Problem 1: Block Diagram Reduction (3 points) Consider the following block diagram:

Using block diagram algebra, reduce the block diagram to find the transfer function $T = \frac{Y}{U}$. Simplify the transfer function as much as possible to receive full credit.

$$T = \frac{G_3}{1 + G_1 H_1}$$
Problem 2: Transfer Function (4 points) Find the transfer function \( \frac{y}{r} \) for the following system:

![Block diagram of the system]

The best way to solve this problem is by writing a system of equations involving the signals \( a \) and \( b \). It is probably impossible (or very difficult) to solve this problem using only block diagram manipulations, whereas the use of Mason’s rule is possible but overly complicated.

Equations involving the signals \( a \) and \( b \):

\( a = r - Gb \)  \hspace{1cm} (1)
\( b = a + Gy \)  \hspace{1cm} (2)
\( y = a + b \)  \hspace{1cm} (3)

Plugging (2) in (1) and (3), we have

\( a = r - Ga - G^2y \)
\( y = 2a + Gy \)

which lead to

\[ y = 2 \frac{r - G^2y}{1 + G} + Gy = \frac{2}{1+G}r \frac{G - G^2}{1+G} y \]
\[ \Rightarrow \left( 1 - \frac{G - G^2}{1+G} \right) y = \frac{2}{1+G} \]
\[ \Rightarrow \frac{y}{r} = \frac{2}{1 + G^2} \]
Problem 3: Time Domain Specifications and Sensitivity (8 points) The following block diagram models an armature-controlled DC motor.

(a) (3 points) Find the transfer function \( G(s) \) from \( T_d(s) \) to \( \Omega(s) \), and the transfer function \( H(s) \) from \( V_a(s) \) to \( \Omega(s) \).

\[
G(s) = -\frac{\frac{1}{Js+F}}{1 + \frac{\frac{k_1 k_5}{k_4 k_5}}{(Js+F)(L_a s+R_a)}} = -\frac{L_a s + R_a}{(Js+F)(L_a s+R_a) + k_4 k_5}
\]

\[
H(s) = \frac{\frac{k_5}{k_4 k_5}}{1 + \frac{\frac{k_5}{k_4 k_5}}{(Js+F)(L_a s+R_a)}} = \frac{k_5}{(Js+F)(L_a s+R_a) + k_4 k_5}
\]

(b) (2 points) Given \( L_a = 1, R_a = 1, J = 2, F = 3, k_4 = 5 \), calculate the sensitivity of the closed-loop transfer function \( H(s) \) with respect to changes in \( k_5 \).

Define \( A(s) \triangleq L_a s^2 + (R_a J + L_a F)s + R_a F \). Then, \( H(s) = \frac{k_5}{A(s) + k_4 k_5} \).

\[
S_{k_5}^H = \frac{dH}{dk_5} \left( \frac{k_5}{H} \right) = \frac{(A(s) + k_4 k_5) - k_5 k_4}{(A(s) + k_4 k_5)^2} (A(s) + k_4 k_5) = \frac{A(s)}{A(s) + k_4 k_5}
\]

\[
= \frac{L_a J s^2 + (R_a J + L_a F)s + R_a F}{L_a J s^2 + (R_a J + L_a F)s + R_a F + k_4 k_5}
\]

\[
= \frac{2s^2 + 5s + 3}{2s^2 + 5s + 3 + 5k_5} = \frac{s^2 + \frac{5}{2}s + \frac{3}{2}}{s^2 + \frac{5}{2}s + \frac{3}{2} + \frac{5}{2}k_5}
\]

(c) (1 points) Determine \( k_5 \) such that the overshoot is \( M_p = e^{-\frac{\pi}{\sqrt{3}}} \).

From \( M_p = e^{-\frac{\pi}{\sqrt{1 - \zeta^2}}} = e^{-\frac{\pi}{\sqrt{3}}} \), we have

\[
\frac{\zeta}{\sqrt{1 - \zeta^2}} = \frac{1}{\sqrt{3}} \Rightarrow \frac{\zeta^2}{1 - \zeta^2} = \frac{1}{3} \Rightarrow 4\zeta^2 = 1 \Rightarrow \zeta = \frac{1}{2}
\]

3
From
\[2\zeta \omega_n = \frac{5}{2} \Rightarrow \omega_n = \frac{5}{2}\]
we have
\[\frac{3}{2} + \frac{5}{2} k_5 = \omega_n^2 = \frac{25}{4} \Rightarrow k_5 = \frac{19}{10}\]

(d) (2 points) What are the settling time, rise time, and peak time for \(H(s)\) determined in (c)?

\[t_s = \frac{4.6}{\zeta \omega_n} = \frac{4.6}{(1/2)(5/2)} = 3.68\]
\[t_r = \frac{1.8}{\omega_n} = \frac{1.8}{5/2} = 0.72\]
\[t_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} = \frac{\pi}{\frac{5}{2} \sqrt{1 - (1/2)^2}} = \frac{4\pi}{5\sqrt{3}} = \frac{4\sqrt{3}}{15}\]

**Problem 4: PID Control (7 points)** Consider the following system:

(a) (2 points) Using Routh’s criterion, find the gain \(K\) that would make the feedback system marginally stable.

\[\frac{Y}{U} = \frac{K}{s^3 + 2s^2 + 2s + 1} = \frac{K}{s^3 + 2s^2 + 2s + 1 + K}\]

This closed-loop system is asymptotically stable. \(\Leftrightarrow 1 + K > 0\) and \(4 > 1 + K\)

\(\Leftrightarrow -1 < K < 3\)

Therefore, the closed-loop system becomes marginally stable when \(K = -1\) or 3.

(b) (1 points) Using Ziegler-Nichols tuning, find the gain \(k_p\) needed for P (proportional) control.

As we increase \(K\) from 0, the closed-loop system becomes marginally stable when \(K = 3\), which means that the ultimate gain is \(K_u = 3\). Thus, the ultimate sensitivity method leads us to \(k_p = \frac{K_u}{2} = \frac{3}{2}\).

(c) (2 points) The ultimate period corresponding to the Ziegler-Nichols tuning in (b) can be shown to be \(P_u = \sqrt{2}\pi\). What should be the PI, PID control gains?

With the ultimate gain \(K_u = 3\) in (b), we have

\[PI : \quad k_p = 0.45 K_u = 1.35, \quad T_I = \frac{P_u}{1.2} = \frac{\sqrt{2}}{1.2} \pi = \frac{5\sqrt{2}}{6} \pi\]

\[PID : \quad k_p = 0.6 K_u = 1.8, \quad T_I = \frac{P_u}{2} = \frac{\sqrt{2}}{2} \pi, \quad T_D = \frac{P_u}{8} = \frac{\sqrt{2}}{8} \pi\]
(d) (2 points) Show that indeed $P_u = \sqrt{2\pi}$.

When $K = 3$, we have

$$\frac{Y}{U} = \frac{K}{s^3 + 2s^2 + 2s + 4} = \frac{K}{s^2(s + 2) + 2(s + 2)} = \frac{K}{(s^2 + 2)(s + 2)}$$

Thus, $\omega_n^2 = 2$, which leads to

$$P_u = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{2}} = \sqrt{2\pi}$$

**Problem 5: Stability (3 points)** Are the following polynomials stable? If not, how many eigenvalues in the right-half plane do they have?

(a) (1 points) $p_a(s) = s^3 + 2s^2 + 2s + 5$

$$\begin{array}{c|ccccc}
s^3 & 1 & 2 \\
s^2 & 2 & 5 \\
s^1 & -\frac{1}{2} \\
s^0 & 5 \\
\end{array}$$

Unstable. There are 2 poles in the right-half plane.

(b) (2 points) $p_b(s) = s^6 + s^5 + 2s^4 + 4s^3 + 3s^2 + 3s + 2$

$$\begin{array}{c|ccccc}
s^6 & 1 & 2 & 3 & 2 \\
s^5 & 1 & 4 & 3 \\
s^4 & -2 & 0 & 2 \\
s^3 & 4 & 4 & \\
s^2 & 2 & 2 \\
s^1 & 0 & a_1(s) = 2s^2 + 2 \\
s^0 & 4 & a'_1(s) = 4s^1 \\
s^0 & 2 \\
\end{array}$$

Unstable. There are 2 poles in the right-half plane.