AMES 141
LINEAR CONTROL

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MIDTERM EXAM

November 9, 1999

NAME: SOLUTIONS

- Open books and notes.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 30.
- Time: 4–5:15 (2.5 minutes/point)

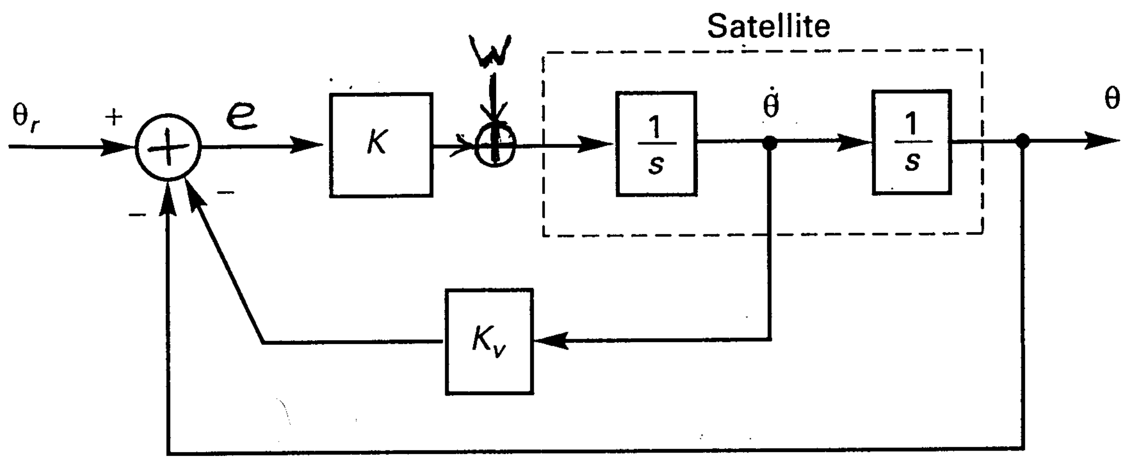
Problem 1.

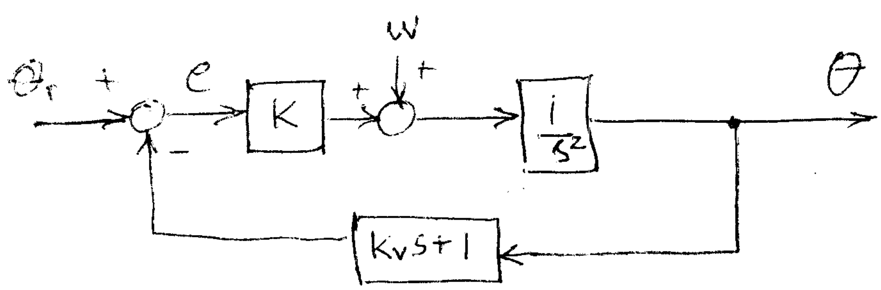
The block diagram given below represents a satellite attitude control system (employing both angle and angular velocity measurements). Derive the following transfer functions (3 points each):

(a) $\dfrac{\theta(s)}{\theta_r(s)}$

(b) $\dfrac{e(s)}{\theta_r(s)}$

(c) $\dfrac{\theta(s)}{w(s)}$





(a) $\dfrac{\theta}{\theta_r} = \dfrac{K/s^2}{1+(K_v s+1)K/s^2} = \dfrac{K}{s^2+KK_v s+K}$

(b) $\dfrac{e}{\theta_r} = \dfrac{1}{1+(K_v s+1)K/s^2} = \dfrac{s^2}{s^2+KK_v s+K}$

(c) $\frac{\Theta}{W} = \frac{1/s^2}{1 + K(k_v s+1)\frac{1}{s^2}} = \frac{1}{s^2 + KK_v s + K}$

Problem 2.

Consider the transfer function:

$$\frac{1}{s^2 + as + 4}$$

(a) (3 points) Find its step response for $a = 5$.

(b) (3 points) Find its step response for $a = 4$.

(c) (3 points) Find its *impulse* response for $a = 2\sqrt{3}$.

(a) $\mathcal{L}^{-1}\left\{ \frac{1}{(s^2+5s+4)} \frac{1}{s} \right\}$

$= \mathcal{L}^{-1}\left\{ \frac{1}{(s+4)(s+1)\,s} \right\}$

$= \mathcal{L}^{-1}\left\{ \frac{1}{(-3)(-4)} \frac{1}{s+4} + \frac{1}{3(-1)} \frac{1}{s+1} + \frac{1}{4\cdot 1} \frac{1}{s} \right\}$

$= \mathcal{L}^{-1}\left\{ \frac{1/12}{s+4} + \frac{-1/3}{s+1} + \frac{1/4}{s} \right\}$

$= \left(\frac{1}{12} e^{-4t} - \frac{1}{3} e^{-3} + \frac{1}{4} \right) 1(t)$

(b) $\mathcal{L}^{-1}\left\{ \frac{1}{s^2+4s+4} \frac{1}{s} \right\}$

$= \mathcal{L}^{-1}\left\{ \frac{1}{(s+2)^2 s} \right\}$

$= \mathcal{L}^{-1}\left\{ \frac{C}{s} + \frac{D_1}{s+2} + \frac{D_2}{(s+2)^2} \right\}$

$= (*)$ $\longrightarrow$

$$C = \frac{1}{4}$$

$$D_1 = \frac{1}{1!}\frac{d}{ds}\left(\frac{1}{s}\right)\Big|_{s=-2} = -\frac{1}{s^2}\Big|_{s=-2} = -\frac{1}{4}$$

$$D_2 = \frac{1}{0!}\frac{d^0}{ds^0}\left(\frac{1}{s}\right)\Big|_{s=-2} = \frac{1}{s}\Big|_{s=-2} = -\frac{1}{2}$$

$$(*) = \mathcal{L}^{-1}\left\{\frac{1/4}{s} - \frac{1/4}{s+2} - \frac{1/2}{(s+2)^2}\right\}$$

$$= \left(\frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}\right)1(t)$$

(c) $$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2\sqrt{3}s+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+2\sqrt{3}s+\sqrt{3}^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+\sqrt{3})^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{\sigma^2+1}\right\} \qquad \sigma = s+\sqrt{3}$$

$$= e^{-\sqrt{3}t}\sin t \qquad \text{(by freq. shift thm)}$$

Problem 3. (4 points)

Consider the transfer function:

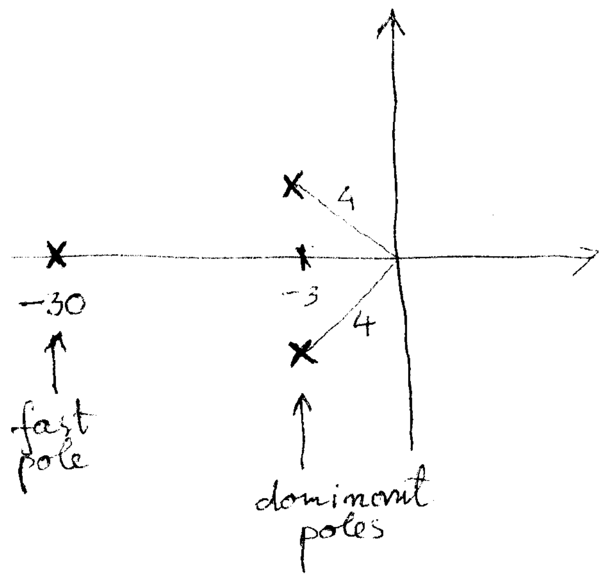
$$\frac{480}{(s+30)(s^2+6s+16)}$$

Calculate the overshoot and the peak time.

$$\frac{16}{\left(\frac{s}{30}+1\right)\left(s^2+2\cdot\frac{3}{4}\cdot 4s+4^2\right)}$$

$\omega_n = 4$

$\zeta = 0.75$



The fast pole has negligible influence on the response. Thus we apply the formulae to

$$\frac{16}{s^2+6s+16}$$

$$M_p = e^{-\pi\frac{\zeta}{\sqrt{1-\zeta^2}}} = 0.0284$$

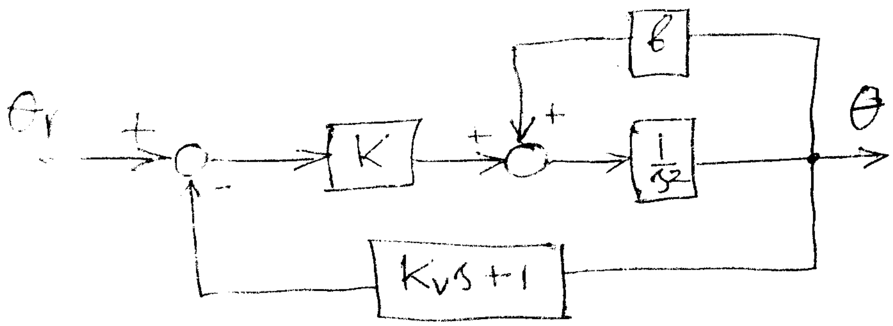
$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 1.19$$

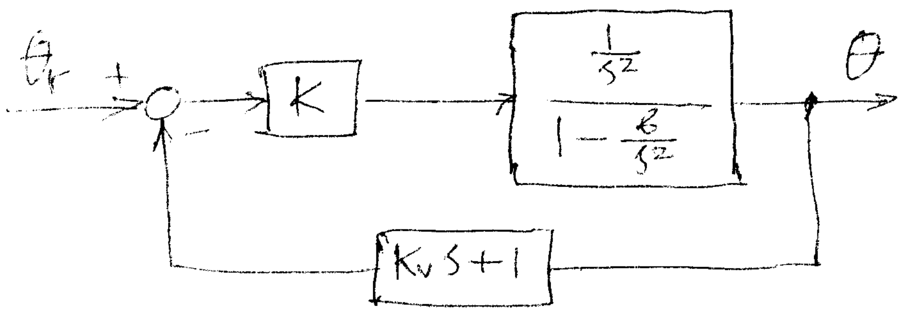
Problem 4. (5 points)

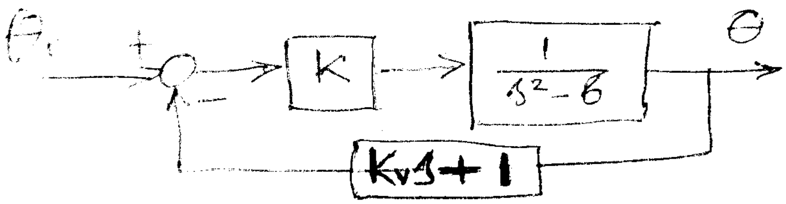
In the system from Problem 1 with an additional loop

$$w(s) = b\theta(s),$$

where b is a constant gain, find the closed loop transfer function $\theta(s)/\theta_r(s)$. (Draw the diagram first!)







$$\frac{\theta}{\theta_r} = \frac{\frac{K}{s^2-b}}{1+(K_v s+1)\frac{K}{s^2-b}} = \frac{K}{s^2-b+K(K_v s+1)}$$

$$= \frac{K}{s^2+KK_v s+K-b}$$

Problem 5. (3 points)

Apply a step input to the system from Problem 4 and determine the steady state value of the ouput for $b < \mathbf{K}$.

step

$$\theta(\infty) = \lim_{s \to 0} s \frac{K}{s^2 + KK_v s + K - b} \frac{1}{s}$$ (final value thm.)

$$= \lim_{s \to 0} \frac{K}{s^2 + KK_v s + K - b}$$

$$= \frac{K}{K - b}$$

Note that the result is valid only for $b < K$ because otherwise $\frac{K}{s^2 + KK_v s + K - b}$ has a pole in the RHP.