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Derive the sensitivity equations for the system

\[ \begin{align*}
\dot{x}_1 &= \tan^{-1}(ax_1) - x_1x_2, \\
\dot{x}_2 &= bx_2^2 - cx_2
\end{align*} \]  \tag{1}

as the parameters \(a\), \(b\), and \(c\) vary from their nominal values \(a_0 = 1\), \(b_0 = 0\), and \(c_0 = 1\).

Solution

Let the state variable and parameter be \(x = (x_1, x_2)^T\), \(\lambda = (a, b, c)^T\), and the nominal value be \(\lambda_0 = (1, 0, 1)^T\). Writing the nonlinear dynamics \(f(x, \lambda) = (f_1(x, \lambda), f_2(x, \lambda))^T = (\tan^{-1}(ax_1) - x_1x_2, bx_2^2 - cx_2)^T\), we obtain its Jacobian matrices \(A(t, \lambda)\) and \(B(t, \lambda)\) as

\[ A(t, \lambda) = \frac{\partial f}{\partial x} = \left( \begin{array}{cc}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{array} \right) = \left( \begin{array}{cc}
a\left(1 + a^2x_1^2\right) - 1 & -x_1 \\
2bx_1 & -c
\end{array} \right) \]  \tag{3}

\[ B(t, \lambda) = \frac{\partial f}{\partial \lambda} = \left( \begin{array}{ccc}
\frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} \\
\frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c}
\end{array} \right) = \left( \begin{array}{ccc}
x_1 & 0 & 0 \\
0 & x_1 & -x_2
\end{array} \right) \]  \tag{4}

Let \(S(t)\) be a sensitivity function s.t. \(S(t) = \frac{\partial x}{\partial \lambda}|_{\lambda = \lambda_0}\), then the sensitivity equation is given by

\[ \dot{S}(t) = A(t, \lambda_0)S(t) + B(t, \lambda_0), \quad S(0) = 0 \]  \tag{5}

Substituting the nominal value into (3) and (4), we obtain

\[ \dot{S}(t) = \left( \begin{array}{cc}
\frac{1}{1 + x_1^2} & -x_2 \\
0 & -1
\end{array} \right) S(t) + \left( \begin{array}{cc}
x_1 & 0 \\
0 & x_1^2
\end{array} \right), \quad S(0) = 0 \]  \tag{6}

which is the derivation of sensitivity equation.
Calculate exactly (in closed form) the sensitivity function at $\lambda_0 = 1$ for the system

$$\dot{x} = -\lambda x^3$$

(7)

What is the approximation $x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$ for $\lambda = 7/2$?

**Solution:**

The explicit solution of (7) is obtained by

$$x(t, \lambda) = \frac{x_0}{(2\lambda x_0^2 t + 1)^{1/2}}.$$  

(8)

Let $S(t)$ be a sensitivity function s.t. $S(t) = \frac{\partial x}{\partial \lambda}|_{\lambda=\lambda_0}$. Then, taking derivative of (8) in $\lambda$, it is given by

$$S(t) = \left. \frac{\partial}{\partial \lambda} \frac{x_0}{(2\lambda x_0^2 t + 1)^{1/2}} \right|_{\lambda=\lambda_0} = \left. -\frac{x_0^3 t}{(2\lambda x_0^2 t + 1)^{3/2}} \right|_{\lambda=1}$$

(9)

Therefore, the approximation for $\lambda = 7/2$ is

$$x(t, 7/2) \approx x(t, 1) + S(t)(7/2 - 1)$$

$$= \frac{x_0}{(2x_0^2 t + 1)^{1/2}} - \frac{5}{2} \frac{x_0^3 t}{(2x_0^2 t + 1)^{3/2}}$$

$$= \frac{x_0(-x_0^2 t + 2)}{2(2x_0^2 t + 1)^{3/2}}.$$  

(10)