Problem 1. (5 points)

Find the *step* response of the system

$$H(s) = 2\frac{s^2+s+1}{s^2+3s+2}$$

$$s^2+3s+2=0 \longrightarrow s_1 = -1\ ,\ s_2=-2$$

$$H(s) = 2\,\frac{s^2+s+1}{(s+1)(s+2)}$$

Step response

$$Y(s) = H(s)\,\frac{1}{s}$$

$$= 2\,\frac{s^2+s+1}{s(s+1)(s+2)}$$

$$= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+2}$$

$$C_1 = sY(s)\Big|_{s=0} = 1$$

$$C_2 = (s+1)Y(s)\Big|_{s=-1} = -2$$

$$C_3 = (s+2)Y(s)\Big|_{s=-2} = \cancel{2}\,\frac{4-2+1}{-\cancel{2}(-2+1)} = 3$$

So

$$Y(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{3}{s+2}$$

$$\longrightarrow y(t) = \left[1-2e^{-t}+3e^{-2t}\right]\mathbb{1}(t)$$

Problem 2. (4 points)

Find the *impulse* response of the system

$$H(s) = e^{-2s}\frac{s+1}{(s+1)^2+9}$$

Denote $G(s) = \frac{s}{s^2+9}$. Recall $g(t) = \mathcal{L}^{-1}\{G(s)\} = \cos 3t$.

We have

$$H(s) = e^{-2s}\, G(s+1)$$

$$\mathcal{L}^{-1}\{G(s+1)\} = e^{-t}\cos 3t$$

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{e^{-2s} G(s+1)\}$$
$$= e^{-(t-2)}\cos 3(t-2)\, 1(t-2)$$

Since the impulse response of $H(s)$ is $h(t) = \mathcal{L}^{-1}\{H(s)\}$, the impulse response is

$$h(t) = e^{-(t-2)}\cos 3(t-2)\, 1(t-2)$$

Problem 3. (5 points)

Let the persistent forcing signal $u(t) = (1 - \cos t)1(t)$ drive the system

$$Y(s) = \frac{s^2+1}{s^2+2s+3}U(s).$$

Does this system, despite persistent forcing, reach a steady state? If so, what is $\lim_{t\to\infty} y(t)$?

$$\lim_{t\to\infty} y(t) = y(\infty)$$
$$= \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{s^2+1}{s^2+2s+3}\underbrace{\mathcal{L}\{(1-\cos t)1(t)\}}_{U(s)}$$

$$U(s) = \mathcal{L}\{1(t)\} - \mathcal{L}\{\cos t\ 1(t)\}$$
$$= \frac{1}{s} - \frac{s}{s^2+1} = \frac{1}{(s^2+1)s}$$

So

$$y(\infty) = \lim_{s\to 0} \cancel{s}\frac{\cancel{s^2+1}}{s^2+2s+3}\,\frac{1}{\cancel{(s^2+1)}\,\cancel{s}}$$
$$= \lim_{s\to 0} \frac{1}{s^2+2s+3} = \frac{1}{3}$$

In, summary, despite persistent forcing, the system does reach a steady state

$y(\infty) = 1/3$.

Problem 4. (7 points)

Consider the system
$$H(s) = \frac{16000}{21}\frac{ks + 21}{(s+20)(s^2+40s+800)}.$$

(a) For $k = 1$ estimate its overshoot and peak time.
(b) Is there anything you can say about the case $k = 0$?

a) For $k=1$,

$$H(s) = \frac{16000}{21}\frac{s+21}{(s+20)(s^2+40s+800)}$$

Since $\frac{s+21}{s+20} \approx 1$,

$$H(s) \approx \frac{16000}{21}\frac{1}{s^2+40s+800}$$

$$\approx \frac{800}{s^2+40s+800}$$

$$\omega_n^2 = 800 \longrightarrow \omega_n = 20\sqrt{2}$$

$$2\zeta\omega_n = 40 \longrightarrow \zeta = \frac{1}{\sqrt{2}}$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.157$$

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.043$$

(6) For $k=0$

$$H(s) = \frac{16000}{\cancel{21}} \frac{\cancel{21}}{(s+20)(s^2+40s+800)}$$

$$= \frac{16\,000}{(s+20)(\underbrace{s^2+40s+400}_{=(s+20)^2}+400)}$$

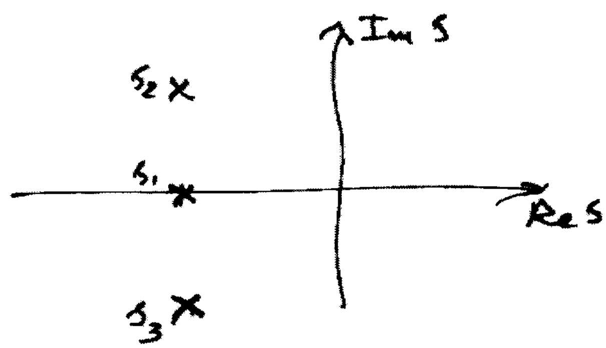
$$= \frac{16000}{(s+20)\left[(s+20)^2+20^2\right]}$$

The poles are

$s_1 = -20$

$s_2 = -20 + j20$

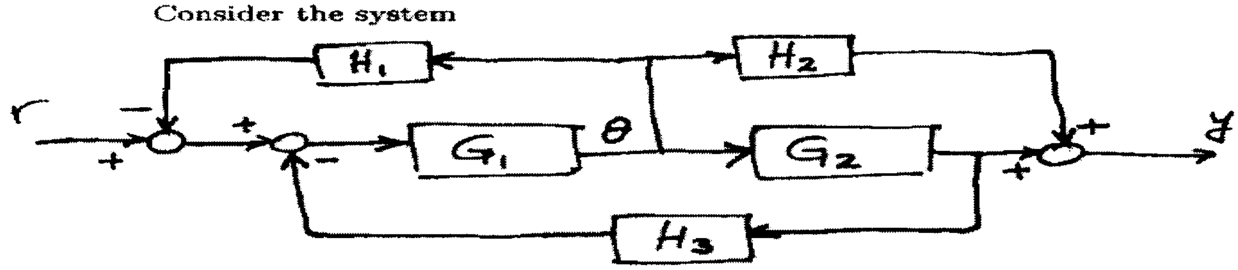
$s_3 = -20 - j20$



No dominant poles $\longrightarrow$ we canot say anything about t_p and M_p.

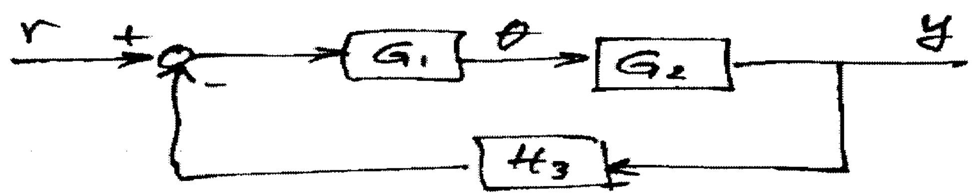
Problem 5. (9 points)

Consider the system



(a) For $H_1 = H_2 = 0$, what is the transfer function from r to θ?
(b) For general $H_1 \neq 0, H_2 \neq 0$, what is the transfer function from r to y?

(a)

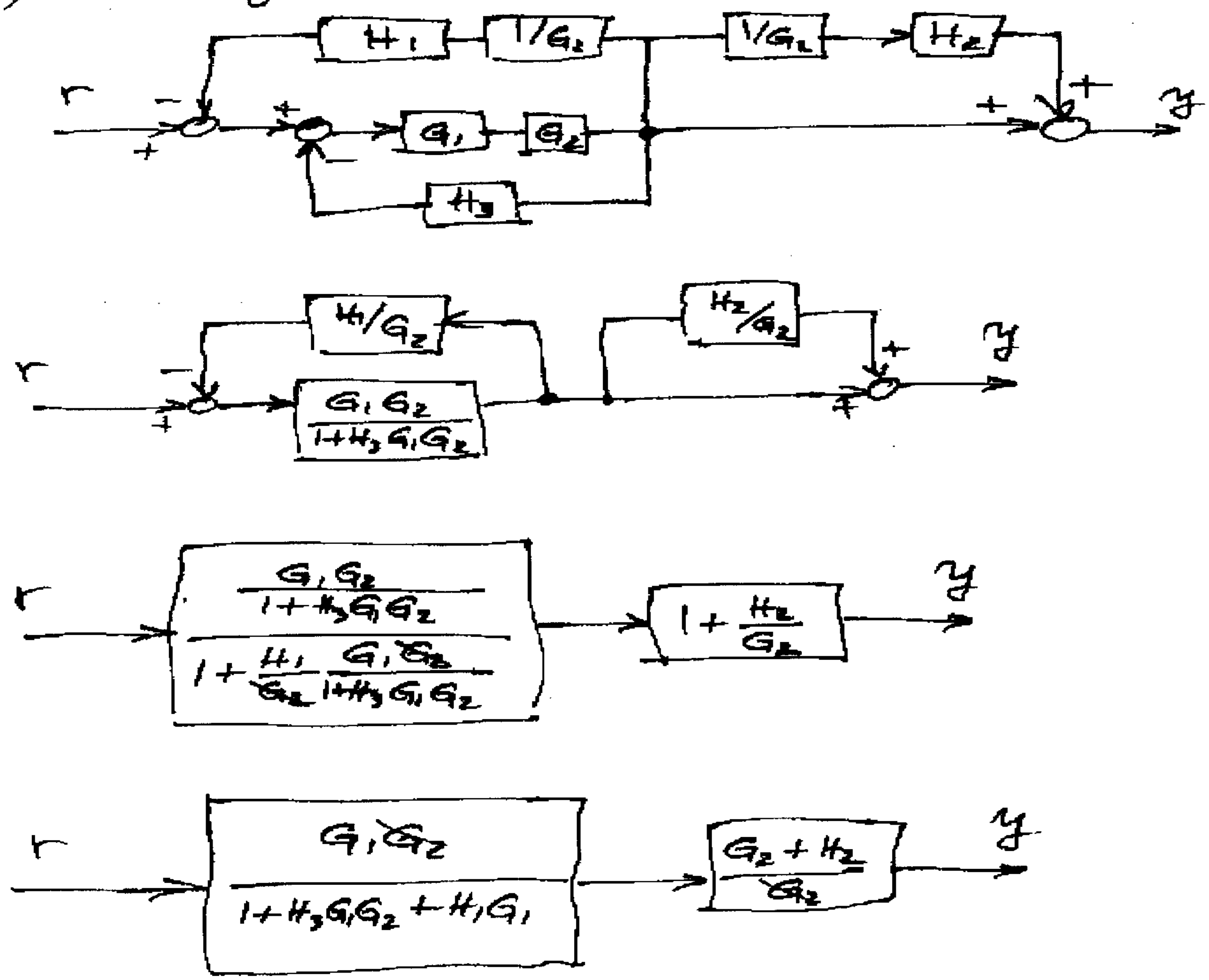


$$\frac{Y}{R} = \frac{G_1 G_2}{1 + H_3 G_1 G_2}$$

Since $Y = G_2 \theta$,

$$\frac{\theta}{R} = \frac{G_1}{1 + H_3 G_1 G_2}$$

(b) The system can be redrawn as



$$\frac{Y}{R} = \frac{G_1(G_2+H_2)}{1+H_3G_1G_2+H_1G_1}$$