of periodic solutions and the possible frequency and amplitude of oscillation for each of the following nonlinearities.

(1) The nonlinearity of Exercise 10.33, part (3), with $A = 1$ and $K = 2$.

(2) The nonlinearity of Exercise 10.33, part (4), with $A = 1$ and $B = 1$.

(3) The nonlinearity of Exercise 10.33, part (5), with $A = 1$, $B = \frac{3}{2}$, and $K = 2$.

Exercise 10.39 Consider the feedback connection of Figure 10.17 with

$$G(s) = \frac{2bs}{s^2-bs+1}, \quad \psi(y) = \text{sat}(y)$$

Using the describing function method, show that for sufficiently small $b > 0$ the system has a periodic solution. Confirm your conclusion by applying Theorem 10.9 and estimate the frequency and amplitude of oscillation.