Problem 1: Root Locus (9 points) For the following, sketch the root locus with respect to $K \geq 0$ for the equation $1 + KG(s) = 0$. Is the sum of the closed-loop poles constant regardless of $K \geq 0$? If yes, what is the value?

(a) (3 points) $G(s) = \frac{(s^2 - 2s + 2)(s + 10)}{(s + 1)^2(s + 2)^3}$

Solution. (1) The open-loop zeros are at $s = 1 + j, 1 - j, -10$
(2) The open-loop poles are at $s = -1, -1, -2, -2, -2$
(3) The relative degree is $(2 + 3) - (2 + 1) = 2$. Thus, the sum of the closed-loop poles is constant and the value is $-1 - 1 - 2 - 2 - 2 = -8$. And, there are two asymptotes.
(4) The center of two asymptotes is $\frac{(-2 - 10) - 2}{2} = 0$.

Note that the root locus crosses the asymptotes.
(b) (3 points) \( G(s) = \frac{2}{(s + 1)(s + 7)(s^2 + 2s + 2)} \)

**Solution.**

(1) There is no open-loop zero

(2) The open-loop poles are at \( s = -1, -7, -1 + j, -1 - j \)

(3) The relative degree is \((1 + 1 + 2) - 0 = 4\). Thus, the sum of the closed-loop poles is constant and the value is \(-1 - 7 - 2 = -10\). And, there are four asymptotes.

(4) The center of four asymptotes is \( \frac{0 - (1 + 7 + 2)}{4} = -\frac{5}{2} \).

\[ \text{Root Locus} \]

(c) (3 points) \( G(s) = \frac{(s + 2)^2(s + 3)(s + 7)^2(s + 8)}{s(s + 1)^2(s + 4)(s + 5)^2(s + 6)} \)

**Solution.**

(1) The open-loop zeros are at \( s = -2, -2, -3, -7, -7, -8 \)

(2) The open-loop poles are at \( s = 0, -1, -1, -4, -5, -5, -6 \)

(3) The relative degree is \((1 + 2 + 1 + 2 + 1) - (2 + 1 + 2 + 1) = 1\). Thus, the sum of the closed-loop poles is not constant. And, there is one asymptote.
**Problem 2:** Root Locus (7 points) Sketch the trace of the closed-loop poles of the following system with respect to $K > 0$. Moreover, find the range of $K > 0$ that makes the closed-loop system unstable.

![Root Locus Diagram]

**Solution.** The characteristic equation for the transfer function $\frac{y}{r}$ is

\[
1 + \frac{s(s - 3)}{(s + K)(3s + 1)(s + 1)} = 0 \\
\iff \frac{s(3s^2 + 4s + 1) + s(s - 3) + K(3s + 1)(s + 1)}{(s + K)(3s + 1)(s + 1)} = 0 \\
\iff \frac{s(3s - 1)(s + 2) + K(3s + 1)(s + 1)}{(s + K)(3s + 1)(s + 1)} = 0
\]

Then, the root locus for the following equation shows the trace of the closed-loop poles:

\[
1 + K \frac{(3s + 1)(s + 1)}{s(3s - 1)(s + 2)} = 0
\]
(1) The open-loop zeros are at \( s = -\frac{1}{3}, -1 \)
(2) The open-loop poles are at \( s = -2, 0, \frac{1}{3} \).
(3) The relative degree is \( 3 - 2 = 1 \). Thus, there is one asymptote.

From (1), the characteristic polynomial is
\[
s(3s - 1)(s + 2) + K(3s + 1)(s + 1) = 3s^3 + (3K + 5)s^2 + (4K - 2)s + K
\]

From the Routh’s criterion, it follows that
\[
(3K + 5)(4K - 2) < 3K \quad (\Leftrightarrow 12K^2 + 11K - 10 < 0)
\]

if, and only if, the closed-loop system is unstable. This condition and \( K > 0 \) imply that the closed-loop system is unstable if, and only if,
\[
0 < K < \frac{-11 + \sqrt{601}}{24}
\]
Problem 3: Bode Plots (9 points) Sketch the Bode plots for the following open-loop transfer functions:

(a) (3 points) \( G(s) = \frac{0.1(s - 0.1)(s + 10)(s + 100)}{(s + 1)^3} \)

(b) (3 points) \( G(s) = \frac{10(s^2 + 0.2s + 1)}{s(s + 10)^2} \)
(c) (3 points) \[ G(s) = \frac{10^3(s - 0.1)^2}{(s + 1)(s^2 + 16s + 100)} \]