MAE 143B

Linear Control — Prof. M. Krstic

MIDTERM

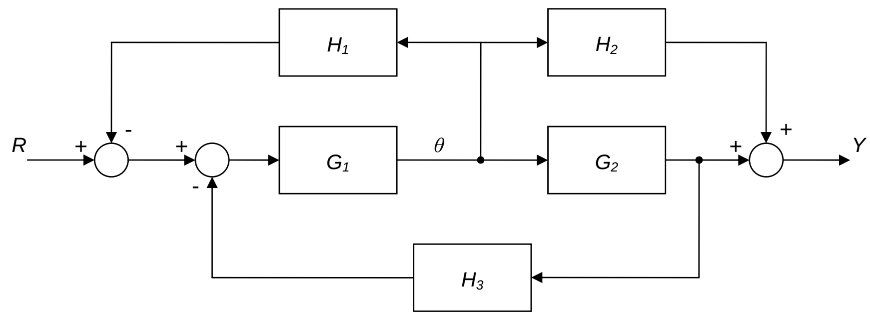
August 15, 2005

NAME: Solution

- Open books and notes.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 30.
- Time: 6:30–7:50

Problem 1. (7 points)

Consider the system

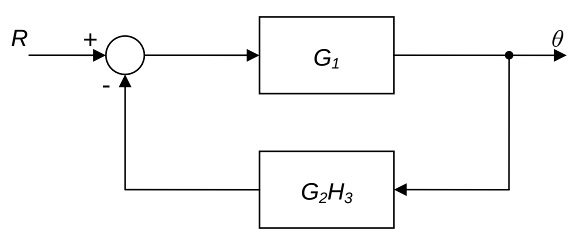


(a) For $H_1 = H_2 = 0$, what is the transfer function from R to θ?

(b) For $H_1 \neq 0, H_2 \neq 0$, what is the transfer function from R to Y?

Solution

(a) Redraw the block diagram with $H1 = H2 = 0$, R as the input and θ as the output. Therefore, the transfer function from R to θ is

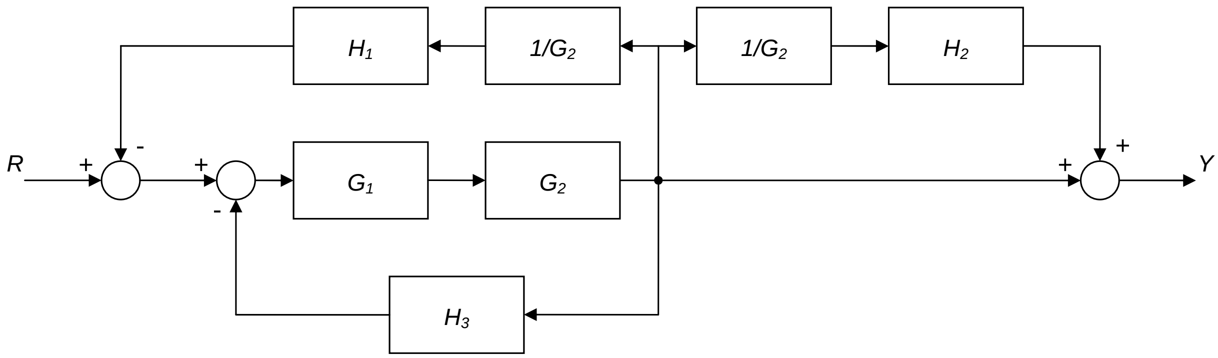


$$\frac{\theta}{R} = \frac{G_1}{1 + G_1 G_2 H_3}.$$

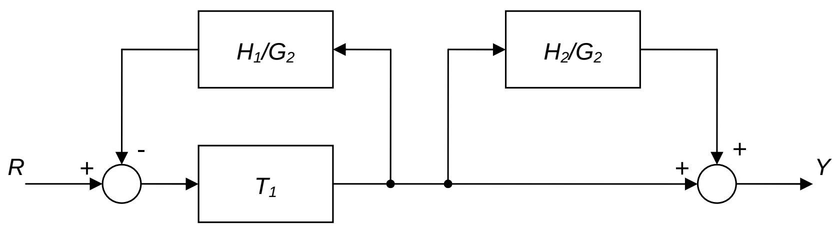
The same result can be found by noting that $\frac{Y}{R} = \frac{G_1 G_2}{1+G_1 G_2 H_3}$ where $Y = G_2 \theta$.

(b) The block diagram can be reduced in the following manner.

(i) Move the G_2 block ahead of the pick-off point.



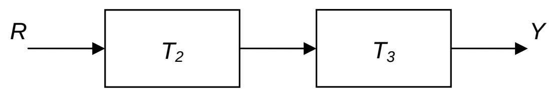
(ii) Combine the blocks in the lower left of the diagram in feedback



where

$$T_1 = \frac{G_1 G_2}{1 + G_1 G_2 H_3}.$$

(iii) Combine the blocks in feedback (left) and feedforward (right)



where

$$\begin{aligned} T_2 &= \frac{T_1}{1 + \frac{T_1 H_1}{G_2}} \\ T_3 &= 1 + \frac{H_2}{G_2} = \frac{G_2 + H_2}{G_2} \end{aligned}$$

Therefore

$$\frac{Y}{R} = T_2 T_3 = \frac{T_1(G_2 + H_2)}{G_2 + T_1 H_1} = \frac{G_1 G_2 + G_1 H_2}{1 + G_1 G_2 H_3 + G_1 H_1}.$$

(b) The transfer function from R to Y can also be found using Mason's rule.

Forward Paths

$$\begin{array}{lll} \text{Path 1} & P_1 = & G_1 G_2 \\ \text{Path 2} & P_2 = & G_1 H_2 \end{array}$$

Loops

$$\begin{array}{lll} \text{Loop 1} & L_1 = & -G_1 H_1 \\ \text{Loop 2} & L_2 = & -G_1 G_2 H_3 \end{array}$$

$$\begin{aligned} \triangle &= 1 - (L_1 + L_2) = 1 + G_1 H_1 + G_1 G_2 H_3 \\ \triangle_1 &= 1 \text{ (both loops touch Path 1)} \\ \triangle_2 &= 1 \text{ (both loops touch Path 2)} \end{aligned}$$

$$\frac{Y}{R} = \frac{1}{\triangle} \sum_{i=1}^{2} P_i \triangle_i = \frac{G_1 G_2 + G_1 H_2}{1 + G_1 G_2 H_3 + G_1 H_1}$$

Problem 2. (5 points)

Consider the following step response:

$$y(t) = 4\left(1 - e^{-2(t-3)}\right) 1(t-3)\,.$$

Find the PID controller parameters (K, T_I, T_D) using the Ziegler-Nichols *transient-response method* with a 0.25 decay ratio (Method 1).

Solution Begin by taking the Laplace transform of $y(t)$.

$$\begin{aligned} Y(s) &= 4\left(\frac{1}{s} - \frac{1}{s+2}\right)e^{-3s} \\ &= \frac{8e^{-3s}}{s(s+2)} \\ &= \frac{8e^{-3s}}{s+2}R(s) \\ &= \frac{4e^{-3s}}{\frac{1}{2}s+1}R(s) \end{aligned}$$

Now compare $\frac{Y(s)}{R(s)}$ to the standard form of a first order system with a time delay.

$$\frac{Y(s)}{R(s)} = \frac{4e^{-3s}}{\frac{1}{2}s+1} = \frac{Ae^{-t_d s}}{\tau s+1}$$

$$\Rightarrow t_d = 3, \tau = \frac{1}{2}, A = 4$$

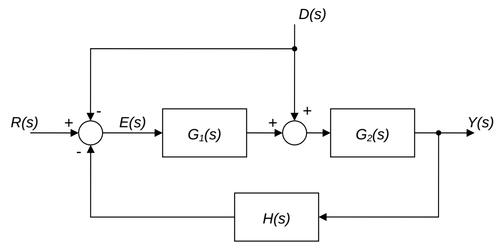
The problem is completed by computing K, T_I and T_D using the equations in *Table 4.1*, where $R = \frac{A}{\tau} = 8$ and $L = t_d = 3$.

$$\begin{aligned} K &= \frac{1.2}{RL} \\ T_I &= 2L \\ T_D &= \frac{L}{2} \end{aligned}$$

$$\Rightarrow K = 0.05, T_I = 6, T_D = 1.5$$

Problem 3. (7 points)

Consider the system



where

$$
\begin{aligned}
G_1(s) &= \frac{k}{2s} \\
G_2(s) &= \frac{1}{s+a} \\
H(s) &= k\,.
\end{aligned}
$$

Compute the transfer function

$$T(s) = \frac{Y(s)}{R(s)}$$

and find its sensitivity to the coefficient k,

$$S_k^T(s) = \frac{dT(s)}{dk}\frac{k}{T(s)}\,.$$

Solution Let $D(s) \equiv 0$, in order to compute $T(s)$.

$$
\begin{aligned}
\Rightarrow T(s) = \frac{Y(s)}{R(s)} &= \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} \\
&= \frac{\left(\frac{k}{2s}\right)\left(\frac{1}{s+a}\right)}{1+\left(\frac{k}{2s}\right)\left(\frac{1}{s+a}\right)k} \\
&= \frac{k}{2s^2+2as+k^2} = \frac{\frac{k}{2}}{s^2+as+\frac{k^2}{2}}
\end{aligned}
$$

Now compute the sensitivity of $T(s)$ to the coefficient k.

$$
\begin{aligned}
S_k^T(s) &= \frac{dT(s)}{dk}\frac{k}{T(s)} \\
&= \left(\frac{(1)\left(2s^2+2as+k^2\right)-(k)(2k)}{\left(2s^2+2as+k^2\right)^2}\right)\frac{k\left(2s^2+2as+k^2\right)}{k} \\
&= \frac{2s^2+2as-k^2}{2s^2+2as+k^2} = \frac{s^2+as-\frac{k^2}{2}}{s^2+as+\frac{k^2}{2}}
\end{aligned}
$$

Problem 4. (4 points)

For the system form Problem 3 find k and a such that the closed-loop step response with respect to the input R exhibits an overshoot of 10% and a settling time of $0.75s$.

Solution The system in problem 3, in standard second-order form, was found to be

$$T(s) = \frac{\frac{k}{2}}{s^2 + as + \frac{k^2}{2}} = C\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\,.$$

The problem is completed by relating the system parameters (k,a) with those of the standard second-order system (ζ, ω_n)and the given performance specifications (M_p, t_s).

From *Figure 3.28*

$$M_p = 10\% \Rightarrow \zeta = 0.61$$

Parameter ζ can also be computed using *Equation 3.50* ($\zeta = 0.59$).

Using *Equation 3.51*

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.61(0.75)} = 10.05\ rad/s$$

The plant parameters k and a are found to be

$$\begin{aligned} k &= \sqrt{2}\omega_n \\ a &= 2\zeta\omega_n \end{aligned}$$

$$\Rightarrow k = 14.2, a = 12.3\,.$$

Problem 5. (7 points)

Consider the system from Problem 3 with $k = 2$ and $a = 1$.

(a) Find the transfer function from $D(s)$ to $Y(s)$.

(b) Find the time-domain output response to a step in D.

Solution

(a) Let $R(s) \equiv 0$ to find $\frac{Y(s)}{R(s)}$, then write the equations corresponding to the block diagram.

$$\begin{aligned} Y(s) &= G_2(s)\left(D(s) + G_1(s)E(s)\right) \quad (1) \\ E(s) &= -H(s)Y(s) - D(s) \quad (2) \end{aligned}$$

Substitute (1) into (2) and compute $\frac{Y(s)}{D(s)}$.

$$Y(s) = G_2(s)D(s) - G_1(s)G_2(s)\left(H(s)Y(s) - D(s)\right)$$
$$\left(1 + G_1(s)G_2(s)H(s)\right)Y(s) = \left(G_2(s) - G_1(s)G_2(s)\right)D(s)$$

$$\begin{aligned} \Rightarrow \frac{Y(s)}{D(s)} &= \frac{G_2(s) - G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \\ &= \frac{\frac{1}{s+1} - \frac{1}{s}\left(\frac{1}{s+1}\right)}{1 + \frac{1}{s}\left(\frac{1}{s+1}\right)2} \\ &= \frac{\frac{s-1}{s(s+1)}}{1 + \frac{2}{s(s+1)}} \\ &= \frac{s-1}{s^2 + s + 2} \end{aligned}$$

(b) For a unit step input $D(s) = \frac{1}{s}$,

$$\begin{aligned}
Y(s) &= \frac{s-1}{s^2+s+2} D(s) \\
&= \frac{1}{s}\left(\frac{s-1}{s^2+s+2}\right) \\
&= \frac{1}{s^2+s+2} - \left(\frac{1}{s}\right)\frac{1}{s^2+s+2} \\
&= F_1(s) - F_2(s)\,.
\end{aligned}$$

$$\Rightarrow y(t) = (f_1(t) - f_2(t))\,1(t)$$

$$\begin{aligned}
F_1(s) &= \frac{1}{s^2+s+2} \\
&= \frac{1}{\left(s+\frac{1}{2}\right)^2 + \frac{7}{4}} \\
&= \frac{2}{\sqrt{7}} \frac{\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \\
&= c_1 \frac{b}{(s+a)^2 + b^2}
\end{aligned}$$

$$\Rightarrow f_1(t) = c_1 e^{-at} \sin bt 1(t) \ \text{(*Table A.2* entry 20)}$$

$$\begin{aligned}
F_2(s) &= \frac{1}{s(s^2+s+2)} \\
&= \frac{1}{2} \frac{2}{s\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \\
&= c_2 \frac{a^2+b^2}{s\left((s+a)^2+b^2\right)}
\end{aligned}$$

$$\Rightarrow f_2(t) = c_2\left\{1 - e^{-at}\left(\cos bt + \frac{a}{b}\sin bt\right)\right\} 1(t) \ \text{(*Table A.2* entry 21)}$$

Therefore,

$$
\begin{aligned}
y(t) &= f_1(t) - f_2(t) \\
&= c_1 e^{-at} \sin bt 1(t) - c_2 \left\{ 1 - e^{-at} \left(\cos bt + \frac{a}{b} \sin bt \right) \right\} \\
&= -\left\{ c_2 + e^{-at} \left[\left(c_1 + \frac{c_2 a}{b} \right) \sin bt + c_2 \cos bt \right] \right\} 1(t) \\
&= -\frac{1}{2} \left\{ 1 - e^{-\frac{t}{2}} \left[\frac{5}{\sqrt{7}} \sin \frac{\sqrt{7}}{2} t + \cos \frac{\sqrt{7}}{2} t \right] \right\} 1(t)
\end{aligned}
$$