SOLUTIONS
Problem 1. (5 points) Using block diagram reduction techniques, find $\frac{Y}{R}$.

Simplify the transfer function as much as possible to receive full credit.

1) $\frac{G_1 G_3}{1 + G_3 H_1}$

2) $\frac{G_1 H_3}{1 + G_3 H_1}$

3) $\frac{G_1 H_3 (1 + G_3 H_1)}{G_1 G_2 G_3}$

4) $\frac{G_4 + H_3 (1 + G_3 H_1)}{G_2 G_3}$

5) $\frac{G_1 G_2 G_3 G_4 + G_1 H_3 + G_1 G_3 H_1 H_3}{1 + G_3 H_1 + H_2 G_1 G_2 G_3}$
Problem 2. (7 points) Using Mason’s rule, find the transfer function \( \frac{Y}{R} \) in the following system:

![System Diagram]

<table>
<thead>
<tr>
<th>Forward Path</th>
<th>Gain</th>
<th>( \Delta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) 1234567</td>
<td>( F_1 = G_1 G_2 G_3 G_4 G_5 G_6 )</td>
<td>( \Delta_1 = 1 )</td>
</tr>
<tr>
<td>2) 123567</td>
<td>( F_2 = G_1 G_2 G_8 G_5 G_6 )</td>
<td>( \Delta_2 = 1 )</td>
</tr>
<tr>
<td>3) 14567</td>
<td>( F_3 = G_7 G_4 G_5 G_6 )</td>
<td>( \Delta_3 = 1 )</td>
</tr>
<tr>
<td>4) 1423567</td>
<td>( F_4 = G_7 H_1 G_2 G_8 G_5 G_6 )</td>
<td>( \Delta_4 = 1 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Loop</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>( l_1 = G_2 G_3 H_1 )</td>
</tr>
<tr>
<td>2)</td>
<td>( l_2 = G_4 G_5 H_2 )</td>
</tr>
<tr>
<td>3)</td>
<td>( l_3 = G_5 G_6 H_4 )</td>
</tr>
<tr>
<td>4)</td>
<td>( l_4 = G_3 G_4 G_5 G_6 H_3 )</td>
</tr>
<tr>
<td>5)</td>
<td>( l_5 = G_5 G_6 H_3 )</td>
</tr>
<tr>
<td>6)</td>
<td>( l_6 = G_2 G_8 G_5 H_2 H_1 )</td>
</tr>
</tbody>
</table>

loops 1) and 3) do not touch

\[
\frac{Y}{R} = \frac{F_1 \Delta_1 + F_2 \Delta_2 + F_3 \Delta_3 + F_4 \Delta_4}{1 - (l_1 + l_2 + l_3 + l_4 + l_5 + l_6) + l_4 l_3}
\]

\[
= \frac{G_5 G_6 \left( G_1 G_2 G_3 G_4 + G_1 G_2 G_8 + G_7 G_4 + G_7 H_1 G_2 G_8 \right)}{1 - G_2 G_3 H_1 - G_4 G_5 H_2 - G_5 G_6 H_4 - G_3 G_4 G_5 G_6 H_3 - G_5 G_6 H_3 - G_2 G_3 G_5 G_6 H_1 + G_2 G_3 G_5 G_6 H_1 H_4}
\]
Problem 3. (5 points) Consider the feedback system for controlling the yaw of a fighter jet:

\[ Y(s) = \frac{1}{s(s+a)(s^2+s+1) + 1} \]

Find the sensitivity of the closed-loop transfer function \( T(s) \) to a small change in the parameter \( a \).

\[
\frac{Y(s)}{R(s)} = T(s) = \frac{1}{s(s+a)(s^2+s+1) + 1}
\]

\[
S_T = \frac{a}{T} \frac{dT}{da} = \frac{a}{T} \left( -\frac{s(s^2+s+1)}{(s(s+a)(s^2+s+1) + 1)^2} \right)
\]

\[
= -\frac{aS(s^2+s+1)}{s(s+a)(s^2+s+1) + 1}
\]
Problem 4. (7 points) Consider a DC motor with PD control:

(a) (4 points) Find $K_1$ and $K_2$ such that the peak time is 1 second and the settling time is 3 seconds.

(b) (3 points) What is the overshoot and the rise time for $K_1$ and $K_2$ determined in (a)?

\[
\frac{Y}{R} = \frac{K_1}{0.5s^2 + s + K_1 (1 + K_2 s)} = \frac{2K_1}{s^2 + 2(1 + K_1 K_2)s + 2K_1}
\]

\[
2K_1 = \omega_n^2, \quad 2(1 + K_1 K_2) = 2\frac{s}{\omega_n}
\]

\[
t_s = \frac{4.6}{5\omega_n} \Rightarrow 5\omega_n = \frac{4.6}{2} = 1.533
\]

\[
l_p = \frac{\pi}{\omega_n \sqrt{1 - \frac{s}{\omega_n}}} \Rightarrow \omega_n = \frac{\pi}{\sqrt{1 + (\sqrt{1 - \frac{s}{\omega_n}})^2}} = 1.533^2 = 12.2197
\]

\[
\omega_n = \sqrt{12.2197} = 3.496
\]

\[
\xi = \frac{1.533}{\omega_n} = 0.4385
\]

\[
K_1 = \frac{1}{2} \omega_n^2 = 6.11
\]

\[
K_2 = \frac{\xi \omega_n - 1}{K_1} = 0.087
\]

\[
t_r = \frac{1.8}{\omega_n} = 0.515
\]

\[
M_p = e^{-\xi/\sqrt{1 - \xi^2}} = 0.216
\]
Problem 5. (6 points) Determine the number of unstable poles for the following polynomials:

(a) (3 points) \( p_1 = s^5 + 2s^4 + 2s^3 + 2s^2 + 3s + 4 \)

(b) (3 points) \( p_2 = s^6 + s^5 + 3s^4 + 4s^3 + s^2 + s + 1 \)

(a) \[
\begin{array}{c|ccc}
  s^5 & 1 & 2 & 3 \\
  s^4 & 2 & 2 & 4 \\
  s^3 & 1 & 1 \\
  s^2 & 0 & 4 & \leftarrow \text{zero element in first column} \\
  s^1 & e & 4 \\
  s^0 & 1 - \frac{4}{e} \rightarrow -\infty \text{ as } e \rightarrow 0^+ \\
\end{array}
\]

Two sign changes \( \Rightarrow \) 2 unstable poles

(b) \[
\begin{array}{c|ccc}
  s^6 & 1 & 3 & 1 \\
  s^5 & 1 & 4 & 1 \\
  s^4 & -1 & 0 & 1 \\
  s^3 & 4 & 2 \\
  s^2 & \frac{1}{2} & 1 \\
  s^1 & -6 \\
  s^0 & 1 \\
\end{array}
\]

Four sign changes \( \Rightarrow \) 4 unstable poles