• Open books and notes.

• Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.

• Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate “Continued on page X.”

• The problems are not ordered by difficulty.

• Total points: 30.

• Time: 6:30–7:50
Problem 1. (7 points)

Consider the system

\[
\begin{array}{c}
H_2 \rightarrow \ H_1 \rightarrow \ G_1 \rightarrow \ G_2 \rightarrow \ Y \\
\end{array}
\]

(a) For \( H_1 = H_2 = 0 \), what is the transfer function from \( R \) to \( \theta \)?
(b) For \( H_1 \neq 0, H_2 \neq 0 \), what is the transfer function from \( R \) to \( Y \)?

Solution

(a) Redraw the block diagram with \( H1 = H2 = 0 \), \( R \) as the input and \( \theta \) as the output. Therefore, the transfer function from \( R \) to \( \theta \) is

\[
\frac{\theta}{R} = \frac{G_1}{1 + G_1 G_2 H_3}.
\]

The same result can be found by noting that

\[
\frac{Y}{R} = \frac{G_1 G_2}{1 + G_1 G_2 H_3}
\]

where \( Y = G_2 \theta \).
(b) The block diagram can be reduced in the following manner.

(i) Move the $G_2$ block ahead of the pick-off point.

(ii) Combine the blocks in the lower left of the diagram in feedback

(iii) Combine the blocks in feedback (left) and feedforward (right)

Therefore

$$\frac{Y}{R} = T_2T_3 = \frac{T_1(G_2 + H_2)}{G_2 + T_1H_1} = \frac{G_1G_2 + G_1H_2}{1 + G_1G_2H_3 + G_1H_1}.$$
(b) The transfer function from $R$ to $Y$ can also be found using Mason’s rule.

**Forward Paths**

Path 1 $P_1 = G_1G_2$
Path 2 $P_2 = G_1H_2$

**Loops**

Loop 1 $L_1 = -G_1H_1$
Loop 2 $L_2 = -G_1G_2H_3$

$$\triangle = 1 - (L_1 + L_2) = 1 + G_1H_1 + G_1G_2H_3$$

$\triangle_1 = 1$ (both loops touch Path 1)
$\triangle_2 = 1$ (both loops touch Path 2)

$$\frac{Y}{R} = \frac{1}{\triangle} \sum_{i=1}^{2} P_i \triangle_i = \frac{G_1G_2 + G_1H_2}{1 + G_1G_2H_3 + G_1H_1}$$
Problem 2. (5 points)

Consider the following step response:

\[ y(t) = 4 \left( 1 - e^{-2(t-3)} \right) 1(t-3). \]

Find the PID controller parameters \((K, T_I, T_D)\) using the Ziegler-Nichols transient-response method with a 0.25 decay ratio (Method 1).

**Solution** Begin by taking the Laplace transform of \(y(t)\).

\[
Y(s) = 4 \left( \frac{1}{s} - \frac{1}{s + 2} \right) e^{-3s}
\]

\[
= \frac{8e^{-3s}}{s(s + 2)}
\]

\[
= \frac{8e^{-3s}}{s + 2} R(s)
\]

\[
= \frac{4e^{-3s}}{\frac{1}{2}s + 1} R(s)
\]

Now compare \(\frac{Y(s)}{R(s)}\) to the standard form of a first order system with a time delay.

\[
\frac{Y(s)}{R(s)} = \frac{4e^{-3s}}{\frac{1}{2}s + 1} = \frac{Ae^{-tds}}{\tau s + 1}
\]

\[\Rightarrow t_d = 3, \tau = \frac{1}{2}, A = 4\]

The problem is completed by computing \(K, T_I\) and \(T_D\) using the equations in Table 4.1, where \(R = \frac{A}{\tau} = 8\) and \(L = t_d = 3\).

\[
K = \frac{1.2}{RL}
\]

\[
T_I = 2L
\]

\[
T_D = \frac{L}{2}
\]

\[\Rightarrow K = 0.05, T_I = 6, T_D = 1.5\]
Problem 3. (7 points)

Consider the system

\[ D(s) \quad G_1(s) \quad G_2(s) \quad H(s) \quad E(s) + \quad R(s) \quad Y(s) + \quad + \quad - \]

where

\[ G_1(s) = \frac{k}{2s} \]
\[ G_2(s) = \frac{1}{s + a} \]
\[ H(s) = k. \]

Compute the transfer function

\[ T(s) = \frac{Y(s)}{R(s)} \]

and find its sensitivity to the coefficient \( k \),

\[ S_T^k(s) = \frac{dT(s)}{dk} \frac{k}{T(s)}. \]

**Solution** Let \( D(s) \equiv 0 \), in order to compute \( T(s) \).

\[ \Rightarrow T(s) = \frac{Y(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \]
\[ = \frac{\left( \frac{k}{2s} \right) \left( \frac{1}{s + a} \right)}{1 + \left( \frac{k}{2s} \right) \left( \frac{1}{s + a} \right) k} \]
\[ = \frac{k}{2s^2 + 2as + k^2} = \frac{k}{s^2 + as + \frac{k^2}{2}} \]

Now compute the sensitivity of \( T(s) \) to the coefficient \( k \).

\[ S_T^k(s) = \frac{dT(s)}{dk} \frac{k}{T(s)} \]
\[ = \left( \frac{1}{(2s^2 + 2as + k^2)^2} \right) \frac{k}{(2s^2 + 2as + k^2)} \]
\[ = \frac{2s^2 + 2as - k^2}{2s^2 + 2as + k^2} = \frac{s^2 + as - \frac{k^2}{2}}{s^2 + as + \frac{k^2}{2}} \]
Problem 4. (4 points)

For the system form Problem 3 find $k$ and $a$ such that the closed-loop step response with respect to the input $R$ exhibits an overshoot of 10% and a settling time of 0.75 s.

**Solution** The system in problem 3, in standard second-order form, was found to be

$$T(s) = \frac{k}{s^2 + as + \frac{k^2}{2}} = C \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$ 

The problem is completed by relating the system parameters ($k,a$) with those of the standard second-order system ($\zeta, \omega_n$) and the given performance specifications ($M_p, t_s$).

From *Figure 3.28*

$$M_p = 10\% \Rightarrow \zeta = 0.61$$

Parameter $\zeta$ can also be computed using *Equation 3.50* ($\zeta = 0.59$).

Using *Equation 3.51*

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.61(0.75)} = 10.05 \text{ rad/s}$$

The plant parameters $k$ and $a$ are found to be

$$k = \sqrt{2}\omega_n$$
$$a = 2\zeta\omega_n$$

$$\Rightarrow k = 14.2, a = 12.3.$$
Problem 5. (7 points)

Consider the system from Problem 3 with \( k = 2 \) and \( a = 1 \).

(a) Find the transfer function from \( D(s) \) to \( Y(s) \).

(b) Find the time-domain output response to a step in \( D \).

Solution

(a) Let \( R(s) \equiv 0 \) to find \( \frac{Y(s)}{R(s)} \), then write the equations corresponding to the block diagram.

\[
Y(s) = G_2(s) (D(s) + G_1(s)E(s)) \tag{1}
\]

\[
E(s) = -H(s)Y(s) - D(s) \tag{2}
\]

Substitute (1) into (2) and compute \( \frac{Y(s)}{D(s)} \).

\[
Y(s) = G_2(s)D(s) - G_1(s)G_2(s)(H(s)Y(s) - D(s))
\]

\[
(1 + G_1(s)G_2(s)H(s))Y(s) = (G_2(s) - G_1(s)G_2(s))D(s)
\]

\[\Rightarrow \frac{Y(s)}{D(s)} = \frac{G_2(s) - G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\]

\[= \frac{1 - \frac{1}{s+1}}{1 + \frac{1}{s+1}} \cdot \frac{1}{2} \]

\[= \frac{s-1}{s(s+1)} \cdot \frac{2}{s(s+1)} \]

\[= \frac{s-1}{s^2 + s + 2} \]
(b) For a unit step input \( D(s) = \frac{1}{s} \),

\[
Y(s) = \frac{s-1}{s^2+s+2} D(s) \\
= \frac{1}{s} \left( \frac{s-1}{s^2+s+2} \right) \\
= \frac{1}{s^2+s+2} - \left( \frac{1}{s} \right) \frac{1}{s^2+s+2} \\
= F_1(s) - F_2(s).
\]

\[
\Rightarrow y(t) = (f_1(t) - f_2(t)) \ 1(t)
\]

\[
F_1(s) = \frac{1}{s^2+s+2} \\
= \frac{1}{\left( s + \frac{1}{2} \right)^2 + \frac{7}{4}} \\
= \frac{2}{\sqrt{7}} \frac{\sqrt{7}}{\left( s + \frac{1}{2} \right)^2 + \left( \frac{\sqrt{7}}{2} \right)^2} \\
= c_1 \frac{b}{(s+a)^2+b^2}
\]

\[
\Rightarrow f_1(t) = c_1 e^{-at} \sin bt 1(t) \ (Table \ A.2 \ entry \ 20)
\]

\[
F_2(s) = \frac{1}{s(s^2+s+2)} \\
= \frac{1}{2} \frac{2}{s \left( s + \frac{1}{2} \right)^2 + \left( \frac{\sqrt{7}}{2} \right)^2} \\
= \frac{a^2+b^2}{c_2 s ((s+a)^2+b^2)}
\]

\[
\Rightarrow f_2(t) = c_2 \left\{ 1 - e^{-at} \left( \cos bt + \frac{a}{b} \sin bt \right) \right\} 1(t) \ (Table \ A.2 \ entry \ 21)
\]
Therefore,

\[
y(t) = f_1(t) - f_2(t)
\]
\[
= c_1 e^{-at} \sin bt 1(t) - c_2 \left\{ 1 - e^{-at} \left( \cos bt + \frac{a}{b} \sin bt \right) \right\}
\]
\[
= - \left\{ c_2 + e^{-at} \left[ (c_1 + \frac{c_2 a}{b}) \sin bt + c_2 \cos bt \right] \right\} 1(t)
\]
\[
= -\frac{1}{2} \left\{ 1 - e^{-\frac{t}{2}} \left[ \frac{5}{\sqrt{7}} \sin \frac{\sqrt{7}}{2} t + \cos \frac{\sqrt{7}}{2} t \right] \right\} 1(t)
\]