MAE 281A Prof. M. Krstic

NONLINEAR SYSTEMS Winter 2019

FINAL EXAM

Open book and class notes. **Collaboration not allowed.**

Total points: 70

Due Thursday, March 21, 11 a.m. at 2101 EBU1.

Problem 1. (15 pts) Consider the system

$$\dot{x}_1 = -x_1 + \tanh(x_1) - \tanh(x_2) \quad (1)$$
$$\dot{x}_2 = -x_2 + \tanh(x_1) + \tanh(x_2) \quad (2)$$

(a) Find all equilibrium points of the system.
(b) Determine the type of the equilibrium points.
(c) Using comparison principle, show that the solution of the state equations satisfies the following inequality

$$||x(t)|| \leq e^{-t}||x(0)|| + 2\sqrt{2}(1 - e^{-t}) \quad (3)$$

Hint: $|\tanh(x)| < 1, \forall x \in \mathbb{R}$ and consider the inequality $a + b \leq \sqrt{2a^2 + 2b^2}$, for $a, b > 0$.

Problem 2. (15 pts) Consider the following system

$$\dot{x}_1 = x_1^2 x_2 \quad (4)$$
$$\dot{x}_2 = -x_1^3. \quad (5)$$

(a) Find all equilibrium points of the system.
(b) Using the Lyapunov function $V = x_1^2 + x_2^2$ to prove the global stability of the origin and prove that every circle centered at the origin is a compact positively invariant set.
(c) Consider the Lyapunov-like function $U = \tan^{-1}(x_2/x_1)$. Note that this function is inspired by the representation of the state in the polar coordinates, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, which gives $U = \theta$. Is $U(x_1, x_2)$ positive definite?
(d) What is the largest invariant set within the set $\left\{\dot{U} = 0\right\}$?
(e) With LaSalle's invariance theorem, prove that $x_1(t) \to 0$ as $t \to \infty$.

Problem 3. (15 pts) Show that the following system is ISS with respect to input u.

$$\dot{x}_1 = -x_1 + x_1 x_2 + u \quad (6)$$
$$\dot{x}_2 = -x_2 \quad (7)$$

Give explicitly gain function $\gamma(r)$.
Hint: Take the Lyapunov function in the form $V = \phi(x_1) + x_2^2$.

Problem 4. (15 pts) Consider a second-order system

$$\dot{x}_1 = x_2 - 2\epsilon x_1 \quad (8)$$
$$\dot{x}_2 = -x_1 + 2\epsilon\cos(t)x_2 \quad (9)$$

Analyze the system for small ϵ using the averaging method.
Hint:
(a) Consider the system $\dot{x} = Ax + \epsilon g(t, x)$, where matrix $A_{n\times n}$ has only imaginary eigenvalues. The change of variables $x = \exp(At)y$ transforms the system into the form $\dot{y} = \epsilon f(t, y)$, where $f(t, y) = \exp(-At)g(t, \exp(At)y)$. The transformation is bounded and invertible.
(b) The following functions have a zero mean over the interval $[0, 2\pi]$: $\sin(\tau), \cos(\tau)$, $\sin(\tau)\cos(\tau)$, $\sin^3(\tau)$, $\cos^3(\tau), \sin^2(\tau)\cos(\tau), \sin(\tau)\cos^2(\tau)$.

Problem 5. (10 pts) Consider a second-order system

$$\dot{x}_1 = -x_1 + x_2 \tag{10}$$

$$\dot{x}_2 = -2x_2 + \epsilon x_1 \sin(x_2) \tag{11}$$

where the constant $0 < \epsilon < 1$.
(a) Determine whether the $\mathcal{O}(\epsilon)$ approximation is valid on the infinite interval.
(b) Determine stability of the origin of the above perturbed system.
Hint:
(a) Read perturbation theory in Chapter 10.1, 10.2 and Theorem 10.2 in Khalil's book.