MAE 143B

LINEAR CONTROL

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MIDTERM

August 20, 2007

- One page (front and back) of your own handwritten notes.
- No graphing calculators.
- Present your reasoning and calculations clearly. Inconsistent etchings will not be graded.
- Write answers only in the blue book.
- Total points: 35. Time: 1 hour 30 minutes.

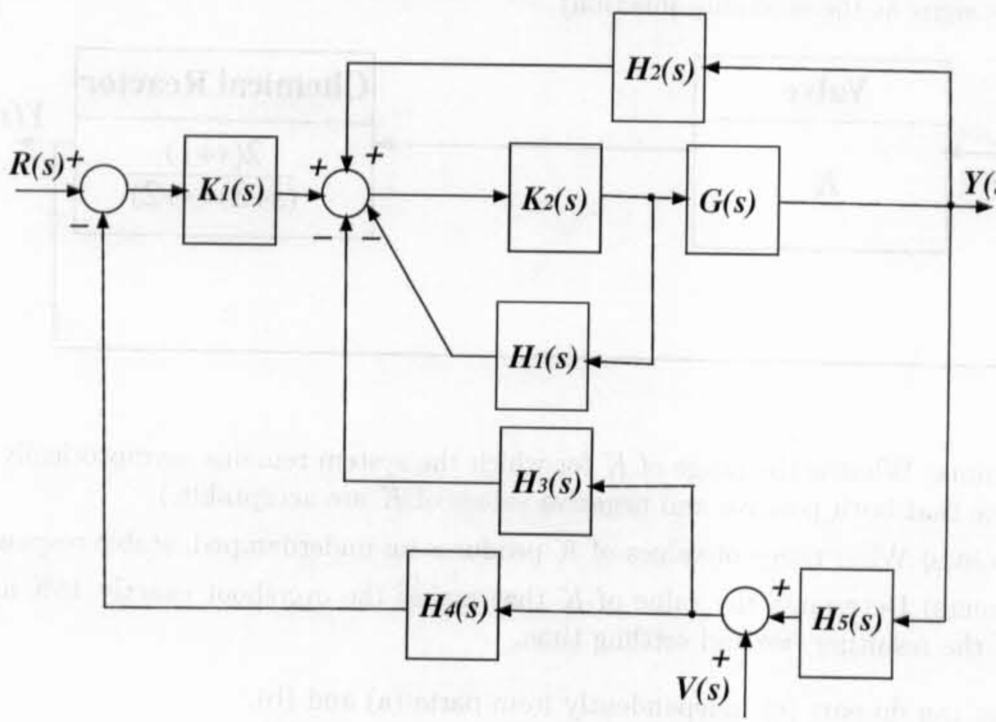
Problem 1. (6 points)

For the system in the following block diagram, compute the following transfer functions:

(a) (3 points) $T_1(s) = \dfrac{Y(s)}{R(s)}$

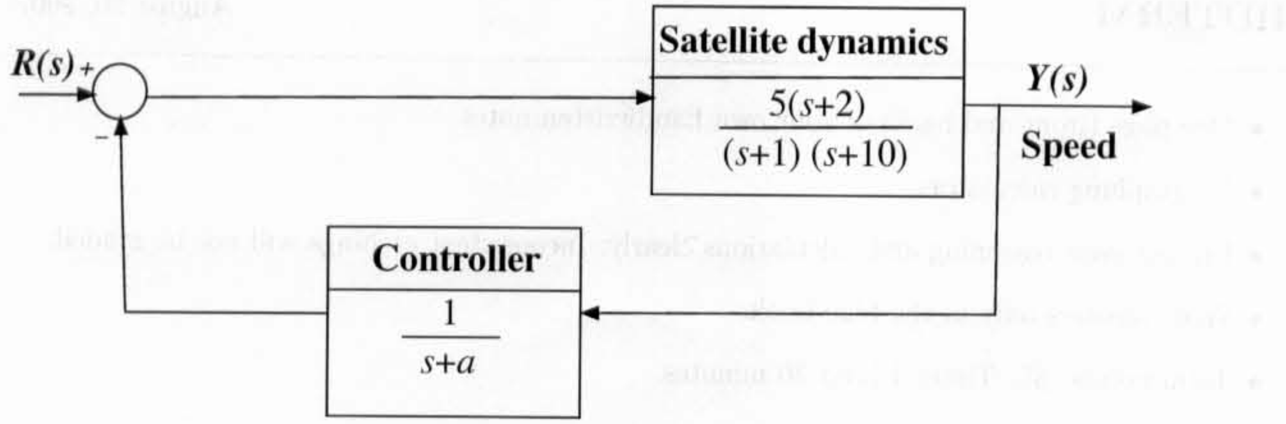
(b) (3 points) $T_2(s) = \dfrac{Y(s)}{V(s)}$

To get full credit simplify the resulting fraction as much as possible.



Problem 2. (5 points)

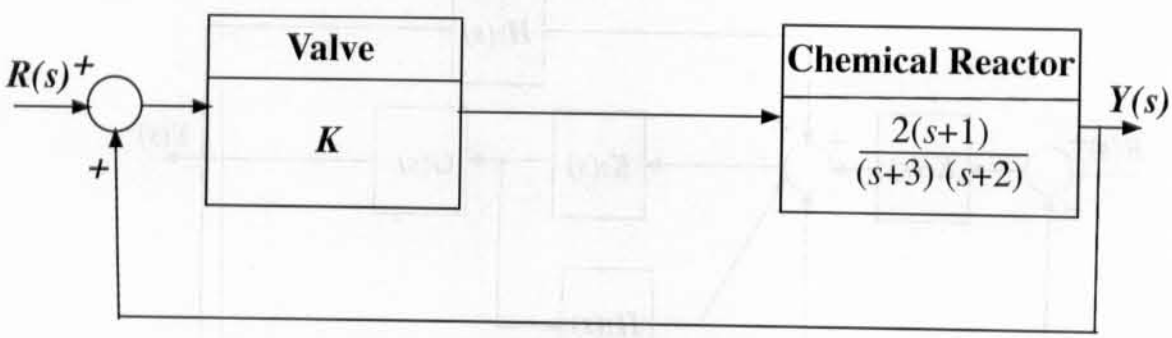
The following feedback system is used to control the rotational speed of an orbiting satellite.



Find the sensitivity of the closed-loop transfer function $T(s)$ to a small change in the parameter a.

Problem 3. (6 points)

An automatic system for controlling the flow of a certain reactant is shown in the figure. By mistake, an operator has connected the system in a "positive feedback" configuration (note the two + signs at the summing junction).



(a) (1 points) What is the range of K for which the system remains asymptotically stable? (Note that both positive and negative values of K are acceptable.)

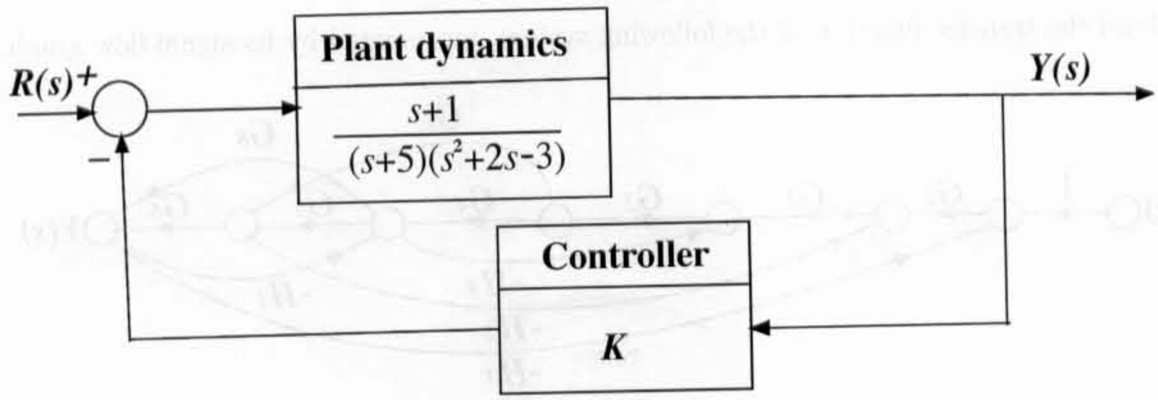
(b) (2 points) What range of values of K produces an underdamped, stable response?

(c) (3 points) Determine the value of K that makes the overshoot exactly 15% and then find the resulting rise and settling time.

Note: You can do part (c) independently from parts (a) and (b).

Problem 4. (6 points)

Consider the following plant with proportional feedback K.



(a) (3 points) What is the range of gains K for asymptotic stability?

(b) (3 points) Study how the number of unstable pole changes as K varies from $-\infty$ to $+\infty$. Determine the critical values of K at which the number of unstable poles changes and state the number of unstable poles between these critical values of K.

Problem 5. (6 points)

Are the following characteristic polynomials asymptotically stable? Explain why. Indicate how many (if any) right-half plane eigenvalues the polynomials (b) and (c) have.

(a) (2 points)

$$p_1(s) = s^{10} + s^9 + 3s^8 + 5s^7 - 4s^6 + s^5 + s^4 + 2s^3 + s^2 + 2s + 1$$

(b) (2 points)

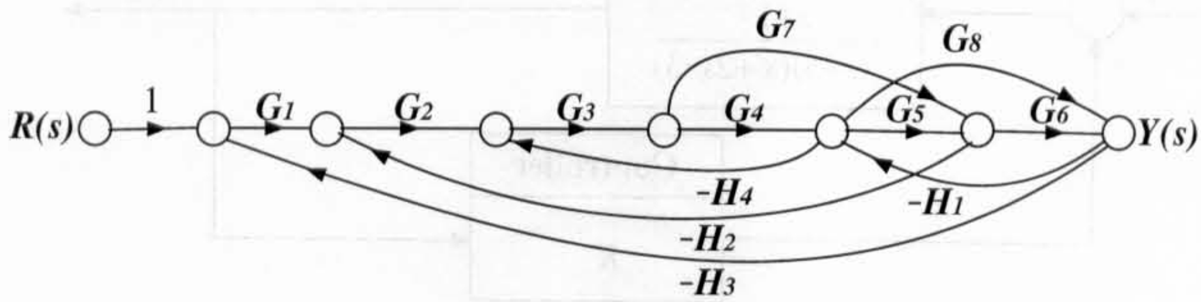
$$p_2(s) = s^5 + 4s^4 + 3s^3 + 2s^2 + 4s + 1$$

(c) (2 points)

$$p_3(s) = 2s^4 + 4s^3 + 6s^2 + 4s + 2$$

Problem 6. (6 points)

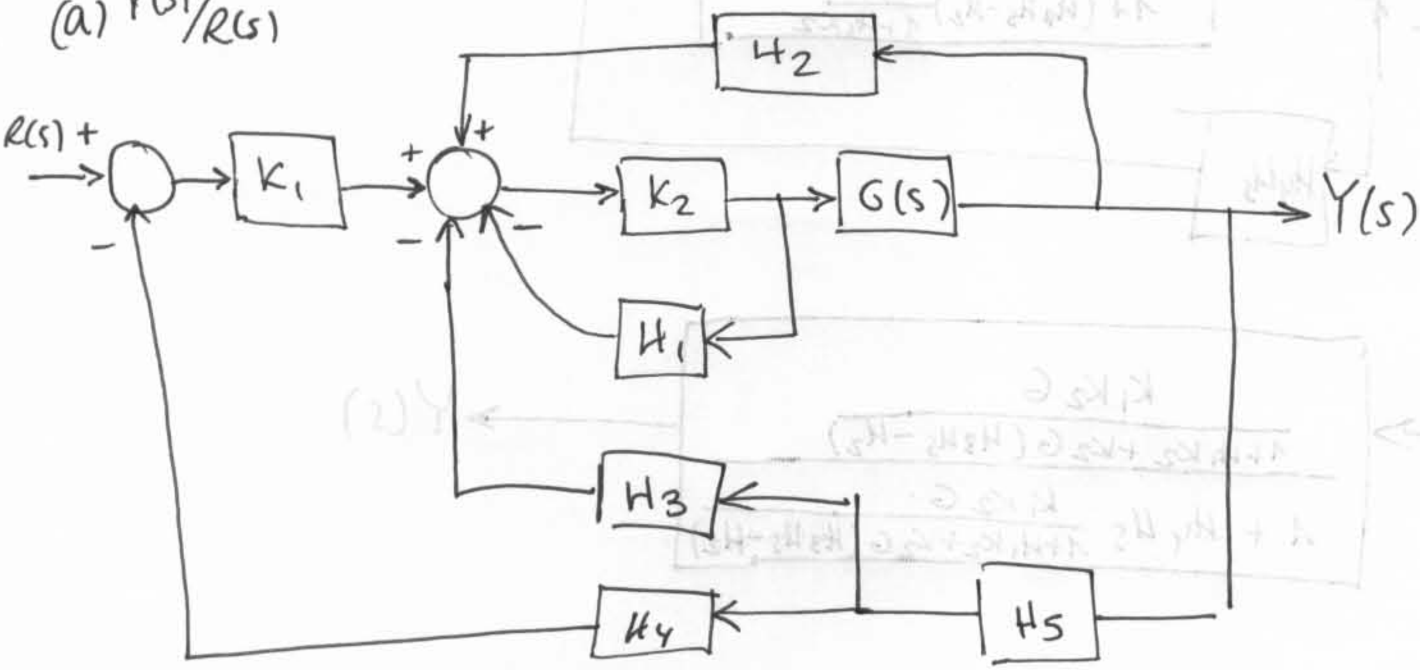
Find the transfer function of the following system, represented by its signal-flow graph.

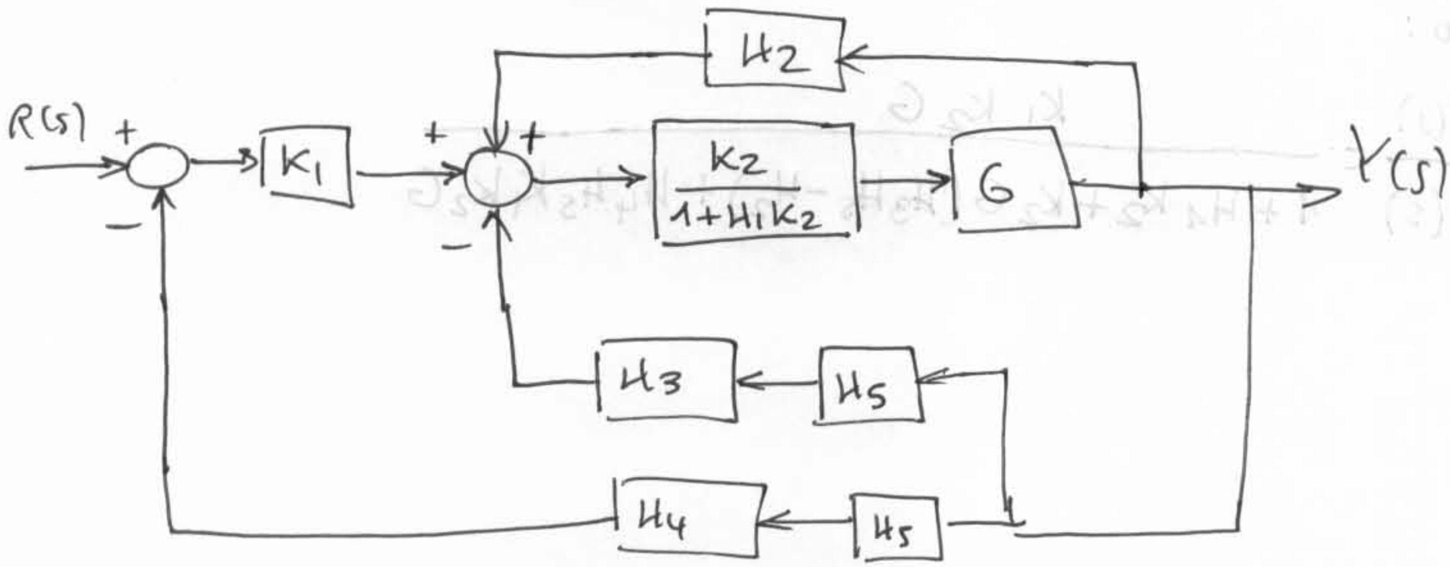


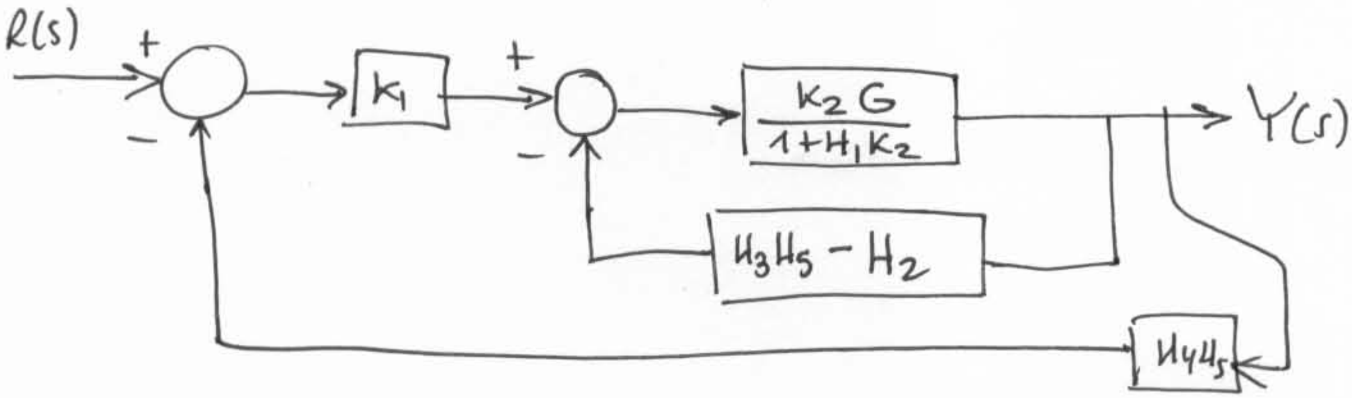
Problem 1.

We simplify the block diagrams to obtain the transfer function.

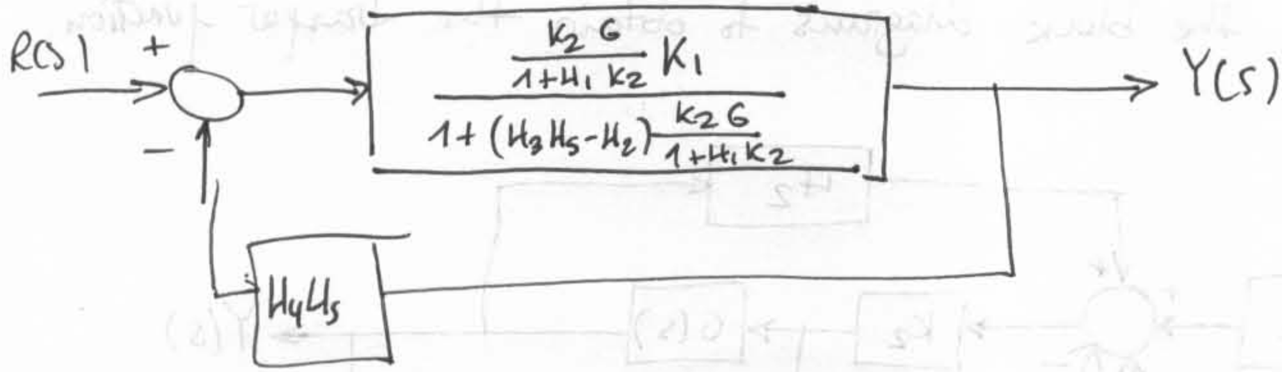
(a) $Y(s)/R(s)$

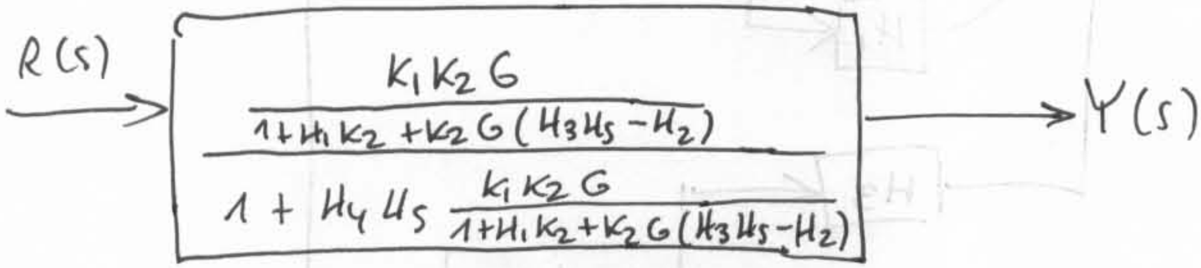






$$\frac{K_1 K_2 G}{1 + H_1 K_2 + K_2 G (H_3 H_5 - H_2)}$$

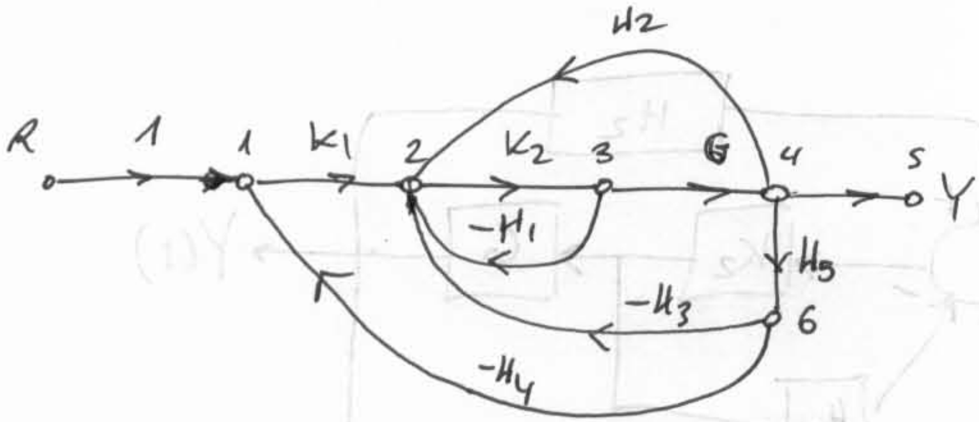




So:

$$\frac{Y(s)}{R(s)} = \frac{K_1 K_2 G}{1 + H_1 K_2 + K_2 G (H_3 H_5 - H_2) + H_4 H_5 K_1 K_2 G}$$

Alternative: Mason's Rule



FORWARD PATH	PATH GAIN
12345	$K_1 K_2 G$

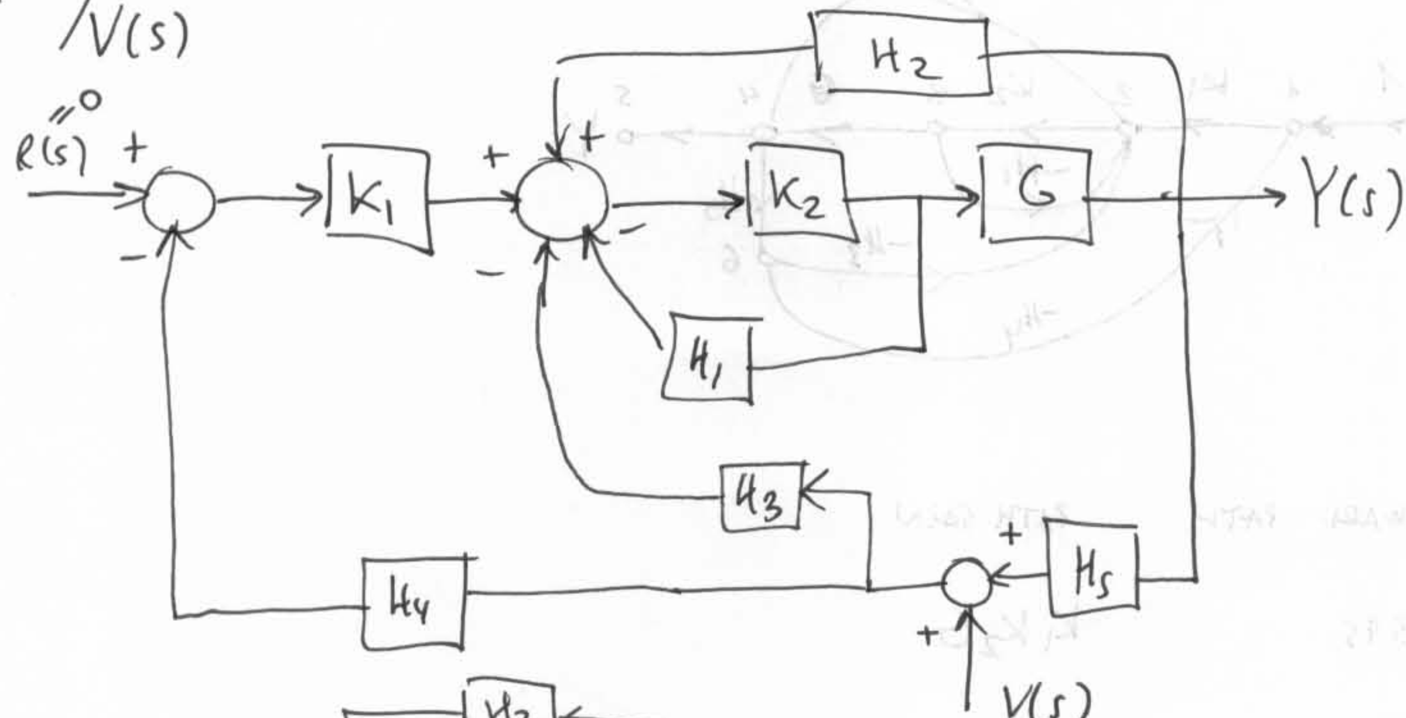
LOOP PATH	PATH GAIN
2342	$K_2 G H_2$
232	$-K_2 H_1$
23462	$-K_2 G H_5 H_3$
123462	$-K_1 K_2 G H_5 H_4$

$$\Delta = 1 - \left(K_2 G H_2 - K_2 H_1 - K_2 G H_5 H_3 - K_1 K_2 G H_5 H_4\right)$$

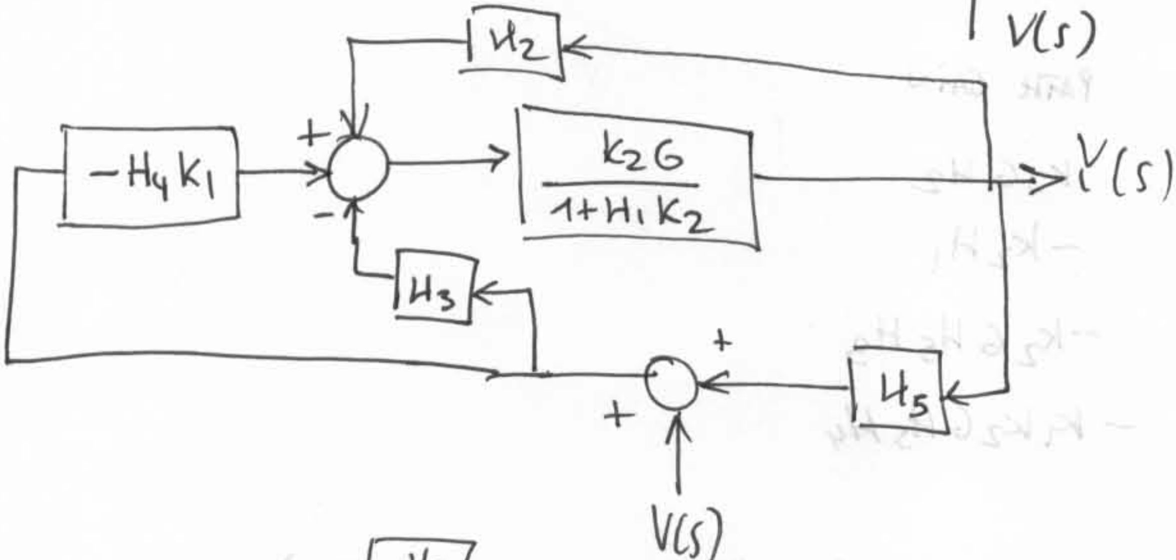
$$Y/R = \frac{K_1 K_2 G}{1 - K_2 G H_2 + K_2 H_1 + K_2 G H_5 H_3 + K_1 K_2 G H_5 H_4}$$

(b)

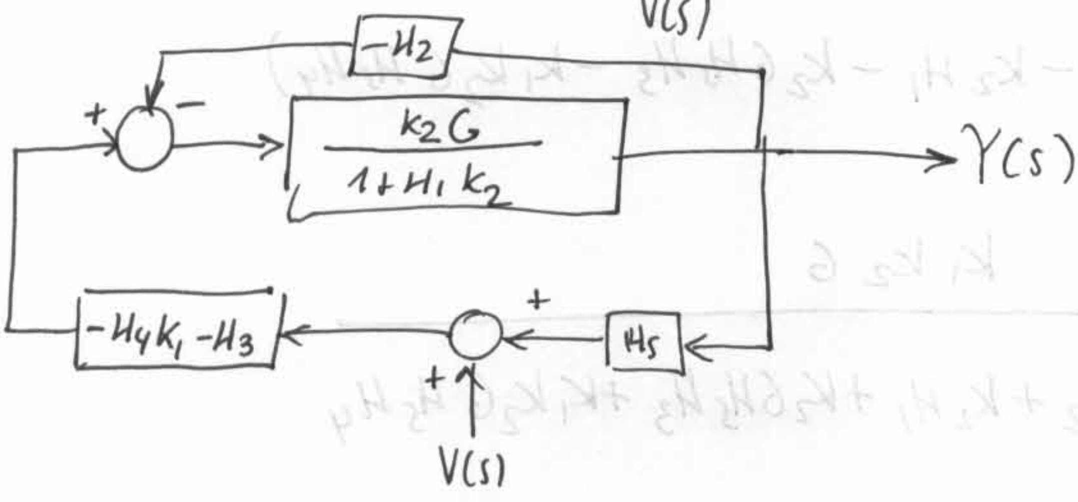
$Y(s)/V(s)$



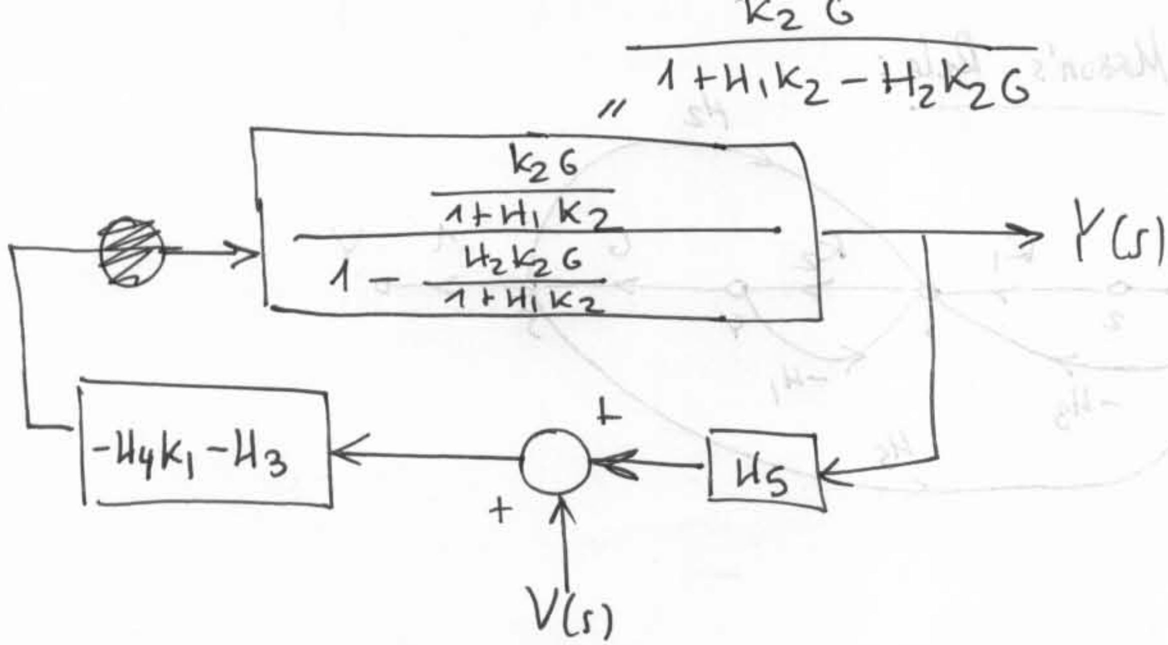
R(s) = 0
+
-
K1
+
+
-
-
K2
G
Y(s)
H2
H1
H3
H4
H5
+
+
V(s)

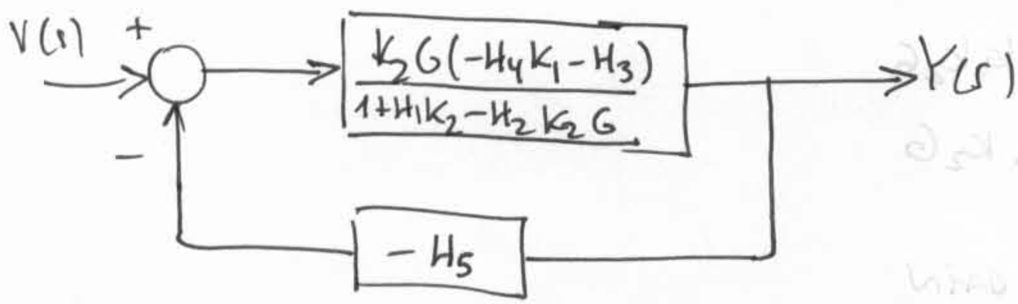


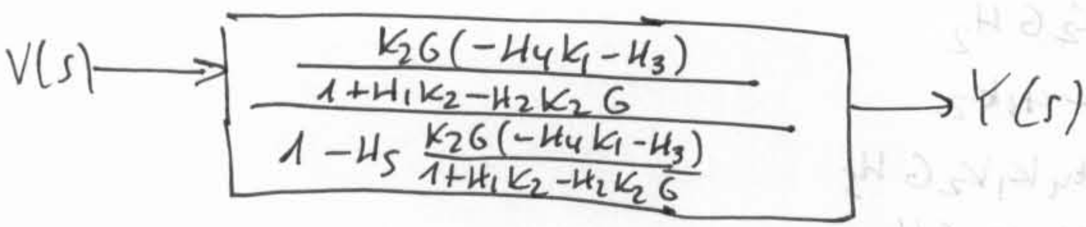
H2
-H4K1
+
-
K2G / 1+H1K2
Y(s)
H3
H5
+
+
V(s)



-H2
+
-
K2G / 1+H1K2
Y(s)
-H4K1 -H3
H5
+
+
V(s)

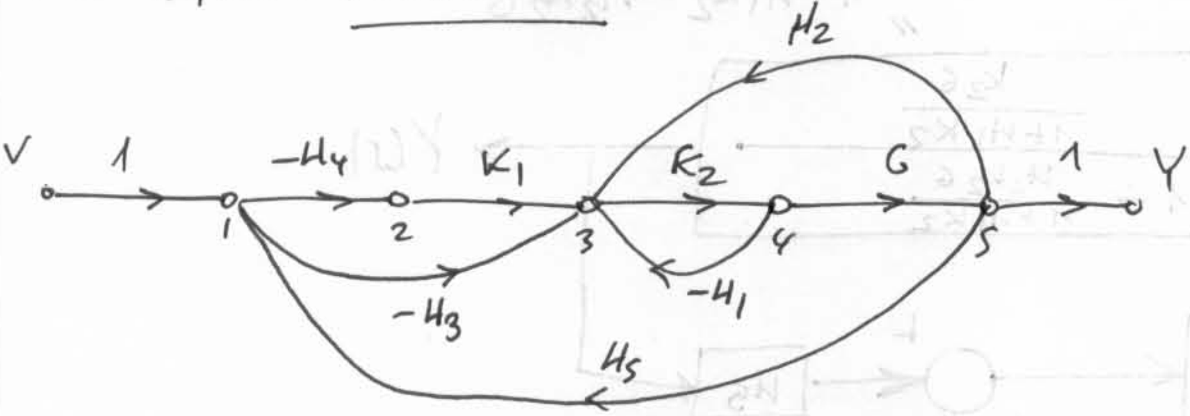






$$\frac{Y(s)}{V(s)} = \frac{-k_2 G (H_4 k_1 + H_3)}{1 + H_1 k_2 - H_2 k_2 G + H_5 k_2 G (H_4 k_1 + H_3)}$$

Alternative: Mason's Rule:



FORWARD PATH	PATH GAIN
12345	$-H_4 K_1 K_2 G$
1345	$-H_3 K_2 G$

LOOP PATH	PATH GAIN
3453	$K_2 G H_2$
343	$-H_1 K_2$
123451	$-H_4 K_1 K_2 G H_5$
13451	$-H_3 K_2 G H_5$

$$\Delta = 1 - (K_2 G H_2 - H_1 K_2 - H_4 K_1 K_2 G H_5 - H_3 K_2 G H_5)$$

$$\frac{Y(s)}{V(s)} = \frac{-H_4 K_1 K_2 G - H_3 K_2 G}{1 - K_2 G H_2 + H_1 K_2 + H_4 K_1 K_2 G H_5 + H_3 K_2 G H_5}$$

Problem 2

First derive the transfer function $T = \frac{Y(s)}{R(s)}$

$$T = \frac{\frac{5(s+2)}{(s+1)(s+10)}}{1 + \frac{1}{s+a}\frac{5(s+2)}{(s+1)(s+10)}} \overset{\text{simplifying}}{=} \frac{5(s+2)(s+a)}{(s+a)(s+1)(s+10)+5(s+2)}$$

The formula for the sensitivity is:

$$S_a^T = \frac{a}{T(s)} \frac{dT(s)}{da}$$

since

$$\frac{dT(s)}{da} = \frac{5(s+2)\left((s+a)(s+1)(s+10)+5(s+2)\right) - (s+1)(s+10)5(s+2)(s+a)}{\left((s+a)(s+1)(s+10)+5(s+2)\right)^2}$$

$$= \frac{25(s+2)^2}{\left((s+a)(s+1)(s+10)+5(s+2)\right)^2}$$

we get

$$S_a^T = \frac{a}{T(s)}\frac{dT(s)}{da} = a\,\frac{(s+a)(s+1)(s+10)+5(s+2)}{5(s+2)(s+a)} \cdot \frac{25(s+2)^2}{\left((s+a)(s+1)(s+10)+5(s+2)\right)^2}$$

$$= \frac{5 \cdot a \cdot (s+2)}{\left((s+a)(s+1)(s+10)+5(s+2)\right)(s+a)}$$

Problem 3

First derive the transfer function $T = \frac{Y(s)}{R(s)}$

$$T = \frac{k\frac{2(s+1)}{(s+3)(s+2)}}{1 - k\frac{2(s+1)}{(s+3)(s+2)}} = \frac{2k(s+1)}{(s+3)(s+2) - 2k(s+1)}$$

positive fbk!

(a) Look at the denominator of T:

$$(s+3)(s+2) - 2k(s+1) = s^2 + (5-2k)s + (6-2k)$$

since it is a 2nd order polynomial, it will have stable poles if and only if all the coefficients are positive (this can be seen also using Routh's Criterion).

Hence the conditions are:

$5 - 2k > 0 \Rightarrow k < 5/2$

$6 - 2k > 0 \Rightarrow k < 3$

if $k < 5/2$ both conditions are verified, hence we require $k \in (-\infty, 5/2)$ for stability.

(b) There will be an underdamped response if the denominator has complex solutions.

The solutions of $s^2 + (5-2k)s + (6-2k) = 0$

are: $$s = \frac{-(5-2k) \pm \sqrt{(5-2k)^2 - 4\cdot(6-2k)}}{2}$$

they will be complex if

$$(5-2k)^2 - 4(6-2k) < 0$$

hence

$$25 + 4k^2 - 20k - 24 + 8k < 0$$

$$k^2 - 3k + 1/4 < 0$$

the solutions of $k^2 - 3k + 1/4 = 0$ are

$$k = \frac{3 \pm \sqrt{9 - 4 \cdot 1/4}}{2} = \frac{3 \pm \sqrt{8}}{2} = 3/2 \pm \sqrt{2}$$

hence for $k \in (3/2 - \sqrt{2}, 3/2 + \sqrt{2})$ the solutions are underdamped

since they are stable if $k < 5/2 < 3/2 + \sqrt{2}$, the desired rage for K is then:

$$K \in \left(3/2 - \sqrt{2}\,,\, 5/2\right)$$

Alternative: identify the denominator $s^2+(5-2K)s+(6-2k)$ with $s^2+2\zeta\omega_n s+\omega_n^2$.

Then: $\begin{cases} 2\zeta\omega_n = 5-2K \\ \omega_n^2 = 6-2K \end{cases}$

Solve for ζ: $\zeta = \dfrac{5-2K}{2\omega_n} = \dfrac{5-2K}{2\sqrt{6-2K}}$

The response will be underdamped if $\zeta < 1$. Let's compute the values of K for which $\zeta = 1$:

$$1 = \frac{5-2K}{2\sqrt{6-2K}} \Rightarrow 4(6-2K) = (5-2K)^2 \Rightarrow 24-8K = 25+4K^2-20K$$

$$4K^2-12K+1=0 \Rightarrow K^2-3K+1/4=0$$

$\Rightarrow$ (as we saw in the previous page) $K = 3/2 \pm \sqrt{2}$

hence for $K \in \left(3/2 - \sqrt{2}, 3/2 + \sqrt{2}\right)$ the response will be underdaped. Combining with the stability result of (a) we get the same answer:

$$K \in \left(3/2 - \sqrt{2}, 5/2\right)$$

(c) The overshoot formula is:

$$M_p = e^{-\pi \zeta / \sqrt{1-\zeta^2}}$$

hence $\ln M_p = \dfrac{-\pi \zeta}{\sqrt{1-\zeta^2}} \Rightarrow \dfrac{-\ln M_p}{\pi} = \dfrac{\zeta}{\sqrt{1-\zeta^2}}$

squaring both sides: $\left(\dfrac{\ln M_p}{\pi}\right)^2 = \dfrac{\zeta^2}{1-\zeta^2}$

solving for ζ:

$$\zeta^2 = \frac{\left(\frac{\ln M_p}{\pi}\right)^2}{1+\left(\frac{\ln M_p}{\pi}\right)^2} \Rightarrow \zeta = \sqrt{\frac{\left(\ln (M_p)/\pi\right)^2}{1+\left(\frac{\ln M_p}{\pi}\right)^2}} = \frac{-\ln M_p/\pi}{\sqrt{1+\left(\frac{\ln M_p}{\pi}\right)^2}}$$

Hence $\zeta_{0.15} = \dfrac{-\frac{\ln 0.15}{\pi}}{\sqrt{1+\left(\frac{\ln 0.15}{\pi}\right)^2}} = 0.5169$

Since $\zeta = \dfrac{5-2k}{2\sqrt{6-2k}}$, squaring we get

$$4\zeta^2(6-2k) = 25-20k+4k^2 \Rightarrow k^2 + (2\zeta^2-5)k + \frac{25-24\zeta^2}{4} = 0$$

substituting $\zeta_{0.15} = 0.5169$:

$$k^2 - 4.4656k + 4.6467 = 0 \Rightarrow k = \begin{cases} 2.8147 \\ 1.6509 \end{cases}$$

since $k = 2.8147 > 5/2$ from (a) we know it produces an unstable solution. Hence we pick $\underline{\underline{k = 1.6569}}$

Since $w_n = \sqrt{6-2k}$ we get $w_n = 1.6426$ rad/s

Hence $\boxed{t_s = \dfrac{4.6}{\zeta w_n} = 5.4174 \text{ s}}$ and $\boxed{t_r \approx 1.8/w_n = 1.0958 \text{ s}}$

Problem 4.

First find the closed-loop transfer function $T(s)=\frac{Y(s)}{R(s)}$

$$T(s)=\frac{\frac{s+1}{(s+5)(s^2+2s-3)}}{1+k\frac{s+1}{(s+5)(s^2+2s-3)}}=\frac{s+1}{(s+5)(s^2+2s-3)+k(s+1)}$$

(a) Study the denominator of $T(s)$:

$$(s+5)(s^2+2s-3)+k(s+1)=s^3+7s^2+(7+k)s-15+k$$

Using the Routh's criterion:

s^3	1	$7+k$
s^2	7	$-15+k$
s	$\frac{49+7k+15-k}{7}=\frac{64+6k}{7}$	
1	$-15+k$	

from the last row: $-15+k>0 \Rightarrow k>15$

from the third row: $\frac{64+6k}{7}>0 \Rightarrow k>-\frac{64}{6}$

hence if $\underline{\underline{k>15}}$ both conditions are verified.

(b) For $K > 15$, from (a), we get 0 unstable poles

- if $K \in (-\frac{64}{6}, 15)$, then the Routh table looks like:

s^3 +
s^2 +
s +
1 −

$\Rightarrow$ 1 unstable pole (1 sign change)

- if $K \in (-\infty, -64/6)$ the Routh table looks like

s^3 +
s^2 +
s −
1 −

$\Rightarrow$ 1 unstable pole (1 sign change)

Hence the number of poles only changes at $K = 15$ and we get:

$K \in (15, \infty) \Rightarrow$ 0 unstable poles

$K \in (-\infty, 15) \Rightarrow$ 1 unstable pole

Problem 5.

(a) There is one negative coefficient ($-4s^6$) and the hence the <u>necessary</u> condition is violated and the characteristic polynomial has unstable poles (no need to do Routh's criterion).

(b)

s^5	1	3	4	+
s^4	4	2	1	+
s^3	$\frac{12-2}{4} = \frac{5}{2}$	$\frac{16-1}{4} = \frac{15}{4}$		+
s^2	$\frac{5-15}{5/2} = -4$	1		−
s^1	[illegible] $\leftarrow \frac{-15-5/2}{-4} = \frac{35}{8}$			+
1	1			+

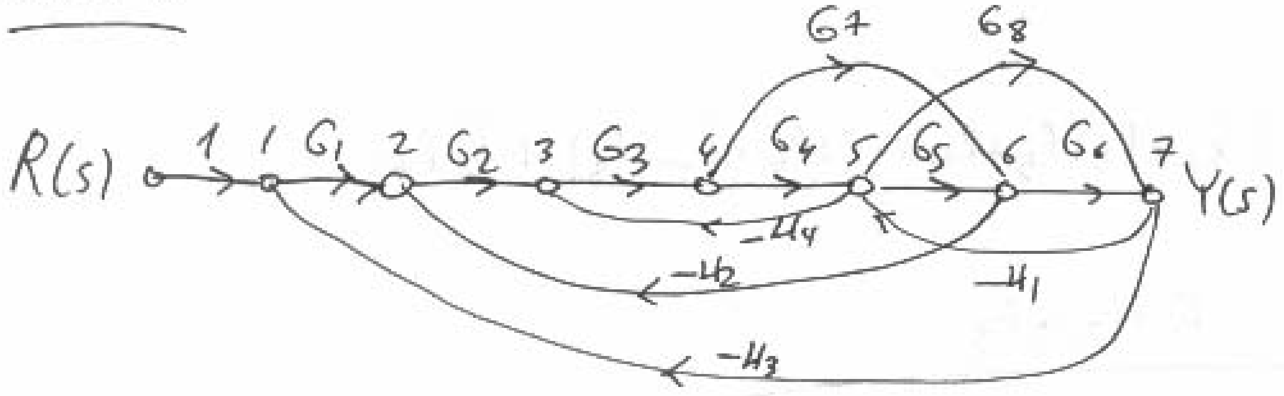
$\Rightarrow$ 2 <u>unstable</u> poles (2 sign changes)

(c)

s^4	2	6	2	+
s^3	4	4		+
s^2	$\frac{24-8}{4} = 4$	2		+
s^1	$\frac{16-8}{4} = 2$			+
1	2			+

$\Rightarrow$ stable (no sign changes)

Problem 6



FORWARD LOOPS	LOOP GAIN	
1234567	$F_1 = G_1 G_2 G_3 G_4 G_5 G_6$	$\Delta_1 = 1$
123467	$F_2 = G_1 G_2 G_3 G_7 G_6$	$\Delta_2 = 1$
123457	$F_3 = G_1 G_2 G_3 G_4 G_8$	$\Delta_3 = 1$

LOOP PATH	LOOP GAIN
12345671	$l_1 = -G_1 G_2 G_3 G_4 G_5 G_6 H_3$
1234671	$l_2 = -G_1 G_2 G_3 G_7 G_6 H_3$
1234571	$l_3 = -G_1 G_2 G_3 G_4 G_8 H_3$
234562	$l_4 = -G_2 G_3 G_4 G_5 H_2$
23462	$l_5 = -G_2 G_3 G_7 H_2$
3453	$l_6 = -G_3 G_4 H_4$
575	$l_7 = -G_8 H_1$
5675	$l_8 = -G_5 G_6 H_1$

l_7 and l_5 don't touch!

Hence

$$\Delta = 1 - (l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7 + l_8) + l_5 l_7$$

$$\frac{Y(s)}{R(s)} = \frac{F_1 + F_2 + F_3}{\Delta}$$

$$= \frac{G_1G_2G_3G_4G_5G_6 + G_1G_2G_3G_7G_6 + G_1G_2G_3G_4G_8}{1 + G_1G_2G_3G_4G_5G_6H_3 + G_1G_2G_3G_7G_6H_3 + G_1G_2G_3G_4G_8H_3 + G_2G_3G_4G_5H_2} \cdots$$

$$\cdots \frac{}{+ G_2G_3G_7H_2 + G_3G_4H_4 + G_8H_1 + G_5G_6H_1 - G_8H_1G_2G_3G_7H_2}$$