AMES 207 Prof. M. Krstic

Nonlinear Systems February 12, 1998

MIDTERM EXAM

Take home. Open books and notes.
Total points: 30
Due February 13, 1998, at 5:00 p.m. in Professor Krstic's office.

Late submissions will not be accepted. Collaboration not allowed.

Problem 1.

Consider the system:

$$\begin{aligned}\dot{x} &= -cx + y^{2m} x \cos^2 x \\ \dot{y} &= -y^3\end{aligned}$$

Using either Gronwall's inequality or the comparison principle, show that

a) (5 pts) $x(t)$ is bounded for all $t \geq 0$ whenever $c = 0$ and $m > 1$.

b) (5 pts) $x(t) \to 0$ as $t \to \infty$ whenever $c > 0$ and $m = 1$.

Problem 2.

Consider the system:

$$\begin{aligned}\dot{x} &= -x + yx + z \cos x \\ \dot{y} &= -x^2 \\ \dot{z} &= -x \cos x\end{aligned}$$

a) (2 pts) Determine all the equilibria of the system.

b) (2 pts) Show that the equilibrium $x = y = z = 0$ is globally stable.

c) (2 pts) Show that $x(t) \to 0$ as $t \to \infty$.

d) (2 pts) Show that $z(t) \to 0$ as $t \to \infty$.

Problem 3.

With Chetaev's theorem, show that the equilibrium at the origin of the following two systems is unstable:

a)(4pts)

$$\begin{aligned} \dot{x} &= x^3 + xy^3 \\ \dot{y} &= -y + x^2 \end{aligned}$$

b)(4pts)

$$\begin{aligned} \dot{\xi} &= \eta + \xi^3 + 3\xi\eta^2 \\ \dot{\eta} &= -\xi + \eta^3 + 3\eta\xi^2 \end{aligned}$$

Problem 4.

(4 pts) Calculate exactly (in closed form) the sensitivity function at $\lambda_0 = 0$ for the system

$$\dot{x} = -x + \tan^{-1}(\lambda x).$$

How accurate is the approximation $x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$ for large λ, say $\lambda > 1$?

AMES 207 Prof. M. Krstic

NONLINEAR SYSTEMS Winter 1998

FINAL EXAM

Take home. Open books and notes.

Total points: 35

Due Saturday, March 14, 1998, at 12:00 noon in Professor Krstic's office.

Late submissions will not be accepted. Collaboration not allowed.

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1. (8 pts) Consider the system

$$\dot{x}_i = x_{i+1} - c_i x_i - k_i s_i(x) x_i + w_i(x) d,$$

$i = 1, \ldots, n, x_{n+1} = 0$, where $c_i, k_i > 0$ and $|w_i(x)| \leq s_i(x)$. Show that the system is ISS w.r.t. d. What is the type of the gain function (linear, quadratic, exponential, . . .)?

2. (9 pts) Using the center manifold theorem, determine whether the origin of the following system is asymptotically stable:

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1 x_3 \\ \dot{x}_2 &= x_1 + x_2 x_3 \\ \dot{x}_3 &= -x_3 - (x_1^2 + x_2^2) + x_3^2 \end{aligned}$$

3. (9 pts) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \sin \omega t \left((x_1 + \sin \omega t)^2 + x_3 \right) \\ \dot{x}_3 &= -x_3^n - (x_1 + \sin \omega t)^2 + \frac{1}{2}. \end{aligned}$$

a) For $n = 1$, show that for sufficiently large ω there exists an exponentially stable periodic orbit in an $O\left(\frac{1}{\omega}\right)$ − neighborhood of the origin.

b) What can you claim for $n = 3$?

4. (9 pts) Show that, for sufficiently small ε, the origin of the system

$$\begin{aligned} \dot{x} &= x^2 + z + \cos(\varepsilon y) - 1 \\ \varepsilon \dot{y} &= -y + x^2 - x \\ \varepsilon^2 \dot{z} &= -z + \sin y + \varepsilon x^3 \end{aligned}$$

is exponentially stable. (Hint: treat $\mu = \varepsilon^2$ as a separate small parameter.) Since the system has three (rather than two) time scales, it has three levels of invariant manifolds – slow, medium, and fast. Without going into high accuracy, give the approximate expressions for these manifolds and discuss the trajectories of the system.

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS February 15, 2000

MIDTERM EXAM

Take home. Open books and notes.
Total points: 25
Due February 16, 2000, at 5:00 p.m. in Professor Krstic's office.

Late submissions will not be accepted. Collaboration not allowed.

Problem 1.

(6 pts) Let g, h, and y be three positive functions on $(0, \infty)$ such that

$$\begin{aligned} \int_0^\infty g(t)dt &\leq C_1 \\ \int_0^\infty \mathrm{e}^{\delta t} h(t)dt &\leq C_2 \\ \int_0^\infty \mathrm{e}^{\delta t} y(t)dt &\leq C_3 \,, \end{aligned}$$

where δ, C_1, C_2, C_3 are positive constants. Assuming that

$$\dot{y} \leq g(t)y + h(t)\,, \qquad \forall t \geq 0$$

using Gronwall's lemma show that

$$y(t) \leq [C_2 + \delta C_3 + y(0)]\mathrm{e}^{C_1 - \delta t}\,.$$

Problem 2.

(6 pts) Consider the system:

$$\begin{aligned} \dot{x} &= A(x,y)x + B(x)y \\ \dot{y} &= -GB(x)^{\mathrm{T}} x\,, \end{aligned}$$

where $x(t), y(t)$ are vectors of arbitrary dimensions, $A(x,y)$ is a matrix valued function that satisfies

$$A(x,y) + A(x,y)^{\mathrm{T}} \leq -qI\,, \qquad q > 0$$

and G is a positive definite symmetric matrix. Show that the equilibrium $x = 0, y = 0$ is globally stable, that $x(t)$ converges to zero, and that $y(t)$ converges to the null space of $B(0)$. If you can't solve the problem for general G, solve it for $G = I$ to receive partial credit.

Problem 3.

(6 pts) With Chetaev's theorem, show that the equilibrium at the origin of the following system is unstable:

$$\begin{aligned} \dot{x} &= |x|x + xy\sqrt{|y|} \\ \dot{y} &= -y + |x|\sqrt{|y|}\,. \end{aligned}$$

Don't worry about uniqueness of solutions.

Problem 4.

(7 pts) Calculate exactly (in closed form) the sensitivity function at $\lambda_0 = 1$ for the system

$$\dot{x} = -\lambda x^3.$$

What is the approximation $x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$ for $\lambda = 7/2$?

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2001

FINAL EXAM

Take home. Open books and notes.

Total points: 50

Due Thursday, March 22, 2001, at 1:00 pm in Professor Krstic's office.

Late submissions will not be accepted. Collaboration not allowed.

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1. (9 pts) Using the Lyapunov function candidate $V = \frac{1}{2}(x^2 + y^2 + z^2)$, study stability of the origin of the system

$$\begin{aligned} \dot{x} &= -x + x^2 z \\ \dot{y} &= z \\ \dot{z} &= -y - z - x^3 \,. \end{aligned}$$

2. (9 pts) Show that the following system is ISS

$$\begin{aligned} \dot{x} &= -x + x^{1/3} y + p^2 \\ \dot{y} &= -y - x^{4/3} + p^3 \\ \dot{p} &= -p + u \,. \end{aligned}$$

3. (8 pts) Using the center manifold theorem, determine whether the origin of the following system is asymptotically stable:

$$\begin{aligned} \dot{y} &= yz + 2y^3 \\ \dot{z} &= -z - 2y^2 - 4y^4 - 2y^2 z \,. \end{aligned}$$

4. (8 pts) Using averaging theory, analyze the following system:

$$\begin{aligned} \dot{x} &= \epsilon[-x + 1 - 2(y + \sin t)^2] \\ \dot{y} &= \epsilon z \\ \dot{z} &= \epsilon \left\{ -z - \sin t \left[\frac{1}{2} x + (y + \sin t)^2 \right] \right\} \,. \end{aligned}$$

5. (8 pts) Using singular perturbation theory, study local exponential stability of the origin of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -z \\ \epsilon \dot{z} &= -z + \sin x + y \,. \end{aligned}$$

Is the origin globally exponentially stable?

6. (8 pts) Consider the feedback system with a linear block $\dfrac{1}{s(s+1)(s+2)}$ (like in class) and a nonlinearity $\mathrm{sgn}(y) + |y|y$ (note that it is an odd nonlinearity, so the describing functions method applies, and note that the first term was already studied in class). First, find the describing function for the nonlinearity. Then, determine if the feedback system is likely to have any periodic solutions.

MAE 281A Prof. M. Krstic

Nonlinear Systems Winter 2002

FINAL EXAM

Take home. Open books and notes.

Total points: 65

Due Friday, March 22, 2002, at 4:00 pm in Professor Krstic's office.

Late submissions will not be accepted. Collaboration not allowed.
Each problem is worth 13 points

Problem 1. Consider the system

$$\begin{aligned}\dot{x} &= -x + yx\sin x\\ \dot{y} &= -y + zy\sin y\\ \dot{z} &= -z\,.\end{aligned}$$

Using Gronwall's lemma (twice), show that

$$|x(t)| \le |x_0|\mathrm{e}^{|y_0|\mathrm{e}^{|z_0|}}\mathrm{e}^{-t}, \qquad \forall t \ge 0.$$

Problem 2. Analyze uniform stability of the origin of the linear time-varying system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -y-(2+\sin t)x\end{aligned}$$

using the Lyapunov function

$$V = x^2 + \frac{y^2}{2+\sin t}.$$

Does your analysis guarantee that $y(t) \to 0$ as $t \to \infty$.

Problem 3. Using the Lyapunov function candidate

$$V = \frac{x^4}{4} + \frac{y^2}{2} + \frac{z^4}{4},$$

study stability of the origin of the system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x^3-y^3-z^3\\ \dot{z} &= -z+y\,.\end{aligned}$$

Problem 4. Using the Chetaev function

$$V = xz$$

prove that the origin of the system

$$\begin{aligned}
\dot{x} &= yz + az \\
\dot{y} &= -xz \\
\dot{z} &= xy + ax,
\end{aligned}$$

where $a > 0$ is a constant, is unstable. (This problem is related to instability of rigid body spinning motion around the "intermediate" axis.)

Problem 5. Show that the ISS gain function of the system

$$\begin{aligned}
\dot{x} &= (3 + \cos(u))\mathrm{sgn}(x) \log \frac{1}{1 + |x|} + y \\
\dot{y} &= -(2 + x^2)|y|y + \frac{x^2}{1 + x^2} u
\end{aligned}$$

from u to x is

$$\gamma(r) = \mathrm{e}^{\sqrt{r}} - 1.$$

Hint: Use Lyapunov functions of the form $V_1(x) = |x|$ and $V_2(y) = |y|$.

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2003

FINAL EXAM

Take home. Open books and notes.

Total points: 65

Due Friday, March 14, 2003, at 4:00 pm in Professor Krstic's office. (4pm is a hard deadline, I am leaving the office at 4pm.)

Late submissions will not be accepted. Collaboration not allowed.

Problem 1. Consider the system

$$\begin{aligned}
\dot{p} &= -\mu\epsilon\gamma \sin\phi \left(r^2 \sin^2 t - q\right) \\
\dot{q} &= \mu\epsilon\gamma \left(r^2 \sin^2 t - q\right) \\
\dot{\phi} &= \mu\epsilon \\
\dot{r} &= \mu r \cos^2 t \left(1 + (p + \sin\phi)^2 - r^2 \sin^2 t\right) ,
\end{aligned}$$

where $r(0) \geq 0$ (note that this implies that $r(t) \geq 0$ for all time because $r = 0$ sets $\dot{r} = 0$). Denote

$$x = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \text{and} \qquad x_{\mathrm{r}}(t) = \begin{bmatrix} 0 \\ 3 \\ 2\sqrt{1 + \sin(\mu\epsilon t)^2} \end{bmatrix}$$

Show that, for sufficiently small μ, ϵ, and γ, the solution $x(t, \mu, \epsilon, \gamma)$ locally exponentially converges to $x_{\mathrm{r}}(t) + \mathrm{O}(\mu + \epsilon + \gamma)$, at least on a finite time interval.

<u>Hint.</u> This is a complicated problem that involves *four time scales.* (I hope it will not take you as much time to solve it as it took me to construct it and double check its solvability.) The four time scales are (going from fastest to slowest):

- $\sin t$
- r
- $\sin(\mu\epsilon t)$
- p, q.

Your analysis should apply the following steps:

- One step of averaging for the complete system, treating μ as small.
- One step of singular perturbation, treating r as fast and ϵ as small, and introducing the new time $\tau = \epsilon t$ to put the system (after averaging) into the standard singular perturbation form. To derive the boundary layer model, you will need to introduce "another" time variable $\hat{t} = \tau/\epsilon = t$.

- A second step of averaging on the (p, q) system with $\phi = \mu\tau = \mu\epsilon t$ as time, treating γ as small (again).

Note that the hardest part of the problem is not to mechanically perform the approximations but to connect them all, through appropriate theorems, to draw the final conclusion. Make sure you do quote the theorems as you go from the last step of simplification backwards towards the original system. I will be quite unimpressed to see that you only know how to calculte an average system or how to find a quasi steady state. Note that, in order to draw the final conclusion, exponential stability needs to be satisfied every step of the way.

Problem 2. Consider the system

$$\begin{aligned}\dot{x} &= -x + xz + y(1-y) \\ \dot{y} &= -x(1-y) \\ \dot{z} &= -x^2 .\end{aligned}$$

Give the most precise statement you can on stability and convergence (global and local) of solutions of this system. Note that this is an open ended problem. Since the system has two entire lines of equilibria, analyzing *all* of them might take many days of work, leading you to use not only the Lyapunov, LaSalle invariance, and linearization theorems, but even the center manifold and Chetaev theorems. Go as far as you can with your ideas and these tools.

Hints. If you try to study individual equilibria, the first thing to note is that, since they all belong to continuous sets of equilibria, none of them can be *asymptotically* stable. So, the equilibria fall into one of the two categories: stable or unstable. By taking linearizations around equilibria, you will note that all of them have at least one eigenvalue at zero in their Jacobians. So, unless you find them unstable by linearization, you may need the center manifold theorem. Note that, since none of the equilibria are asymptotically stable, the center manifold theorem should work only for the equilibria that happen to be unstable. Don't immediately look for complicated center manifolds—trivial ones ($h(\cdot) = 0$) will carry you a long way. For those equilibria that happen to be stable you can use Lyapunov functions parametrized by the equilibria. For one of the equilibria, (0,1,1), even center manifold is not enough and the stability question needs to be resolved by direct Lyapunov or Chetaev. I personally have not figured out this one as of this writing.

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2004

FINAL EXAM

Take home. Open books and notes.

Total points: 65

Due Wednesday, March 17, 2004, at 1:00 pm in Professor Krstic's office.

Late submissions will not be accepted. Collaboration not allowed.

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1. (13 pts) Using the Lyapunov function candidate $V = \frac{1}{2}(x^2 + y^2 + z^2)$, study stability of the origin of the system

$$\begin{aligned} \dot{x} &= -x + x^2 z \\ \dot{y} &= z \\ \dot{z} &= -y - z - x^3 \,. \end{aligned}$$

2. (13 pts) Show that the following system is ISS

$$\begin{aligned} \dot{x} &= -x + x^{1/3} y + p^2 \\ \dot{y} &= -y - x^{4/3} + p^3 \\ \dot{p} &= -p + u \,. \end{aligned}$$

3. (13 pts) Using averaging theory, analyze the following system:

$$\begin{aligned} \dot{x} &= \epsilon[-x + 1 - 2(y + \sin t)^2] \\ \dot{y} &= \epsilon z \\ \dot{z} &= \epsilon \left\{ -z - \sin t \left[\frac{1}{2} x + (y + \sin t)^2 \right] \right\} \,. \end{aligned}$$

4. (13 pts) Using singular perturbation theory, study local exponential stability of the origin of the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -z \\ \epsilon \dot{z} &= -z + \sin x + y \,. \end{aligned}$$

Is the origin globally exponentially stable?

5. (13 pts) Consider the feedback system with a linear block $\dfrac{1}{s(s+1)(s+2)}$ (like in class) and a nonlinearity $\mathrm{sgn}(y) + |y|y$ (note that it is an odd nonlinearity, so the describing functions method applies, and note that the first term was already studied in class). First, find the describing function for the nonlinearity. Then, determine if the feedback system is likely to have any periodic solutions.

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2006

FINAL EXAM

Take home. Open books and notes.

Total points: 65

Due Tuesday, March 21, 2006, at 5:00 pm in Professor Krstic's office.

Collaboration not allowed.

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1. (20 pts) Show that the following system has an unstable equilibrium at the origin:

$$\begin{aligned}\dot{x}_1 &= x_1^3 + 2x_2^3 \\ \dot{x}_2 &= x_1 x_2^2 + x_2^3 .\end{aligned}$$

Hint: For second order systems one should always first try Chetaev functions of the form $V = x_1^2 - a x_2^2$, where a is some positive constant which you are free to choose in the analysis.

2. (25 pts) Show that the following system is ISS

$$\begin{aligned}\dot{x} &= -x + y^3 \\ \dot{y} &= -y - f(x) + p^2 \\ \dot{p} &= -p + u ,\end{aligned}$$

where $f(x)$ is a function such that $\int_0^x f(\xi)d\xi$ and $xf(x)$ are positive definite and radially unbounded functions.

[An example of such a function is the function $f(x) = \mathrm{e}^x - 1$—convince yourself that this is so. A simpler, trivial example is the function is $f(x) = x$.]

This is the hardest problem on the exam. You can solve it either by showing that the (x, y)-system is ISS with respect to p and by noting that p is ISS with respect to u, or by directly building an ISS-Lyapunov function for the whole (x, y, p)-system (the latter is more difficult but more impressive if you can do it). In any case you will have to construct some Lyapunov functions. Hints: Use the terms $\int_0^x f(\xi)d\xi$ and $y^4/4$ in those Lyapunov functions. In showing that the (x, y)-system is ISS with respect to p you should use Young's inequality (the version with powers of 4 and 4/3).

3. (20 pts) Using averaging theory, analyze the following system:

$$\begin{aligned}\dot{x}_1 &= [(x_1 - \sin\omega t)^2 - x_2]\sin\omega t \\ \dot{x}_2 &= -(x_1 - \sin\omega t)^2 - x_2 .\end{aligned}$$

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2007

FINAL EXAM

Take home. Open books and notes.

Total points: 75

Tuesday, March 20, 2006

Collaboration not allowed.

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1. (10 pts) Prove global stability of the origin of the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{x_1}{1+x_2^2}\,. \end{aligned}$$

2. (10 pts) Prove global asymptotic stability of the origin of the system

$$\begin{aligned} \dot{x}_1 &= -x_2^3 \\ \dot{x}_2 &= x_1 - x_2\,. \end{aligned}$$

Is the origin exponentially stable (at least locally)?

3. (10 pts) Which of the state variables of the following system are guaranteed to converge to zero from any initial condition?

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 x_3 \\ \dot{x}_2 &= -x_1 - x_2 + x_2 x_3 \\ \dot{x}_3 &= -x_1^2 - x_2^2\,. \end{aligned}$$

4. (10 pts) Using averaging theory, analyze the behavior of the following system for large ω, for both $a = 1$ and $a = -1$:

$$\begin{aligned} \dot{x}_1 &= -\sin x_1 + 2x_2 + (x_1 + 4x_2 \sin\omega t)\sin\omega t \\ \dot{x}_2 &= \left(-2x_1\cos\omega t + x_2^2 \sin\omega t\right)\cos\omega t - a x_2\,. \end{aligned}$$

5. (10 pts) Consider the following control system:

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1 u \\ \varepsilon\dot{z} &= A_{21}x + A_{22}z\,. \end{aligned}$$

Assume that the matrix A_{22} is Hurwitz and that there exists a matrix/vector K (of appropriate dimensions) such that

$$A_{11} - A_{12}A_{22}^{-1}A_{21} + B_1 K$$

is also Hurwitz. Prove that the "partial-state" feedback law

$$u = Kx$$

exponentially stabilizes the equilibrium $(x, z) = (0, 0)$ for sufficiently small ε.

6. (5 pts) Show that the following system is ISS and determine its gain function:

$$\dot{x} = -x^3 + xu\,.$$

7. (5 pts) Show that the following system is ISS and determine its gain function:

$$\dot{x} = -x + u^3\,.$$

8. (5 pts) Show that the following system is ISS and guess its gain function:

$$\begin{aligned}\dot{x} &= -x^3 + xy \\ \dot{y} &= -y + u^3\,.\end{aligned}$$

Michael Madzarevic

65/65

Problem 1

$\dot{x} = x_2$

$\dot{x}_2 = -\frac{x_1}{1+x_2^2}$

Equilibrium

$0 = x_2 \Rightarrow x_2 = 0$

$0 = -\frac{x_1}{1+x_2^2} \Rightarrow x_1 = 0$

$p' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_2^4$ pdf

$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 + x_2^3\dot{x}_2$

$= x_1 x_2 - \frac{x_1 x_2}{1+x_2^2} - \frac{x_1 x_2^3}{1+x_2^2}$

$= \frac{x_1 x_2 + x_1 x_2^3}{1+x_2^2} - \frac{x_1 x_2}{1+x_2^2} - \frac{x_1 x_2^3}{1+x_2^2}$

$\dot{V} = 0$

Stable

10/10

Michael Madzarevic

Problem 2)

$$\dot{x}_1 = -x_2^3$$
$$\dot{x}_2 = x_1 - x_2$$

$$V = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$$

$$\dot{V} = x_1\dot{x}_1 + x_2^3\dot{x}_2 = -x_1x_2^3 + x_2^3x_1 - x_2^4$$
$$= -x_2^4$$

Equilibrium

$$0 = -x_2^3 \Rightarrow x_2 = 0 \qquad e' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$0 = x_1 - x_2 \Rightarrow x_1 = 0$$

From Barbashin - krasovski

V is pdf $\dot{V} \leq 0 \quad \forall x \in D$

and only solution is at $x = e'$

the the origin is g.a.s.

locally E.S. ?

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x_1=0 \\ x_2=0}} = \left.\begin{bmatrix} 0 & -3x_2^2 \\ 1 & -1 \end{bmatrix}\right|_{\substack{x_1=0 \\ x_2=0}} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

Not locally E.S. $\qquad \lambda = -1, 0$

10/10

Michael Madzarevic

Problem 3)

$$\dot{x}_1 = x_2 + x_1 x_3$$
$$\dot{x}_2 = -x_1 - x_2 + x_2 x_3$$
$$\dot{x}_3 = -x_1^2 - x_2^2$$

$$V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3$$
$$= \cancel{x_1 x_2} + \cancel{x_1^2 x_3} - \cancel{x_1 x_2} - x_2^2 + \cancel{x_2^2 x_3} - \cancel{x_1^2 x_3} - \cancel{x_2^2 x_3}$$
$$= -x_2^2$$

Equilibrium

$$0 = x_2 + x_1 x_3$$
$$0 = -x_1 - x_2 + x_2 x_3$$
$$0 = -x_1^2 - x_2^2 \Rightarrow x_1^2 = -x_2^2 \Rightarrow x_1 = x_2 = 0$$

$0 = x_2 + x_1 x_3$ true for any x_3

$0 = -x_1 - x_2 + x_2 x_3$

Let M be the largest invariant set

$M = \{x_1 = 0,\ x_2 = 0\}$

10/10

from LaSalle's rule

$$\boxed{\begin{aligned} x_1(t) &\to 0 \\ x_2(t) &\to 0 \\ x_3(t) &\to C, \quad C = \text{constant} \end{aligned}}$$

Michael Madzarevic

Problem 4 for large ω and $a=1$ + $a=7$

$$\dot{x}_1 = -\sin x_1 + 2x_2 + (x_1 + 4x_2 \sin\omega t)\sin\omega t$$
$$\dot{x}_2 = (-2x_1 \cos\omega t + x_2^2 \sin\omega t)\cos\omega t - a x_2$$

$\tau = \omega t$, $\frac{d}{dt} = \omega \frac{d}{d\tau}$ $\qquad \varepsilon = \frac{1}{\omega}$

$$\left.\begin{aligned} \frac{dx_1}{d\tau} &= \frac{1}{\omega}\left[-\sin x_1 + 2x_2 + (x_1 + 4x_2 \sin\tau)\sin\tau\right] \\ \frac{dx_2}{d\tau} &= \frac{1}{\omega}\left[(-2x_1\cos\tau + x_2^2 \sin\tau)\cos\tau - a x_2\right] \end{aligned}\right\} = \varepsilon f(t, x, \varepsilon)$$

$$f_{AV}(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t,x,0)\,dt = \left\{\begin{bmatrix} -\sin x_1 + 2x_2 + 2x_2 \\ -x_1 - a x_2 \end{bmatrix}\right. = \begin{bmatrix} -\sin x_1 + 4x_2 \\ -x_1 - a x_2 \end{bmatrix}$$

Stability of f_{AV}

$$\left.\frac{\partial f_{AV}}{\partial x}\right|_{x_1=0} = \left.\begin{bmatrix} -\cos x_1 & 4 \\ -1 & -a \end{bmatrix}\right|_{\substack{x_1=0 \\ x_2=0}} = \begin{bmatrix} -1 & 4 \\ -1 & -a \end{bmatrix}$$

Eigen-Values of $\left.\frac{\partial f_{AV}}{\partial x}\right|_{\substack{x_1=0 \\ x_2=0}}$

$$\left|\frac{\partial f_{AV}}{\partial x} - \lambda I\right| = 0 = \left|\begin{bmatrix} -1-\lambda & 4 \\ -1 & -a-\lambda \end{bmatrix}\right| = 0$$

$$(-1-\lambda)(-a-\lambda) + 4 = 0$$
$$\lambda^2 + \lambda(a+1) + (a+4) = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left[-(a+1) \pm \sqrt{(a+1)^2 - 4(a+4)}\right)$$

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for $a=1$

$$\lambda_p = \frac{1}{2}[-2 \pm \sqrt{2^2 - 4\cdot 5}\ ]$$
$$= \frac{1}{2}[-2 \pm 4i]$$
$$= -1 \pm 2i$$

the nominal system is e.s. @ $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = p^*$

If $|\eta(\varepsilon) - p^*|$ is small and ε is small then

$$x(t, \varepsilon) - \sum_{k=0}^{N-1} \varepsilon^k x_k(t) = O(\varepsilon^N)$$

$\therefore$ for all $t \geq 0$ the error from the nominal system is $O(\varepsilon^N)$

for $a=-1$

$$\lambda_{1,2} = \frac{1}{2}[-0 \pm \sqrt{0^2 - 4(3)}$$
$$= \pm\sqrt{3}\ i$$

Stable but not e.s.

nice work

$$x(t) - \sum_{k=0}^{N-1} \varepsilon^N x_k(t) = O(\varepsilon^N)$$

on a time interval inversely prop to ε

10/10

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Problem 5

$$\dot{x} = A_{11}x + A_{12}z + B_1 u = f(t, x, z, \varepsilon)$$
$$\varepsilon \dot{z} = A_{21}x + A_{22}z = g(t, x, z, \varepsilon)$$

A_{22} is Hurwitz and $\exists$ K s.t.

$A_{11} - A_{12}A_{22}^{-1}A_{21} + B_1 K$ is Hurwitz

$u = Kx$

Singular Perturbation

$$\varepsilon \dot{z} = 0 = A_{21}x + A_{22}z$$

$$z = -A_{22}^{-1}A_{21}x = h(t, x) \quad \text{(quasi steady state)}$$

RM

$$\dot{x} = f(t, x, h(t,x), 0)$$

substitute $u = Kx$

$$\dot{x} = A_{11}x + A_{12}A_{22}^{-1}A_{21}x + B_1 u = (A_{11} - A_{12}A_{22}^{-1}A_{21} + B_1 K)x$$

BLM

$$\frac{dy}{d\tau} = g(t, x, y + h(t,x), 0)$$

$$= A_{21}x + A_{22}(y - A_{22}^{-1}A_{21}x)$$

$$\frac{dy}{d\tau} = A_{22}y$$

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RM

$$\dot{x} = (A_{11} + A_{12} A_{22}^{-1} A_{21} + B_1 k)x$$

BLM

$$\frac{dy}{d\tau} = A_{22} y$$

$$f(t, 0, 0, \varepsilon) = A_{11}[0] + A_{12}[0] + B_1 k[0] = 0$$
$$g(t, 0, 0, \varepsilon) = A_{21}[0] + A_{22}[0] = 0$$
$$h(t, x) = -A_{22}^{-1} A_{21}[0] = 0$$

from Tiklonov's Theorem

If the BLM + RM are e.s. at the origin
then for sufficiently small ε

$$x(t,\varepsilon) - \bar{x}(t) = O(\varepsilon)$$
$$z(t,\varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon) + O(e^{-\eta\frac{t-t_0}{\varepsilon}})$$

and if $f(t, 0, 0, \varepsilon) = 0$
$g(t, 0, 0, \varepsilon) = 0$
$h(t, x) = 0$

then the origin of the full system is E.S

In order for RM + BLM to be E.S:
RM is E.S. if $(A_{11} + A_{12} A_{22}^{-1} A_{21} + B_1 k)$ is hurwitz

BLM is E.S. if A_{22} is hurwitz

10/10

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Problem 6

Show the following is ISS and determine its gain function

$\dot{x} = -x^3 + xu$

$V = \frac{1}{2}x^2 , \quad \alpha_1(|x|) \le V \le \alpha_2(|x|)$

$\alpha_1(r) = \alpha_2(r) = \frac{1}{2}x^2$

$\dot{V} = x\dot{x}$

$= -x^4 + x^2 u$

$\dot{V} = -x^2(x^2 - u) \le 0, \ \forall |x| \ge \rho(|u|)$

ISS $\qquad \rho(r) = \sqrt{r}$

$\gamma(r) = \alpha_1^{-1}(\alpha_2(\rho(r))) = \rho(r)$

$\boxed{\gamma(r) = \sqrt{r}}$

5/5

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problem 7)

$$\dot{x} = -x + u^3$$

$V = \frac{1}{2}x^2$, $\alpha_1(|x|) \le V \le \alpha_2(|x|)$

$\alpha_1(r) = \alpha_2(r) = \frac{1}{2} r^2$

$\dot{V} = x\dot{x} = -x^2 + xu^3$

$\dot{V} = -x^2 + xu^3 \le 0$, $\forall |x| \ge \rho(|u|)$

ISS

$\rho(r) = u^3$

$\gamma(r) = \alpha_1^{-1}(\alpha_2(\rho(r))) = \rho(r)$

$\boxed{\gamma(r) = r^3}$

5/5

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Problem 8

$\dot{x} = -x^3 + xy$
$\dot{y} = -y + u^3$

1st show $\dot{y} = -y + u^3$ is ISS to input u

$V_1 = \frac{1}{2}y^2$ $\alpha_{11}(r) = \alpha_{12}(r) = \frac{1}{2}y^2$

$\dot{V}_1 = y\dot{y} = -y^2 + yu^3$

$\dot{V}_1 = -y^2 + yu^3 \leq 0$, $\forall |y| \geq |u|^3$, $\rho_1(r) = |r^3|$

$\gamma_1(r) = \alpha_{11}^{-1}(\alpha_2(\rho_2(r))$
$= r^3$

Show $\dot{x} = -x^3 + xy$ is ISS to input y

$V_2 = \frac{1}{2}x^2$ $\alpha_{21}(r) = \alpha_{22}(r) = \frac{1}{2}x^2$

$\dot{V}_2 = x\dot{x} = -x^4 + x^2 y \leq 0$, $\forall |x| \geq \sqrt{|y|}$

$\rho_2(r) = \sqrt{|r|}$

5/5

$\gamma_2(r) = \alpha_1^{-1}(\alpha_3(\rho_2(r))$

$= \sqrt{r}$

Gain of $u \to x$

from cascade
y is ISS to u
x is ISS to y
x is there for ISS to u

$\gamma_{xu}(r) \approx \gamma_1(r)\gamma_2(r)$

$\boxed{\gamma_{xu} \approx r^{3/2}}$

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2008

FINAL EXAM

Open book and class notes. **Collaboration not allowed.**
Total points: 65

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1. (10 pts) Prove global asymptotic stability of the origin of the system

$$\begin{aligned}\dot{x}_1 &= x_2 - \left(2x_1^2 + x_2^2\right) x_1 \\ \dot{x}_2 &= -x_1 - 2\left(2x_1^2 + x_2^2\right) x_2 .\end{aligned}$$

Is the origin locally exponentially stable and why or why not?

2. (10 pts) Which of the states of the following system are guaranteed to go to zero (provide a proof for your answer):

$$\begin{aligned}\dot{x} &= z^2 \\ \dot{y} &= z \cos z \\ \dot{z} &= -z - zx - y^3 \cos z .\end{aligned}$$

3. (10 pts) Find the power-two (quadratic) approximation of the center manifold of the following system:

$$\begin{aligned}\dot{y}_1 &= y_1 y_2^2 + y_1 z \\ \dot{y}_2 &= -y_2 y_1^2 + y_2 z \\ \dot{z} &= -z + y_1^2 - y_2^2 .\end{aligned}$$

4. (10 pts) Using the center manifold theorem and the suitable Lyapunov or Chetaev theorem, determine asymptotic stability or instability of the origin of the system in Problem 3.

5. (15 pts) Analyze the following system using the method of averaging for large ω:

$$\begin{aligned}\dot{x}_1 &= (x_2 \sin \omega t - 2)\, x_1 - x_3 \\ \dot{x}_2 &= -x_2 + \left(x_2^2 \sin \omega t - 2x_3 \cos \omega t\right) \cos \omega t \\ \dot{x}_3 &= 2x_2 - \sin x_3 + (4x_2 \sin \omega t + x_3) \sin \omega t .\end{aligned}$$

6. (10 pts) Consider the system

$$\dot{x} = -x^3 + u .$$

Show that the following holds

$$\int_0^\infty x(\tau)^4 d\tau \leq \int_0^\infty u(\tau)^{4/3} d\tau + \frac{2}{3} x(0)^2 .$$

Explanation for further learning (not needed to solve the problem): This property is called *integral ISS*. Recall that we proved the standard ISS property in class for the same example. Recall also the discussion of input-output properties of linear systems in class. For the linear example $\dot{x} = -x + u$ we have input-output stability in both L_2 and L_∞ norms, with gains equal to 1 in both cases. While the standard ISS property is the nonlinear analog of L_∞ stability, the 'integral ISS' is the nonlinear analog of L_2 stability. The above inequality says that the integral ISS gain function of the system $\dot{x} = -x^3 + u$ is the the cube root function, $\sqrt[3]{\cdot}$ (when $x(t)$ is penalized in the sense of the L_4 norm). This statement is the nonlinear analog of the statement that the transfer function $1/(s+1)$ has an H_∞ norm of 1, i.e., that the linear example $\dot{x} = -x + u$ has an L_2 induced gain of 1.

MAE 281A Prof. M. Krstic

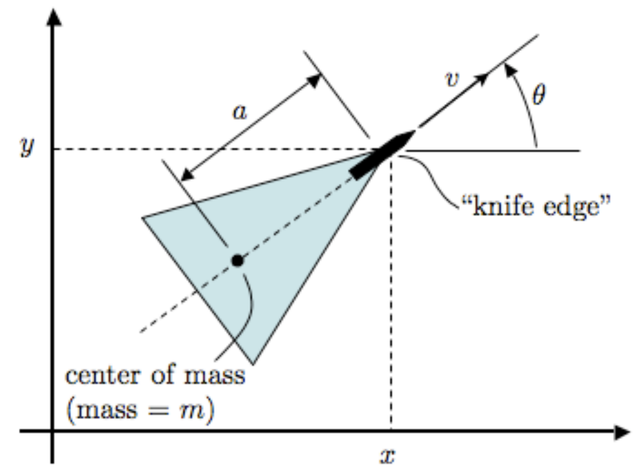
NONLINEAR SYSTEMS Winter 2009

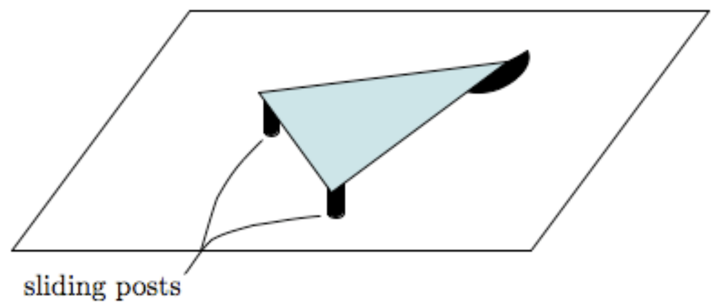
FINAL EXAM

Open book and class notes. **Collaboration not allowed.**

Total points: 65

The first four problems in this exam are dedicated to "Chaplygin's sleigh" (1895):





The nominal/ideal model for this system is given by

$$\dot{x} = v\cos\theta \tag{1}$$

$$\dot{y} = v\sin\theta \tag{2}$$

$$\dot{\theta} = \omega \tag{3}$$

$$\dot{v} = a\omega^2 \tag{4}$$

$$\dot{\omega} = -\frac{ma}{I+ma^2}v\omega\,, \tag{5}$$

where (x, y) is the position of the "knife edge" of the sleigh, θ is the heading angle, v is the surge velocity of the sleigh, ω is the angular velocity, a is the distance from the knife edge to the sleigh's center of mass, m is the mass of the sleigh, and I is the moment of interia of the sleigh.

For those of you familiar with the 'nonholonomic unicycle,' note that the first three equations of the Chaplygin sleigh model (1)–(3) are the same as the unicycle model. Those are the kinematic equations. We are actually not going to use them in this exam, but I provide them to you so that you can physically appreciate the motivation for the problems.

The remaining two equations of the Chaplygin sleigh (4), (5) are the basis of Problems 1–4 of this exam.

To physically imagine the motion of the Chaplygin sleigh, note that the knife edge is not able to slip sideways (like an ice skate) but it can slide forward. Note also that the sliding posts are there only to provide support and impose no friction, so the sleigh can freely rotate around the knife edge as the center of rotation (while at the same time sliding).

Observe that the crucial quantities in the entire problem are the initial conditions for the velocity $v(0)$ and the angular velocity $\omega(0)$. Based on those quantities, the sleigh may go forward (though not straight) while spinning around, creating complex figures in the (x, y) space.

Problem 1. (12 pts) What are the trajectories of the Chaplygin sleigh model in the (v, ω) space? Note that I am not asking you to find the solutions $(v(t), \omega(t))$ as functions of time but only the trajectory curves (phase portrait) in the (v, ω) plane. Note also that this is actually a problem of finding a Lyapunov function for the (v, ω) system.

Problem 2. (6 pts) Suppose now that the posts are subject to friction in the rotational (yaw) degree of freedom, namely, that the (v, ω) equations are given by

$$\begin{aligned} \dot{v} &= a\omega^2 && (6) \\ \dot{\omega} &= -h\omega - \frac{ma}{I + ma^2} v\omega\,, && (7) \end{aligned}$$

where $h > 0$ is the friction coefficient. What are the (v, ω) trajectories in this case?

Problem 3. (10 pts) Suppose now that the blade itself is subject to friction in the surge degree of freedom, namely, that the (v, ω) equations are given by

$$\begin{aligned} \dot{v} &= -gv + a\omega^2 && (8) \\ \dot{\omega} &= -\frac{ma}{I + ma^2} v\omega\,, && (9) \end{aligned}$$

where $g > 0$ is the friction coefficient. What type of stability property holds for the equilibrium $v = \omega = 0$ and what is the physical meaning of this? Is the origin locally exponentially stable?

Problem 4. (12 pts) Now, as the final problem on the Chaplygin sleigh, suppose that friction exists in both the surge and yaw degrees of freedom $(h, g > 0)$, and, more importantly, that the knife edge is allowed to slip sideways slightly, namely, that the friction coefficient on the knife edge in the sideways direction is not infinite but finite. In this case the (v, ω) equations are replaced by the three-equation model (v, ω, σ) given by

$$\begin{aligned} \dot{v} &= -gv + a\omega^2 + \epsilon\omega\sigma && (10) \\ \dot{\omega} &= -h\omega + \sigma && (11) \\ \epsilon\dot{\sigma} &= -\frac{I + ma^2}{ma}\sigma - v\omega\,, && (12) \end{aligned}$$

where $\varepsilon > 0$ is a small parameter which is inversely proportional to the friction coefficient on the sideways motion of the knife edge. Note that $\dot{\sigma}$ is proportional to the derivative of the angular acceleration and, as such, can be referred to as rotational "jerk." Prove local exponential stability of the equilibrium $v = \omega = \sigma = 0$ for small $\epsilon > 0$ (and $g, h > 0$) by using the singular perturbation approach. What can you state in the case $g = h = 0$ using Theorem 11.1 in Khalil?

Problem 5. (10 pts) Consider the system

$$\begin{aligned} \dot{x} &= -x - \sin\omega t\left((z + \sin\omega t)^2 + y\right) && (13) \\ \dot{y} &= -y - (z + \sin\omega t)^2 + \frac{1}{2} && (14) \\ \dot{z} &= x\,. && (15) \end{aligned}$$

Show that for sufficiently large ω there exists an exponentially stable periodic orbit in an $O\left(\frac{1}{\omega}\right)$ neighborhood of the origin.

Problem 6. (15 pts) Consider the system

$$\begin{aligned} \dot{x} &= -x + y^3 && (16) \\ \dot{y} &= -y - \frac{x}{\sqrt{1 + x^2}} + z^2 && (17) \\ \dot{z} &= -z + u\,. && (18) \end{aligned}$$

Show that this system is ISS using the Lyapunov function

$$V = \sqrt{1 + x^2} - 1 + \frac{y^4}{4} + \frac{z^8}{2} \qquad (19)$$

MAE 281A Prof. M. Krstic

Nonlinear Systems Winter 2010

FINAL EXAM

Open book, class notes, and a couple of other additional handwritten pages of notes as summary.

Collaboration not allowed.

Total points: 70

Problem 1. (12 pts) The standard definition of global stability for a system $\dot{x} = f(x)$ at $x = 0$ is that there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ such that $|x(t)|_2 \leq \alpha(|x(0)|_2)$ for all $t \geq 0$, where $|\cdot|_2$ denotes the standard 2-norm, namely, $|x|_2 \triangleq \sqrt{x_1^2 + \cdots + x_n^2}$. Since all p-norms ($p \geq 1$) for vectors are equivalent (not equal but equivalent; recall, for example, that $|x|_2 \leq |x|_1 \leq \sqrt{n}|x|_2$, where $|x|_1 \triangleq |x_1| + \cdots + |x_n|$), the norm on $x(t)$ and $x(0)$ in the above definition of global stability can be replaced, for example, by the 1-norm, as we shall do in this problem.

In this problem we consider the system

$$\dot{x}_1 = x_1 x_2 \tag{1}$$
$$\dot{x}_2 = -x_2 \,. \tag{2}$$

By finding an explicit solution of this system, show that

$$|x_1(t)| + |x_2(t)| \leq \alpha(|x_1(0)| + |x_2(0)|)\,, \tag{3}$$

where

$$\alpha(r) \triangleq r\mathrm{e}^r \,. \tag{4}$$

Problem 2. (12 pts) Consider the system

$$\dot{x}_1 = -x_1 + x_1 x_2 \tag{5}$$
$$\dot{x}_2 = -\frac{x_1^2}{1 + x_1^2} \,. \tag{6}$$

Show that the equilibrium $x_1 = x_2 = 0$ is globally stable and that

$$\lim_{t\to\infty} x_1(t) = 0 \,. \tag{7}$$

Hint: Seek a Lyapunov function in the form

$$V(x_1, x_2) = \phi(x_1) + x_2^2 \,, \tag{8}$$

where the function $\phi(\cdot)$ is to be found. Make sure that your Lyapunov function $V(\cdot,\cdot)$ is positive definite and radially unbounded.

Problem 3. (15 pts) Prove the result from Problem 2 by seeking a different Lyapunov function, in the form

$$U(x_1, x_2) = \left(1 + x_1^2\right) \psi(x_2) - 1 \,, \tag{9}$$

where the function $\psi(\cdot)$ is to be found.

Hint: try to find $\psi(\cdot)$ so that $\dot{U} = -2x_1^2 \psi(x_2)$. Again, it is even more important in this problem that you show that your Lyapunov function $U(\cdot,\cdot)$ is positive definite and radially unbounded.

Problem 4. Show that the system

$$\dot{x}_1 = -x_1 + x_1 x_2 + u \tag{10}$$
$$\dot{x}_2 = -x_2 \tag{11}$$

is ISS with respect to input u by performing the following steps.

a) (3 pts) Take the Lyapunov function

$$V = \ln(1 + x_1^2) + x_2^2 \tag{12}$$

and show that

$$\dot{V} \leq \frac{-x_1^2 + 2x_1 u}{1 + x_1^2} - x_2^2 \,. \tag{13}$$

b) (4 pts) Next, show that

$$\dot{V} \leq \frac{1}{1 + x_1^2} \left[-\frac{1}{2}(x_1^2 + x_2^2) + 2u^2 \right] . \tag{14}$$

c) (2 pts) Next, show that

$$\dot{V} \leq -\frac{1}{4} \frac{x_1^2 + x_2^2}{1 + x_1^2} \,, \qquad \forall |x| \geq \sqrt{8}|u| \,, \tag{15}$$

where $|x| \triangleq \sqrt{x_1^2 + x_2^2}$.

d) (3 pts) Show that

$$\ln(1 + x_1^2 + x_2^2) \leq V \leq x_1^2 + x_2^2 \,. \tag{16}$$

e) (4 pts) With the help of (15) and (16), show that the system (10), (11) is ISS with the gain function

$$\gamma(r) \triangleq \sqrt{e^{8r^2} - 1} \,. \tag{17}$$

Problem 5. (15 pts) Consider the second-order system

$$\dot{x}_1 = \sin(\omega t) y_1 \tag{18}$$
$$\dot{x}_2 = \cos(\omega t) y_2 \tag{19}$$
$$y_1 = [x_1 + \sin(\omega t)] [x_2 + \cos(\omega t) - x_1 - \sin(\omega t)] \tag{20}$$
$$y_2 = [x_2 + \cos(\omega t)] [x_1 + \sin(\omega t) - x_2 - \cos(\omega t)] \,. \tag{21}$$

Show that for sufficiently large ω there exists an exponentially stable periodic orbit in an $O\left(\frac{1}{\omega}\right)$ neighborhood of the origin $x_1 = x_2 = 0$.

Hint: the following functions have a zero mean over the interval $[0, 2\pi]$: $\sin(\tau), \cos(\tau)$, $\sin(\tau)\cos(\tau)$, $\sin^3(\tau)$, $\cos^3(\tau), \sin^2(\tau)\cos(\tau), \sin(\tau)\cos^2(\tau)$.

MAE 281A Prof. M. Krstic

Nonlinear Systems Winter 2011

FINAL EXAM

Take home exam.

Due in Professor Krstic's office (slide under the door if the door is locked) by 5 pm on Thursday, March 17, 2011.

Absolutely no collaboration allowed.

Total points: 70

Problem 1. (25 pts) The "bouncing ball" system (imagine a ball bouncing on a floor or table) is not a conventional dynamical system like those that we studied in this class. The bouncing ball is a "hybrid system," which behaves as a regular continuous system while the ball is in the air and immediately after the impact, but at the time of the impact, it behaves as a discrete system that undergoes a jump in velocity, with a change in sign of the velocity. Between the impacts, the ball behaves like a continuous-time linear system, with a constant input (the force of gravity). At the instant of impact, the ball acts as a discrete-time linear system, where the velocity immediately after the impact is equal to the velocity immediately before the impact times a "coefficient of restitution" (which is smaller than unity) times -1 to account for the reversal in the direction of travel of the ball.

Mathematically, the model of the bouncing ball is given as follows. Denote by x_1 the height of the ball above the floor and by x_2 the velocity of the ball in the upward direction. The state space of the ball is $\{x_1 \geq 0\}$ (meaning that the ball is above or on the floor, with an arbitrary velocity. The dynamic equations of the ball are given as one differential equation (during the non-impact phase of the motion) and one difference equation (during the impact phase):

$$\begin{aligned} \dot{x} &= F(x), \quad \text{while the state is in the set } C = \{x_1 \geq 0\} && (1) \\ x^+ &= G(x), \quad \text{while the state is in the set } D = \{x_1 = 0, x_2 \leq 0\} && (2) \end{aligned}$$

where

$$\begin{aligned} F(x) &= \begin{bmatrix} x_2 \\ -g \end{bmatrix} && (3) \\ G(x) &= \begin{bmatrix} 0 \\ -\gamma x_2 \end{bmatrix}, && (4) \end{aligned}$$

where g is the acceleration of gravity and $\gamma \in [0, 1)$ is the coefficient of restitution. The superscript $^+$ in (2) indicates that the state undergoes a jump from the value x to the value $G(x)$ when $x \in D$.

When $0 < \gamma < 1$, we know that the ball's height gradually decays with each bounce. At each impact, the ball looses a part of its kinetic energy. In the extreme case when $\gamma = 0$, which would be a ball made out of a completely non-elastic material, the ball doesn't bounce even once—it stays on the floor upon the first impact.

Your task in this problem is related to the intuitive knowledge that the ball will eventually stop bouncing. This is the completely obvious part. A slightly less obvious part is that the ball will stop bouncing in finite time—not only in practice, but also according to its mathematical model. The fact that the ball will stop bouncing is related to the asymptotic stability property of the ball at the equilibrium position on the floor. However, the ball is not like a standard asymptotically stable system whose solution decays as the time goes to infinity. The ball's solution decays to zero in finite time, which is for standard systems sometime referred to as "finite-time asymptotic stability." In addition, as a hybrid system, the bouncing ball has the so-called "Zeno property," which is roughly the property that the ball will, in the mathematically idealized model, bounce infinitely many times, with decaying heights of each successive bounce, and with decaying times between bounces, as it comes to rest in finite time. This Zeno property has a long history and it has fascinated people since antiquity (Zeno was a Greek philosopher).

To study stability of the rest state of the bouncing ball, you need to know a basic Lyapunov theory for the so-called "uniform Zeno asymptotic stability" (UZAS) of hybrid systems. Adapted to the bouncing ball problem, the Lyapunov theorem goes as follows.

The equilibrium $x = 0$ of the buncing ball is USAZ if there exists a constant $c > 0$ and a positive definite, radially unbounded, continuously differentiable Lyapunov function $V(x)$ defined on the set $C \cup D = \{x_1 \geq 0\}$ such that

$$\frac{\partial V}{\partial x}(x)F(x) \leq -c\,, \qquad \forall x \in C \setminus 0 \tag{5}$$

$$V(G(x)) \leq V(x)\,, \qquad \forall x \in D \setminus 0\,. \tag{6}$$

Note that the construction of a Lyapunov function for a hybrid system is trickier than for the problems that we have studied in the class because two (rather than one) conditions have to be satisfied by V, in addition to the usual conditions of positive definiteness, radial unboundedness, and continuous differentiability. The Lyapunov function needs to simultaneously satisfy a continuous-time requirement (5) and a discrete-time requirement (6) for a non-increase as the time progresses. Note also that the right-hand side of (5) is unusual, as it is not a negative definite function, but a negative constant. This requirement is related to the finite-time (Zeno) property that the ball possesses.

Task (a) Consider the Lyapunov function candidate

$$V(x) = qx_2 + \sqrt{\frac{1}{2}x_2^2 + gx_1}\,, \tag{7}$$

where

$$q = \frac{1}{\sqrt{2}}\frac{1-\gamma}{1+\gamma} \tag{8}$$

and show that $V(x)$ is positive definite on $C \cup D$.

Task (b) Show that condition (5) is satisfied and clearly state the value of c with which you have satisfied it.

Task (c) Show that condition (6) is satisfied. (This is the trickiest part of the problem.)

Task (d) Noting that, during impact, $V(x(t))$ jumps downward, whereas during the continuous motion $V(x(t))$ decays linearly in time according to the decay function $-ct$, provide an estimate for the largest time in which the ball will come to rest starting from a general initial condition $(x_1(0), x_2(0))$. Then give an estimate for the time in which the ball will come to rest starting from height H with a zero initial speed.

Problem 2. (30 pts) Consider the system

$$\dot{x}_1 = -x_1(1 + x_2) \tag{9}$$

$$\dot{x}_2 = -x_2(1 + x_2)\,. \tag{10}$$

Study the stability of this system at the origin using the Lyapunov function

$$V(x) = \frac{1}{2}\ln\left(1 + x_1^2\right) + x_2 - \ln(1 + x_2)\,. \tag{11}$$

Tasks: Show that this function is positive definite on the set $\{x_2 > -1\}$. Show that $\dot{V}$ is negative definite on the same set. What is the region of attraction of the origin? Using Simulink or any other software, produce a phase portrait of the system in the region $-5 < x_1 < 5, -1 < x_2 < 5$.

Bonus: You can actually find an explicit solution of the system (9), (10). If you can do that, then you don't have to produce the phase portrait numerically. You can simply plot the phase portrait of the analytically obtained solutions via Matlab.

Problem 3. (15 pts) Consider the system

$$\dot{y}_1 = -y_2 \left(y_1 + \sin \omega t\right)^2 \sin \omega t \tag{12}$$

$$\dot{y}_2 = y_2 + 4y_2^2 \left(y_1 + \sin \omega t\right)^2 \cos(2\omega t)\,. \tag{13}$$

Task: Study stability of this system for large ω using averaging theory.

MAE 281A Prof. M. Krstic

Nonlinear Systems Winter 2015

FINAL EXAM

In-class exam.

Open book, class notes, and a couple of other additional handwritten pages of notes as summary.

Absolutely no collaboration allowed.

Total points: 70

Problem 1. (15 pts) Consider the system

$$\dot{x} = -(1+x^2)\mathrm{sgn}(x)\,,$$

where sgn denotes the standard signum function which is given by 1 for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$. Ignoring the question of uniqueness and differentiability of solutions, prove that the origin $x = 0$ is globally stable and $x(t)$ converges to zero in no more than $\pi/2$ seconds, no matter what the value of the initial condition x_0.

Problem 2. (15 pts) Consider the system

$$\begin{aligned}\dot{x} &= 1 - \mathrm{e}^{y}\\ \dot{y} &= 1 - \mathrm{e}^{y-x}\,.\end{aligned}$$

Prove global stability of this system at the origin using the Lyapunov function

$$V(x) = x + e^{-x} - 1 + e^{y} - y - 1\,.$$

Then, using the Barbashin-Krasovskii or LaSalle's theorem, prove global asymptotic stability of the origin.

Problem 3. (15 pts) Consider the system

$$\begin{aligned}\dot{x} &= -(1+z^2)z\\ \varepsilon\dot{z} &= -(1+z^2)(z-x)\,, \qquad \varepsilon > 0\,.\end{aligned}$$

Using a suitably weighed quadratic Lyapunov function, prove global stability of the origin $x = z = 0$. Then, using the Barbashin-Krasovskii theorem, prove global asymptotic stability of the origin. Finally, for small ε, using the singular perturbation theory, sketch the trajectories of the system.

Problem 4. (10 pts) Show that the following system is ISS and determine its gain function:

$$\dot{x} = -x^3 + xu\,.$$

Problem 5. (15 pts) Consider the system

$$\begin{aligned}\dot{x} &= -x - \sin\omega t\left((z+\sin\omega t)^2 + y\right)\\ \dot{y} &= -y - (z+\sin\omega t)^2 + \frac{1}{2}\\ \dot{z} &= x\,.\end{aligned}$$

Show that for sufficiently large ω there exists an exponentially stable periodic orbit in an $O\left(\frac{1}{\omega}\right)$ neighborhood of the origin.

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2016

FINAL EXAM

Open book and class notes.

Collaboration not allowed.

Total points: 70

Problem 1. (12 pts) Consider the system

$$\dot{x} = -(|x| + 1)^2 \mathrm{sgn}(x) \tag{1}$$

where sgn denotes the standard signum function which is given by 1 for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$.

a) (6pts) Consider a Lyapunov candidate function $V = |x|$. Ignoring the question of uniqueness and differentiability of solutions, prove that the origin $x = 0$ is globally stable. (Hint : $\frac{\partial |x|}{\partial x} = \mathrm{sgn}(x)$)

b) (6pts) Prove $x(t)$ converges to zero in no more than 1 second no matter what the value of the initial condition x_0.

Solution

a) Because $\mathrm{sgn}(0) = 0$, the equilibrium of (1) is only $x = 0$. Consider the Lyapunov candidate $V(x) = |x|$. Then, $V(x) > 0$ for $\forall x \neq 0$ and $V(0) = 0$. Thus V is "pdf". In addition, $V \to \infty$ as $|x| \to \infty$, thus V is radially unbounded. Taking the time derivative along with (1), we obtain

$$\dot{V} = \dot{x}\frac{\partial |x|}{\partial x} = -(|x| + 1)^2 \mathrm{sgn}(x)^2 \tag{2}$$

Because $\dot{V}(x) < 0$ for $\forall x \neq 0$ and $\dot{V}(0) = 0$, $\dot{V}$ is "ndf". Therefore, we can state that the origin $x = 0$ is globally asymptotically stable by Lyapunov theorem.

b) Consider (2) for $x \neq 0$. Because $V = |x|$, we can write the differential equation of V as

$$\dot{V} = -(1 + V)^2 \tag{3}$$

Using separation of variable method, we obtain the explicit solution of (3) as

$$V = -1 + \left(t + \frac{1}{V_0 + 1}\right)^{-1} \tag{4}$$

where $V_0 = V(x_0) = |x_0|$. By the solution (4), we can see that V arrives at 0 in finite time t^* such that $t^* = 1 - (|x_0| + 1)^{-1}$. Therefore, for any initial condition x_0, we have a bound of t^* as $0 < t^* < 1$, which shows that $x(t)$ converges to zero in no more than 1 second.

Problem 2. (15 pts+Bonus 3pts) Consider the system

$$\dot{x}_1 = \frac{1}{2}x_2(x_1 - x_2)^2 \tag{5}$$

$$\dot{x}_2 = -\frac{1}{2}x_1(x_1 - x_2)^2 . \tag{6}$$

a) (2pts) Find a set of equilibrium of the system.

b) (3pts) Using the Lyapunov function $V = x_1^2 + x_2^2$ prove the global stability of the origin and prove that every circle centered at the origin is a compact positively invariant set.

c) (2pts) Consider the Lyapunov-like function $U = \tan^{-1}(x_2/x_1)$. Note that this function is inspired by the representation of the state in the polar coordinates, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, which gives $U = \theta$. Is $U(x_1, x_2)$ positive definite?

d) (3pts) Show that $\dot{U} = -\frac{1}{2}(x_1 - x_2)^2$.

e) (3pts) What is the largest invariant set within the set $\left\{\dot{U} = 0\right\}$?

f) (2pts) With LaSalle's invariance theorem, prove that $(x_1(t), x_2(t)) \to \{x_1 = x_2\}$ as $t \to \infty$.

g) (Bonus : 3pts) Show that $\dot{\theta} = -\left(x_1^2(0) + x_2^2(0)\right)\cos^2(\theta + \pi/4)$ and sketch the trajectories of the system.

Solution

a) By (5) and (6), the equilibrium (x_1^*, x_2^*) satisfies $x_2^*(x_1^* - x_2^*)^2 = 0$ and $x_1^*(x_1^* - x_2^*)^2 = 0$. Thus the set of equilibrium is $\{x_1^* = x_2^*\}$.

b) By a) the origin is one equilibrium point of the system. Let $V = x_1^2 + x_2^2$ be a Lyapunov candidate. Then, obviously V is "pdf" and radially unbounded. Taking the time derivative along with (5), (6), we obtain

$$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = x_1x_2(x_1 - x_2)^2 - x_1x_2(x_1 - x_2)^2 = 0 \tag{7}$$

which is "nsdf". Therefore, the origin is globally stable. Due to (7), we have $x_1(t)^2 + x_2(t)^2 = x_1(0)^2 + x_2(0)^2$ for $\forall t \geq 0$. Thus the trajectory keeps a circle with a radius of initial condition, which implies that if $x(0) \in B_r$ then $x(t) \in B_r$ for any radius r and any time t. Therefore, every circle centered at the origin is a compact positively invariant set.

c) In the polar coordinate, we can define the range of θ to keep one round 2π. Define $0 \leq \theta < 2\pi$. Then, $U \geq 0$ for any $\forall x_1, x_2$, but $U = 0$ for $\forall(x_1, 0)$, which is "psdf", but not "pdf".

d) Taking time derivative of $U$and using a formula, we have

$$\dot{U} = \dot{\theta} = \frac{x_1\dot{x}_2 - \dot{x}_1x_2}{x_1^2 + x_2^2} = -\frac{x_1^2(x_1 - x_2)^2 + x_2^2(x_1 - x_2)^2}{2(x_1^2 + x_2^2)} = -\frac{1}{2}(x_1 - x_2)^2. \tag{8}$$

e) Let $E := \{(x_1, x_2) \in R^2 | \dot{U} = 0\}$ and $M \subset E$ be the largest invariant set. Then, by (8) we have $E = \{x_1 = x_2\}$. By a) the set of equilibrium is the same as E. Therefore, we have $M = E = \{x_1 = x_2\}$.

f) By LaSalle's invariance theorem, it concludes $(x_1(t), x_2(t)) \to M = \{x_1 = x_2\}$ as $t \to \infty$.

g) Taking polar coordinate in (8), we have

$$\begin{aligned}\dot{U} &= -\frac{1}{2}(x_1 - x_2)^2 = -\frac{r^2}{2}(\cos(\theta) - \sin(\theta))^2 = -r^2\left(\frac{1}{\sqrt{2}}\cos(\theta) - \frac{1}{\sqrt{2}}\sin(\theta)\right)^2 \\ &= -r^2\cos^2\left(\theta + \frac{\pi}{4}\right) \end{aligned} \tag{9}$$

Because $\dot{U} = \dot{\theta}$ and $r^2 = x_1(0)^2 + x_2(0)^2$ by b), we arrive at a differential equation of θ

$$\dot{\theta} = -\left(x_1^2(0) + x_2^2(0)\right)\cos^2\left(\theta + \frac{\pi}{4}\right) \tag{10}$$

Let $\phi = \theta + \pi/4$, then it satisfies $\dot{\phi} = -r^2\cos^2(\phi)$. By separation of variables method, we obtain its explicit solution as

$$\phi(t) = \arctan\left(-r_0^2 t + \tan(\phi_0)\right) \tag{11}$$

Note $\pi/4 \leq \phi < 9\pi/4$. Therefore, if $\pi/4 \leq \phi_0 < \pi/2$ or $3\pi/2 < \phi_0 < 9\pi/4$ then $\phi(t) \to 3\pi/2$, and if $\pi/2 < \phi_0 < 3\pi/2$ then $\phi(t) \to \pi/2$. Switching ϕ to θ, we can write the trajectories as following.

Problem 3. (13 pts) Consider the system

$$\dot{y}_1 = -[y_2 (y_1 + \sin \omega t)^2 + y_2^2] \sin \omega t - \cos^3(\omega t) \tag{12}$$
$$\dot{y}_2 = y_2 + 4y_2^2 (y_1 + \sin \omega t)^2 \cos(2\omega t) . \tag{13}$$

Study stability of this system for large ω using averaging theory.

Solution

Let $y = (y_1, y_2)^T$, $\tau = \omega t$, $\varepsilon = 1/\omega$, and write the system (12)(13) as $dy_1/d\tau = \varepsilon f_1(y, \tau)$ and $dy_2/d\tau = \varepsilon f_2(y, \tau)$. Let $f_{1av}(y)$, $f_{2av}(y)$ be their averaged function. Then, we obtain them as

$$\begin{aligned}
f_{1av}(y) &= \left[-[y_2 (y_1 + \sin \tau)^2 + y_2^2] \sin \tau - \cos^3(\tau)\right]_{ave} \\
&= [-2y_1 y_2 \sin^2(\tau)]_{ave} \\
&= -y_1 y_2 && (14) \\
f_{2av}(y) &= \left[y_2 + 4y_2^2 (y_1 + \sin \tau)^2 \cos(2\tau)\right]_{ave} \\
&= y_2 + 4y_2^2[(y_1 + \sin \tau)^2 (2\cos^2(\tau) - 1)]_{ave} \\
&= y_2 - 4y_2^2(y_1^2 + 1/2) + 8y_2^2[(y_1^2 + 2y_1 \sin(\tau) + \sin^2(\tau)) \cos^2(\tau)]_{ave} \\
&= y_2 - 4y_2^2(y_1^2 + 1/2) + 4y_1^2 y_2^2 + 2y_2^2[\sin^2(2\tau)]_{ave} \\
&= y_2 - y_2^2 && (15)
\end{aligned}$$

Therefore, averaged system (y_{1av}, y_{2av}) is written as

$$\frac{d}{d\tau} \begin{bmatrix} y_{1av} \\ y_{2av} \end{bmatrix} = \varepsilon \begin{bmatrix} -y_{1av} y_{2av} \\ y_{2av} - y_{2av}^2 \end{bmatrix} \tag{16}$$

The equilibrium (y_{1av}^*, y_{2av}^*) of averaged system is given by

$$\begin{bmatrix} y_{1av}^* \\ y_{2av}^* \end{bmatrix} = \begin{bmatrix} y_{1av}^* \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{17}$$

Let J_1 and J_2 be Jacobian matrices of the averaged system (16) evaluated at each equilibrium, then we have

$$J_1 = \begin{pmatrix} -y_{2av} & -y_{1av} \\ 0 & 1 - 2y_{2av} \end{pmatrix} |_{(y_{1av}^*, 0)} = \begin{pmatrix} 0 & -y_{1av} \\ 0 & 1 \end{pmatrix} \tag{18}$$
$$J_2 = \begin{pmatrix} -y_{2av} & -y_{1av} \\ 0 & 1 - 2y_{2av} \end{pmatrix} |_{(0,1)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

Computing each eigenvalues, $\text{eig}(J_1) = 0, 1$, and $\text{eig}(J_2) = -1$. Therefore, J_2 is Hurwitz while J_1 is not. Therefore, the original system (12)(13) has a unique locally exponentially stable periodic solution of period $T = 2\pi$ in an $\mathcal{O}(\epsilon)$ neighborhood of $(0, 1)$.

Problem 4. (15 pts) Show that the system

$$\dot{x}_1 = -x_1 + 2u^2 \tag{20}$$

$$\dot{x}_2 = -x_2 + \frac{x_1^2}{1+x_1^2} \tag{21}$$

is ISS with respect to input u by performing the following steps.

a) (4pts) Take the Lyapunov function

$$V = \ln(1+x_1^2) + x_2^2 \tag{22}$$

and show that

$$\dot{V} \leq \frac{-x_1^2 + 4x_1u^2}{1+x_1^2} - x_2^2\,. \tag{23}$$

b) (3pts) Next, show that

$$\dot{V} \leq \frac{1}{1+x_1^2}\left[-\frac{1}{2}(x_1^2+x_2^2) + 8u^4\right]. \tag{24}$$

c) (2pts) Next, show that

$$\dot{V} \leq -\frac{1}{4}\frac{x_1^2+x_2^2}{1+x_1^2}\,, \qquad \forall |x| \geq 4\sqrt{2}|u|^2\,, \tag{25}$$

where $|x| \triangleq \sqrt{x_1^2+x_2^2}$.

d) (3pts) Show that

$$\ln(1+x_1^2+x_2^2) \leq V \leq x_1^2 + x_2^2\,. \tag{26}$$

e) (3pts) With the help of (25) and (26), show that the system (20), (21) is ISS with the gain function

$$\gamma(r) \triangleq \sqrt{\mathrm{e}^{32r^4} - 1}. \tag{27}$$

Solution

a) Taking time derivative of (22) along with (20)(21), we have

$$\begin{aligned}\dot{V} &= \frac{2x_1\dot{x}_1}{1+x_1^2} + 2x_2\dot{x}_2 = \frac{2x_1(-x_1+2u^2)}{1+x_1^2} + 2x_2\left(-x_2 + \frac{x_1^2}{1+x_1^2}\right) = \frac{-2x_1^2 + 4x_1u_2 + 2x_1^2x_2}{1+x_1^2} - 2x_2^2 \\ &= \frac{-x_1^2+4x_1u^2}{1+x_1^2} - x_2^2 + \frac{-x_1^2(x_2-1)^2 - x_2^2}{1+x_1^2} \\ &\leq \frac{-x_1^2+4x_1u^2}{1+x_1^2} - x_2^2.\end{aligned} \tag{28}$$

b) By Young's inequality, $4x_1u^2 \leq x_1^2/2 + 8u^4$. Continuing the inequalities,

$$\begin{aligned}\dot{V} &\leq \frac{-x_1^2/2 + 8u^4}{1+x_1^2} - \frac{x_2^2(1+x_1^2)}{1+x_1^2} \leq \frac{1}{1+x_1^2}\left[-\frac{1}{2}(x_1^2+x_2^2) + 8u^4\right] - \frac{x_2^2(1/2+x_1^2)}{1+x_1^2} \\ &\leq \frac{1}{1+x_1^2}\left[-\frac{1}{2}(x_1^2+x_2^2) + 8u^4\right]\end{aligned} \tag{29}$$

c) Continuing the inequalities,

$$\begin{aligned}\dot{V} &\le \frac{1}{1+x_1^2}\left[-\frac{1}{4}(x_1^2+x_2^2) + \left(-\frac{1}{4}(x_1^2+x_2^2) + 8u^4\right)\right] \\ &\le -\frac{1}{4}\frac{x_1^2+x_2^2}{1+x_1^2}, \quad \forall |x| \ge 4\sqrt{2}u^2 \end{aligned} \tag{30}$$

d) Firstly, let us show $\ln(1+x_1^2+x_2^2) \le \ln(1+x_1^2) + x_2^2$. Let $a = 1+x_1^2 \ge 1$ and $b = x_2^2 \ge 0$ and $f(a,b)$ be a function $f(a,b) = \ln(a) + b - \ln(a+b)$. Taking partial derivatives, we have

$$\frac{\partial f}{\partial a} = \frac{1}{a} - \frac{1}{a+b} = \frac{b}{a(a+b)} \ge 0, \quad \forall a \ge 1, b \ge 0 \tag{31}$$

$$\frac{\partial f}{\partial b} = 1 - \frac{1}{a+b} \ge 0, \quad \forall a \ge 1, b \ge 0 \tag{32}$$

In addition, we have $f(1,0) = 0$, so we have $f(a,b) \ge 0$ for $\forall a \ge 1, b \ge 0$, which shows the first inequality. Next, we show $\ln(1+x_1^2) \le x_1^2$. Let $g(x) = x - \ln(1+x)$. Then we have $g'(x) = 1 - 1/(1+x) \ge 0$ for $\forall x \ge 0$. In addition $g(0) = 0$, which shows the second inequality. Hence we can state the given inequalities.

e) Continuing the inequalities,

$$\dot{V} \le -\frac{1}{4}\frac{x_1^2+x_2^2}{1+x_1^2} \le -\frac{1}{4}\frac{x_1^2+x_2^2}{1+(x_1^2+x_2^2)}, \quad \forall |x| \ge 4\sqrt{2}u^2 \tag{33}$$

By this inequality and (25) and (26), we can write the class-$\mathcal{K}$ functions as

$$\alpha_1(r) = \ln(1+r^2), \quad \alpha_2(r) = r^2, \quad \alpha_3(r) = \frac{r^2}{4(1+r^2)}, \quad \rho(r) = 4\sqrt{2}r^2. \tag{34}$$

Therefore, the system is ISS w/ gain function $\gamma(r) = \alpha_1^{-1}(\alpha_2(\rho(r)))$. We have $\alpha_2(\rho(r)) = 32r^4$. By the relation $r = \sqrt{e^{\alpha_1(r)} - 1}$, we have $\alpha_1^{-1}(r) = \sqrt{e^r - 1}$. Finally, we can write the gain function as

$$\gamma(r) = \sqrt{e^{32r^4} - 1} \tag{35}$$

Problem 5. (15 pts) Consider the system

$$\begin{aligned} \dot{x} &= -(1+x^2)z & (36) \\ \varepsilon\dot{z} &= -(1+x^2)(z-x)\,, \qquad \varepsilon > 0\,. & (37) \end{aligned}$$

a) (3pts) Using a suitably weighed quadratic Lyapunov function, prove global stability of the origin $x = z = 0$.

b) (2pts) Then, using the Barbashin-Krasovskii theorem, prove global asymptotic stability of the origin.

c) (10pts) Finally, for small ε, using the singular perturbation theory, sketch the trajectories of the system.

Solution

a) Let V be a Lyapunov candidate such that

$$V = \frac{1}{2}x^2 + \frac{\varepsilon}{2}z^2 \tag{38}$$

be a Lyapunov candidate. Then, obviously V is "pdf" and radially unbounded. Taking the time derivative along with (36), (37), we obtain

$$\dot{V} = -(1+x^2)zx - (1+x^2)(z-x)z = -(1+x^2)z^2 \tag{39}$$

$\dot{V} \leq 0$ for $\forall x, z$, so $\dot{V}$ is "nsdf". Therefore, by Lyapunov theorem, the origin is globally stable.

b) Let $S := \{(x,z) \in R^2 | \dot{V} = 0\}$. Then, by (39) we have $S = \{z = 0\}$. Suppose that a solution $(\hat{x}, \hat{z})$ stay in S. then, we have $\hat{z} = 0$. In addition, by (37), we have $(1+\hat{x}^2)\hat{x} = 0$, which deduces $\hat{x} = 0$. Therefore, no solution can stay in S forever other than the origin $(0,0)$. By Barbashin-Krasovskii Theorem, we can state the origin is globally asymptotically stable.

c) (i) Set $\varepsilon = 0$ in (37). Then, we have $z = h(x) = x$(QSS). To obtain RM, substitute QSS into (36). Then,

$$\dot{\bar{x}} = -(1+\bar{x}^2)\bar{x} \tag{40}$$

which is RM. The equilibrium of RM is only at the origin. Jacobian of RM is obtained by $J_{RM} = -1$, so this RM is e.s.
Let $y = z - h(x)$ and $g(x,z) = -(1+x^2)(z-x)$. Then, BLM is written as

$$\frac{dy}{d\tau} = g(x, y + h(x)) = -(1+x^2)y \tag{41}$$

Computing the Jacobian of BLM at the origin, we have $J_{BLM} = -(1+x^2)$. So J_{BLM} is Hurwitz and BLM is e.s.
By singular perturbation theory, we can write the figure as follows.

MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2017

FINAL EXAM

Take home.

Collaboration not allowed.

Total points: 70

Problem 1. (12 pts) Show that the system

$$\dot{x}_1 = -x_1 + \sqrt{|x_1 x_2|} + u \tag{1}$$
$$\dot{x}_2 = -x_2 \tag{2}$$

is input-to-state stable. Use the Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ to show the system has the gain function $\rho(r) = 8r$.

Solution

Considering the given Lyapunov function, and taking a time derivative:

$$\dot{V}(x) = x_1\left(-x_1 + \sqrt{|x_1 x_2|} + u\right) + x_2(-x_2) \tag{3}$$
$$= -x_1^2 - x_2^2 + \sqrt{|x_1||x_2|}x_1 + x_1 u \tag{4}$$

Using Young's inequality on the third term,

$$\dot{V}(x) \leq -x_1^2 - x_2^2 + \frac{3}{4}x_1^2 + \frac{1}{4}x_2^2 + x_1 u \tag{5}$$
$$\leq -\frac{1}{4}x_1^2 - \frac{3}{4}x_2^2 + |x_1||u| \tag{6}$$
$$\leq -\frac{1}{4}x_1^2 - \frac{3}{4}x_2^2 + ||x||\,|u| \tag{7}$$

where $|x_1| \leq ||x||$. Then,

$$\dot{V}(x) \leq -\frac{1}{8}x_1^2 - \frac{5}{8}x_2^2 + ||x||\left(|u| - \frac{1}{8}\,||x||\right) \tag{8}$$

Then we can conclude the system is ISS with gain function $\rho(r) = 8r$.

Problem 2. (13 pts) Analyze the following system using the method of averaging for large ω

$$\dot{y}_1 = -[y_2\,(1+\cos(\omega t))^2 + y_2^2]\cos(\omega t)\sin(y_1) \tag{9}$$
$$\dot{y}_2 = 2y_2^2\,(y_1+\sin(\omega t))^2\cos(2\omega t) + y_2 + y_1^2\sin(\omega t)\,. \tag{10}$$

Solution

Let $y=(y_1,y_2)^T$, $\tau=\omega t$, $\varepsilon=1/\omega$, and write the system (9)(10) as $dy_1/d\tau=\varepsilon f_1(y,\tau)$ and $dy_2/d\tau=\varepsilon f_2(y,\tau)$. Let $f_{1av}(y)$, $f_{2av}(y)$ be their averaged function. Then, we obtain them as

$$\begin{aligned} f_{1av}(y) &= -\left[[y_2\,(1+\cos(\omega t))^2+y_2^2]\cos(\omega t)\sin(y_1)\right]_{ave} \\ &= -\sin(y_1)y_2\left[(1+\cos(\omega t))^2\cos(\omega t)\right]_{ave} \\ &= -\sin(y_1)y_2 && (11) \\ f_{2av}(y) &= \left[y_2+2y_2^2\,(y_1+\sin\tau)^2\cos(2\tau)\right]_{ave} \\ &= y_2+2y_2^2[(y_1+\sin\tau)^2\,(2\cos^2(\tau)-1)]_{ave} \\ &= y_2-2y_2^2(y_1^2+1/2)+4y_2^2[(y_1^2+2y_1\sin(\tau)+\sin^2(\tau))\cos^2(\tau)]_{ave} \\ &= y_2-2y_2^2(y_1^2+1/2)+2y_1^2y_2^2+y_2^2[\sin^2(2\tau)]_{ave} \\ &= y_2-\frac{y_2^2}{2} && (12) \end{aligned}$$

Therefore, averaged system (y_{1av}, y_{2av}) is written as

$$\frac{d}{d\tau}\begin{bmatrix} y_{1av} \\ y_{2av}\end{bmatrix} = \varepsilon\begin{bmatrix} -\sin(y_{1av})y_{2av} \\ y_{2av}-y_{2av}^2/2\end{bmatrix} \tag{13}$$

The equilibrium (y_{1av}^*, y_{2av}^*) of averaged system is given by

$$\begin{bmatrix} y_{1av}^* \\ y_{2av}^*\end{bmatrix} = \begin{bmatrix} y_{1av}^* \\ 0\end{bmatrix}, \begin{bmatrix} 2n\pi \\ 2\end{bmatrix}, \begin{bmatrix} (2n+1)\pi \\ 2\end{bmatrix} \tag{14}$$

Let J_1 and J_2 be Jacobian matrices of the averaged system (13) evaluated at each equilibrium, then we have

$$J_1 = \begin{pmatrix} -\cos(y_{1av})y_{2av} & -\sin(y_{1av}) \\ 0 & 1-y_{2av}\end{pmatrix}|_{(y_{1av}^*,0)} = \begin{pmatrix} 0 & -\sin(y_{1av}) \\ 0 & 1\end{pmatrix} \tag{15}$$
$$J_2 = \begin{pmatrix} -\cos(y_{1av})y_{2av} & -\sin(y_{1av}) \\ 0 & 1-y_{2av}\end{pmatrix}|_{(2n\pi,2)} = \begin{pmatrix} -2 & 0 \\ 0 & -1\end{pmatrix} \tag{16}$$
$$J_3 = \begin{pmatrix} -\cos(y_{1av})y_{2av} & -\sin(y_{1av}) \\ 0 & 1-y_{2av}\end{pmatrix}|_{(2(n+1)\pi,2)} = \begin{pmatrix} 2 & 0 \\ 0 & -1\end{pmatrix} \tag{17}$$

Computing each eigenvalues, $\mathrm{eig}(J_1)=0,1$, $\mathrm{eig}(J_2)=-1,-2$, and $\mathrm{eig}(J_3)=2,-1$. Therefore, J_2 is Hurwitz while J_1 and J_3 are not. Therefore, the original system (9)(10) has a unique locally exponentially stable periodic solution of period $T=2\pi$ in an $\mathcal{O}(\epsilon)$ neighborhood of $(2n\pi,2)$.

Problem 3. (15 pts) Consider the system

$$\dot{x} = -(x^2+1)\text{sgn}(x) - x\sin^2(x) \tag{18}$$

where sgn denotes the standard signum function which is given by 1 for $x > 0$, -1 for $x < 0$, and 0 for $x = 0$.

a) (8pts) Consider a Lyapunov candidate function $V = |x|$. Ignoring the question of uniqueness and differentiability of solutions, prove that the origin $x = 0$ is globally stable. (Hint : $\frac{\partial |x|}{\partial x} = \text{sgn}(x)$)

b) (7pts) Using comparison principle, prove $x(t)$ converges to zero in no more than $\pi/2$ seconds no matter what the value of the initial condition x_0.

Solution

a) Because $\text{sgn}(0) = 0$, the equilibrium of (18) is only $x = 0$. Consider the Lyapunov candidate $V(x) = |x|$. Then, $V(x) > 0$ for $\forall x \neq 0$ and $V(0) = 0$. Thus V is "pdf". In addition, $V \to \infty$ as $|x| \to \infty$, thus V is radially unbounded. Taking the time derivative along with (18), we obtain

$$\begin{aligned}\dot{V} = \dot{x}\frac{\partial |x|}{\partial x} &= -(x^2+1)\text{sgn}(x)^2 - \text{sgn}(x)x\sin^2(x) \\ &= -(x^2+1)\text{sgn}(x)^2 - |x|\sin^2(x)\end{aligned} \tag{19}$$

Because $\dot{V}(x) < 0$ for $\forall x \neq 0$ and $\dot{V}(0) = 0$, $\dot{V}$ is "ndf". Therefore, we can state that the origin $x = 0$ is globally asymptotically stable by Lyapunov theorem.

b) Consider (19) for $x \neq 0$. Since $-|x|\sin^2(x) < 0$ for $\forall x$, we deduce the differential inequality of V as

$$\dot{V} \leq -(1+V^2) \tag{20}$$

Introduce a variable v which satisfies the following differential equation

$$\dot{v} = -(1+v^2) \tag{21}$$

Then, the explicit solution of (21) is obtained by

$$v(t) = \tan(\arctan(v_0) - t) \tag{22}$$

Then, we can see that $v(t)$ arrives at 0 in finite time t^* such that $t^* = \arctan(v_0)$. Since for any initial condition we have $\arctan(v_0) \leq \pi/2$, we can see that $v(t)$ arrives at 0 within $\pi/2$ seconds. Finally, applying Comparison principle to (20) –(22), it yields $V(t) \leq v(t)$, which implies that $x(t)$ arrives at 0 faster than $v(t)$. Hence, for any initial condition x_0, $x(t)$ converges to zero in no more than $\pi/2$ second.

Problem 4. (15 pts+Bonus 3pts) Consider the system

$$\dot{x}_1 = x_2^2\left(\sqrt{x_1^2+x_2^2}-2x_1\right) \tag{23}$$

$$\dot{x}_2 = -x_2\left(x_1\sqrt{x_1^2+x_2^2}-x_1^2+x_2^2\right). \tag{24}$$

a) (5pts) Using the Lyapunov function $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, prove the global stability of the origin.

b) (3pts) Prove that $x_2(t) \to 0$ as $t \to \infty$.

c) (7pts) A cardioid is a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius. The representation of a cardioid in polar coordinates obeys $r = 2a(1-\cos(\theta))$ with the coordinates $x_1 = r\cos(\theta)$ and $x_2 = r\sin(\theta)$, where a is a constant radius of the rolling circle. Prove that the trajectories of the system (23), (24) correspond with a cardioid shape by finding a proper Lyapunov function.

d) (Bonus 3pts) For a given non-zero initial condition $(x_1(0), x_2(0)) \neq (0,0)$, determine the radius of the rolling circle a of the cardioid.

Solution

a) Let $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ be a Lyapunov candidate. Then, obviously V is "pdf" and radially unbounded. Taking the time derivative along with (23), (24), we obtain

$$\begin{aligned}\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 = x_1x_2^2\left(\sqrt{x_1^2+x_2^2}-2x_1\right) - x_2^2\left(x_1\sqrt{x_1^2+x_2^2}-x_1^2+x_2^2\right)\\ &= -x_2^2\left(x_1^2+x_2^2\right) \leq 0\end{aligned} \tag{25}$$

which is "nsdf". Therefore, the origin is globally stable.

b) Let $E := \{(x_1,x_2) \in R^2 | \dot{V} = 0\}$ and $M \subset E$ be the largest invariant set. Then, by (25) we have $E = \{x_2 = 0\}$. Substituting $x_2 = 0$ into (23) and (24), we can see $M = E = \{x_2 = 0\}$. By LaSalle's invariance theorem, it concludes $(x_1(t), x_2(t)) \to M = \{x_2 = 0\}$ as $t \to \infty$.

c) By a), we can see that $r\dot{r} = -x_2^2\left(x_1^2+x_2^2\right) = -r^4\sin^2(\theta)$, thus it yields

$$\dot{r} = -r^3\sin^2(\theta) \tag{26}$$

Additionally, the differential equation of θ follows $\dot{\theta} = (x_1\dot{x}_2 - \dot{x}_1x_2)/r^2$, hence we have

$$\begin{aligned}\dot{\theta} &= \frac{-x_2}{r^2}\left(x_1^2\sqrt{x_1^2+x_2^2}-x_1^3+x_1x_2^2+x_2^2\sqrt{x_1^2+x_2^2}-2x_1x_2^2\right)\\ &= \frac{-x_2(x_1^2+x_2^2)}{r^2}\left(\sqrt{x_1^2+x_2^2}-x_1\right)\\ &= -r^2\sin(\theta)(1-\cos(\theta))\end{aligned} \tag{27}$$

Define a Lyapunov function such that

$$U = \frac{r}{1-\cos(\theta)} \tag{28}$$

Taking time derivative of (28), we have

$$\begin{aligned}\dot{U} &= \frac{\dot{r}(1-\cos(\theta)) - \dot{\theta}\sin(\theta)r}{(1-\cos(\theta))^2} = \frac{-r^3\sin^2(\theta)(1-\cos(\theta)) + r^3\sin^2(\theta)(1-\cos(\theta))}{(1-\cos(\theta))^2}\\ &= 0\end{aligned} \tag{29}$$

Hence, it yields

$$\begin{aligned}\frac{r}{1-\cos(\theta)} &= \frac{r(0)}{1-\cos(\theta(0))} \\ &= \frac{x_1^2(0)+x_2^2(0)}{\sqrt{x_1^2(0)+x_2^2(0)}-x_1(0)} = 2a\end{aligned} \tag{30}$$

Thus, the trajectory of the system corresponds with the cardioid with the radius of the rolling circle

$$a = \frac{x_1^2(0)+x_2^2(0)}{2\left(\sqrt{x_1^2(0)+x_2^2(0)}-x_1(0)\right)} \tag{31}$$

Problem 5. (15 pts) Consider the system

$$\dot{x} = -x - z - 2y \tag{32}$$
$$\varepsilon\dot{y} = -2y + \tan(z) \tag{33}$$
$$\delta\dot{z} = -z - \arctan\left(x^3 - y\right) \tag{34}$$

where $0 < \delta \ll \varepsilon$ are small parameters. Use singular pertubation to show that the origin is exponentially stable. Please be explicit in your conclusions, and show the intermediate steps you take to come to the result.

Hint: Use the fact that $\delta \ll \varepsilon$ to treat the z-subsystem as a faster system than the (x, y)-subsystem, leading to the reduced system and boundary layer model. Then apply singular pertubation once again, but now for ε. Make your conclusions about each system you derive clearly.

Solution

Letting $\delta \to 0$, we can find the first quasi-steady-state as

$$z_{\text{qss}} = -\arctan\left(x^3 - y\right) \tag{35}$$

This admits a boundary layer model $\xi_1 := z + \arctan\left(x^3 - y\right)$ as

$$\frac{d\xi_1}{d\tau_1} = -\xi_1 \tag{36}$$

which is obviously exponentially stable.

The reduced model $(x_{\mathrm{r}}, y_{\mathrm{r}})$ is found as

$$\dot{x}_{\mathrm{r}} = -x_{\mathrm{r}} + \arctan\left(x_{\mathrm{r}}^3 - y_{\mathrm{r}}\right) - 2y_{\mathrm{r}} \tag{37}$$
$$\varepsilon\dot{y}_{\mathrm{r}} = -2y_{\mathrm{r}} - x_{\mathrm{r}}^3 + y_{\mathrm{r}} = -y_{\mathrm{r}} - x_{\mathrm{r}}^3 \tag{38}$$

To show that the reduced model $(x_{\mathrm{r}}, y_{\mathrm{r}})$ is exponentially stable, we will again use singular pertubation. Letting $\varepsilon \to 0$, we find the second quasi-steady-state as

$$y_{\mathrm{r,qss}} = -x_{\mathrm{r}}^3 \tag{39}$$

This admits the second boundary layer model $\xi_2 := y_{\mathrm{r}} + x^3$ as

$$\frac{d\xi_2}{d\tau_2} = -\xi_2 \tag{40}$$

which again is obviously exponentially stable. The second reduced system is derived as

$$\dot{x}_{\mathrm{rr}} = -x_{\mathrm{rr}} + \arctan\left(2x_{\mathrm{rr}}^3\right) + 2x_{\mathrm{rr}}^3 \tag{41}$$

Taking the linearization around $x_{\mathrm{rr}} = 0$, we find the Jacobian of -1, implying local exponential stability of the second reduced model.

Thus, from the exponential stability of x_{rr} and ξ_2, we can conclude exponential stability of $(x_{\mathrm{r}}, y_{\mathrm{r}})$ using singular pertubation theory. Then, from the exponential stability of $(x_{\mathrm{r}}, y_{\mathrm{r}})$ and the exponential stability of ξ_1, we can conclude exponential stability in (x, y, z) by singular pertubation.