Homework set 1

Linearization & phase portrait

1. For each of the following systems, find all equilibrium points and determine the type of each isolated equilibrium. Use Matlab to compute the eigenvalues.

(1) $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + \frac{1}{6}x_1^3 - x_2$

(2) $\dot{x}_1 = -x_1 + x_2$, $\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$

(3) $\dot{x}_1 = -x_1 + x_2(1 + x_1)$, $\dot{x}_2 = -x_1(1 + x_1)$

(4) $\dot{x}_1 = -x_1^3 + x_2$, $\dot{x}_2 = x_1 - x_2^3$

2. The phase portrait of the following four systems are shown in Figure 1: parts (a), (b), (c), and (d), respectively. Mark the arrowheads and discuss the qualitative behavior of each system.

(1) $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1 - x_2(1 - x_1^2 + 0.1x_1^4)$

(2) $\dot{x}_1 = x_2$, $\dot{x}_2 = x_1 + x_2 - 3\tan^{-1}(x_1 + x_2)$

(3) $\dot{x}_1 = x_2$, $\dot{x}_2 = -(0.5x_1 + x_1^3)$

(4) $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_2 - \psi(x_1 - x_2)$

Where $\psi(y) = y^3 + 0.5y$ if $|y| \leq 1$ and $\psi(y) = 2y - 0.5$ if $|y| > 1$.

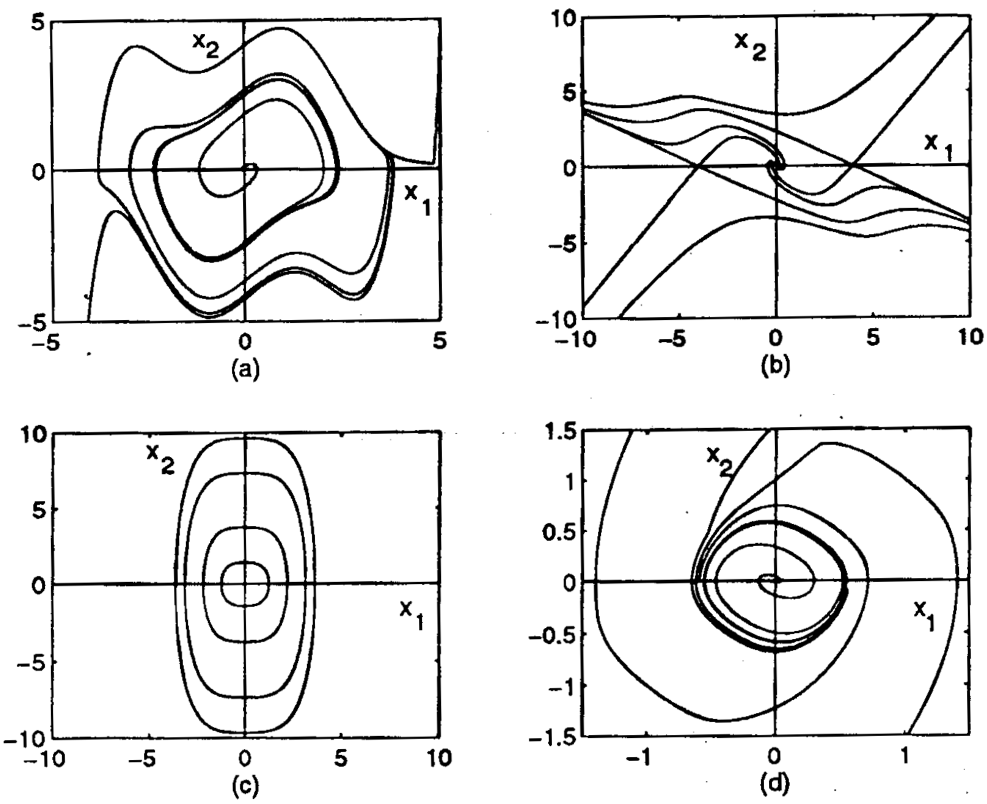

Figure 1: Exercise 2

Homework set 2

Gronwall's inequality & comparison principle

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1. Let $y(t)$ be a nonnegative scalar function that satisfies the inequality

$$y(t) \quad \leq \quad k_1 \mathrm{e}^{-\alpha(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{-\alpha(t-\tau)} \big[k_2 y(\tau) + k_3\big] \mathrm{d}\tau \tag{1}$$

where k_1, k_2, and k_3 are nonnegative constants and α is a positive constant that satisfies $\alpha > k_2$. Using the Gronwall-Bellman inequality, show that

$$y(t) \leq k_1 \mathrm{e}^{-(\alpha-k_2)(t-t_0)} + \frac{k_3}{\alpha - k_2}\big[1 - \mathrm{e}^{-(\alpha-k_2)(t-t_0)}\big] \tag{2}$$

Hint: Take $z(t) = y(t)\mathrm{e}^{\alpha(t-t_0)}$ and find the inequality satisfied by z. This exercise shows how to apply Gronwall lemma to the case: $y(t) < \lambda(t) + \rho(t) \int \mu(\tau) y(\tau) \mathrm{d}\tau$.

2. Using the comparison principle, show that if ν, l_1, and l_2 are functions that satisfy

$$\dot{\nu} \leq -c\nu + l_1(t)\nu + l_2(t), \qquad \nu(0) \geq 0 \tag{3}$$

and if $c > 0$, then

$$\nu(t) \leq \big(\nu(0)\mathrm{e}^{-ct} + \|l_2\|_1\big)\mathrm{e}^{\|l_1\|}, \tag{4}$$

where $\|\cdot\|$ denotes the L_1 norm defined as

$$\|f\|_1 \quad = \quad \int_0^\infty |f(t)| \mathrm{d}t. \tag{5}$$

Using Gronwall's lemma show that

$$\nu(t) \leq \big(\nu(0)\mathrm{e}^{-ct} + \|l_2\|_1\big)\big(1 + \|l_1\|_1 \mathrm{e}^{\|l_1\|_1}\big). \tag{6}$$

Which of the two bounds is less conservative?

3. Consider the system:

$$\dot{x} \quad = \quad -cx + y^{2m} x \cos^2(x) \tag{7}$$

$$\dot{y} \quad = \quad -y^3. \tag{8}$$

Using either Gronwall's inequality or the comparison principle, show that

a) $x(t)$ is bounded for all $t \geq 0$ whenever $c = 0$ and $m > 1$.

b) $x(t) \to 0$ as $t \to \infty$ whenever $c > 0$ and $m = 1$.

4. Consider the system

$$\dot{x} = -x + yx\sin(x) \tag{9}$$

$$\dot{y} = -y + zy\sin(y) \tag{10}$$

$$\dot{z} = -z. \tag{11}$$

Using Gronwall's lemma (twice), show that

$$|x(t)| \leq |x_0| \mathrm{e}^{|y_0|\mathrm{e}^{|z_0|}} \mathrm{e}^{-t}, \qquad \forall t \geq 0. \tag{12}$$

Homework set 3

Sensitivity equations

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1. Derive the sensitivity equations for the system

$$\dot{x}_1 = \tan^{-1}(\alpha x_1) - x_1 x_2 \tag{1}$$

$$\dot{x}_2 = b x_1^2 - c x_2 \tag{2}$$

as the parameters a, b, and c vary from their nominal values $a_0 = 1, b_0 = 0$, and $c_0 = 1$.

2. Calculate exactly (in closed form) the sensitivity function at $\lambda_0 = 1$ for the system

$$\dot{x} = -\lambda x^3. \tag{3}$$

What is the approximation $x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$ for $\lambda = 7/2$?

1. Prove global stability of the origin of the system

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = -\frac{x_1}{1+x_2^2}. \tag{2}$$

2. Prove global asymptotatic stability of the origin of the system

$$\dot{x}_1 = -x_2^3 \tag{3}$$
$$\dot{x}_2 = x_1 - x_2. \tag{4}$$

3. Prove global asymptotatic stability of the origin of the system

$$\dot{x}_1 = x_2 - (2x_1^2 + x_2^2)x_1 \tag{5}$$
$$\dot{x}_2 = -x_1 - 2(2x_1^2 + x_2^2)x_2. \tag{6}$$

Is the origin locally exponentially stable and why or why not?

4. Consider the system

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{7}$$
$$\dot{x}_2 = -\frac{x_1^2}{1+x_1^2}. \tag{8}$$

Show that the equilibrium $x_1 = x_2 = 0$ is globally stable and that

$$\lim_{t\to\infty} x_1(t) = 0 \tag{9}$$

Hint: Seek a Lyapunov function in the form

$$V(x_1, x_2) = \phi(x_1) + x_2^2, \tag{10}$$

where the function $\phi(\cdot)$ is to be found. Make sure that your Lyapunov function $V(\cdot,\cdot)$ is positive definite and radially unbounded.

Homework set 5

Chetaev's theorem

1. With Chetaev's theorem, show that the equilibrium at the origin of the following three systems is unstable:

a)

$$\dot{x} = x^3 + xy^3 \tag{1}$$
$$\dot{y} = -y + x^2 \tag{2}$$

b)

$$\dot{\xi} = \eta + \xi^3 + 3\xi\eta^2 \tag{3}$$
$$\dot{\eta} = -\xi + \eta^3 + 3\eta\xi^2 \tag{4}$$

c)

$$\dot{x} = |x|x + xy\sqrt{|y|} \tag{5}$$
$$\dot{y} = -y + |x|\sqrt{|y|}. \tag{6}$$

Homework set 6

LaSalle's theorem

1. Using the Lyapunov function candidate

$$V = \frac{1}{4}x^4 + \frac{1}{2}y^2 + \frac{1}{4}z^4 \tag{1}$$

study stability of the origin of the system

$$\begin{aligned} \dot{x} &= y & (2)\\ \dot{y} &= -x^3 - y^3 - z^3 & (3)\\ \dot{z} &= -z + y. & (4) \end{aligned}$$

2. Consider the system

$$\begin{aligned} \dot{x} &= -x + yx + z\cos(x) & (5)\\ \dot{y} &= -x^2 & (6)\\ \dot{z} &= -x\cos(x) & (7) \end{aligned}$$

a) Determine all the equilibria of the system.

b) Show that the equilibrium $x = y = z = 0$ is globally stable.

c) Show that $x(t) \to 0$ as $t \to \infty$.

d) Show that $z(t) \to 0$ as $t \to \infty$.

3. Which of the state variables of the following system are guaranteed to converge to zero from any initial condition?

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1x_3 & (8)\\ \dot{x}_2 &= -x_1 - x_2 + x_2x_3 & (9)\\ \dot{x}_3 &= -x_1^2 - x_2^2. & (10) \end{aligned}$$

Homework set 7

Perturbation theory

1. Consider the system

$$\dot{x}_1 = x_2 \tag{1}$$

$$\dot{x}_2 = -x_1 - x_2 + \epsilon x_1^3. \tag{2}$$

a) Find an $O(\epsilon)$ approximation.

b) Find an $O(\epsilon^2)$ approximation.

c) Investigate the validity of the approximation on the infinite interval.

2. Repeat exercise 1 for the system

$$\dot{x}_1 = -x_1 + x_2(1 + x_1) + \epsilon(1 + x_1)^2 \tag{3}$$

$$\dot{x}_2 = -x_1(x_1 + 1). \tag{4}$$

Homework set 8

Averaging theory

1. Using averaging theory, analyze the following system:

$$\dot{x} = \epsilon\big[-x + 1 - 2(y + \sin(t))^2\big] \tag{1}$$

$$\dot{y} = \epsilon z \tag{2}$$

$$\dot{z} = \epsilon\Big[-z - \sin(t)\big[\frac{1}{2}x + (y + \sin(t))^2\big]\Big]. \tag{3}$$

2. Analyze the following system using the method of averaging for large ω:

$$\dot{x}_1 = \big(x_2\sin(\omega t) - 2\big)x_1 - x_3 \tag{4}$$

$$\dot{x}_2 = -x_2 + \big(x_2^2\sin(\omega t) - 2x_3\cos(\omega t)\big)\cos(\omega t) \tag{5}$$

$$\dot{x}_3 = 2x_2 - \sin(x_3) + \big(4x_2\sin(\omega t) + x_3\big)\sin(\omega t) \tag{6}$$

3. Consider the second-order system

$$\dot{x}_1 = \sin(\omega t)y_1 \tag{7}$$

$$\dot{x}_2 = \cos(\omega t)y_2 \tag{8}$$

$$y_1 = \big[x_1 + \sin(\omega t)\big]\big[x_2 + \cos(\omega t) - x_1 - \sin(\omega t)\big] \tag{9}$$

$$y_2 = \big[x_2 + \cos(\omega t)\big]\big[x_1 + \sin(\omega t) - x_2 - \cos(\omega t)\big]. \tag{10}$$

Show that for sufficiently large ω there exists an exponentially stable periodic orbit in an $O(1/\omega)$ neighborhood of the origin $x_1 = x_2 = 0$.

Hint: the following functions have a zero mean over the interval $[0, 2\pi]$: $\sin(\tau), \cos(\tau)$, $\sin(\tau)\cos(\tau), \sin^3(\tau), \cos^3(\tau), \sin^2(\tau)\cos(\tau)$, and $\sin(\tau)\cos^2(\tau)$.

4. Consider Rayleigh's equation

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + ku = \lambda\Big[1 - \alpha\Big(\frac{\mathrm{d}u}{\mathrm{d}t}\Big)^2\Big]\frac{\mathrm{d}u}{\mathrm{d}t} \tag{11}$$

where m, k, λ, and α are positive constants.

a) Using the dimensionless variables $y = \frac{u}{u^*}, \tau = \frac{t}{t^*}$, and $\epsilon = \frac{\lambda}{\lambda^*}$, where $(u^*)^2\alpha k = \frac{m}{3}$, $t^* = \sqrt{\frac{m}{k}}$, and $\lambda^* = \sqrt{km}$, show that the equation can be normalized to

$$\ddot{y} + y = \epsilon(\dot{y} - \frac{1}{3}\dot{y}^3) \tag{12}$$

where $\dot{y}$ denotes the derivative of y with respect to τ.

b) Apply the averaging method to show that the normalized Rayleigh equation has a stable limit cycle. Estimate the location of the limit cycle in the plane $(y, \dot{y})$.

Homework set 9

Singular perturbation

1. Using singular perturbation theory, study local exponential stability of the origin of the system

$$\dot{x} = y \tag{1}$$

$$\dot{y} = -z \tag{2}$$

$$\epsilon\dot{z} = -z + \sin(x) + y. \tag{3}$$

Is the origin globally exponentially stable?

2. Consider the following control system:

$$\dot{x} = A_{11}x + A_{12}z + B_1u \tag{4}$$

$$\epsilon\dot{z} = A_{21}x + A_{22}z. \tag{5}$$

Assume that the matrix A_{22} is Hurwitz and that there exists a matrix/vector K (of appropriate dimensions) such that

$$A_{11} - A_{12}A_{22}^{-1}A_{21} + B_1K \tag{6}$$

is also Hurwitz. Prove that the "partial-state" feedback law

$$u = Kx \tag{7}$$

exponentially stabilizes the equilibrium $(x, z) = (0, 0)$ for sufficiently small ϵ.

3. Find the exact slow manifold of the singularly perturbed system

$$\dot{x} = xz^3 \tag{8}$$

$$\epsilon\dot{z} = -z - x^{4/3} + \frac{4}{3}\epsilon x^{16/3}. \tag{9}$$

Hint: try substitute the quasi-steady state into the (exact) manifold condition.

4. How many slow manifolds does the following system have? Which of these manifolds will attract trajectories of the system?

$$\dot{x} = -xz \tag{10}$$

$$\epsilon\dot{z} = -(z - \sin^2(x))(z - \mathrm{e}^{ax})(z - 2\mathrm{e}^{2ax}), \qquad a > 0. \tag{11}$$

Homework set 10

Input-to-state stability

1. Show that the following system is ISS and determine its gain function:

$$\dot{x} = -x^3 + xu. \tag{1}$$

2. Show that the following system is ISS and determine its gain function:

$$\dot{x} = -x + u^3. \tag{2}$$

3. Show that the following system is ISS and guess its gain function:

$$\dot{x} = -x^3 + xy \tag{3}$$

$$\dot{y} = -y + u^3. \tag{4}$$

4. Consider the system

$$\dot{x} = -x + y^3 \tag{5}$$

$$\dot{y} = -y - \frac{x}{\sqrt{1+x^2}} + z^2 \tag{6}$$

$$\dot{z} = -z + u. \tag{7}$$

Show that this system is ISS using the Lyapunov function

$$V = \sqrt{1+x^2} - 1 + \frac{1}{4}y^4 + \frac{1}{2}z^8. \tag{8}$$

5. Show that the following system is ISS

$$\dot{x} = -x + x^{1/3}y + p^2 \tag{9}$$

$$\dot{y} = -y - x^{4/3} + p^3 \tag{10}$$

$$\dot{p} = -p + u. \tag{11}$$

1. For each of the following systems, investigate the stability of the origin using the center manifold theorem.

a)

$$\dot{x}_1 = -x_2^2 \quad (1)$$

$$\dot{x}_2 = -x_2 + x_1^2 + x_1 x_2 \quad (2)$$

b)

$$\dot{x}_1 = a x_1^2 - x_2^2, \qquad a \neq 0 \quad (3)$$

$$\dot{x}_2 = -x_2 + x_1^2 + x_1 x_2 \quad (4)$$

2. Consider the system

$$\dot{x}_1 = x_1 x_2 + a x_1^3 + b x_1 x_2^2 \quad (5)$$

$$\dot{x}_2 = -x_2 + c x_1^2 + d x_1^2 x_2 \quad (6)$$

Investigate the stability of the origin using the center manifold theorem for each of the following cases

i) $a + c > 0$

ii) $a + c < 0$

iii) $a + c = 0$ and $cd + bc^2 < 0$

iv) $a + c = 0$ and $cd + bc^2 > 0$

3. Using the center manifold theorem, determine whether the origin of the following system is asymptotically stable:

$$\dot{x}_1 = -x_2 + x_1 x_3 \quad (7)$$

$$\dot{x}_2 = x_1 + x_2 x_3 \quad (8)$$

$$\dot{x}_3 = -x_3 - (x_1^2 + x_2^2) + x_3^2. \quad (9)$$

4. Using the center manifold theorem, determine whether the origin of the following system is asymptotically stable:

$$\dot{y} = yz + 2y^3 \quad (10)$$

$$\dot{z} = -z - 2y^2 - 4y^4 - 2y^2 z. \quad (11)$$

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Homework set 12

Describing function

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1. For each of odd nonlinearity $\psi(y)$ on the following list, verify the given expression of the describing function $\Psi(a)$.

i) $\psi(y) = y^5$; $\Psi(a) = 5a^4/8$

ii) $\psi(y) = y^3|y|$; $\Psi(a) = 32a^3/15\pi$

iii) $\psi(y)$: Figure 1; $\Psi(a) = k + 4A/a\pi$

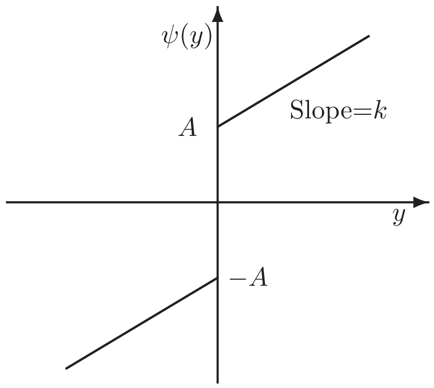

Figure 1: $\psi(y)$

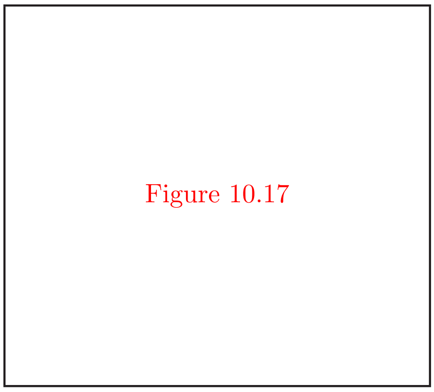

Figure 2: Feedback connection

2. Consider the feedback connection of Figure 2 with

$$G(s) = \frac{1}{(s+1)^6} \qquad \text{and} \qquad \psi(y) = \mathrm{sgn}(y). \tag{1}$$

Using the describing function method, investigate the possibility of existence of periodic solutions and the possible frequency and amplitude of oscillation.

3. Repeat exercise 2 with

$$G(s) = \frac{s}{s^2 - s + 1} \qquad \text{and} \qquad \psi(y) = y^5. \tag{2}$$

4. Consider the feedback system with a linear block $\frac{1}{s(s+1)(s+2)}$ and a nonlinearity $\mathrm{sgn}(y)+|y|y$ (note that it is an odd nonlinearity, so the describing functions method applies, and note that the first term was already studied in class). First, find the describing function for the nonlinearity. Then, determine if the feedback system is likely to have any periodic solutions.