MAE 143B
LINEAR CONTROL — Prof. M. Krstic

MIDTERM EXAM

May 3, 2005

NAME(Last,First): Solutions

Problem 1. (5 points)

Find the impulse response $y(t)$ of the system

$$G(s) = \frac{Y(s)}{R(s)} = \frac{2s+12}{s^2+2s+5}$$

Answer
The Laplace transform of impulse response is the system transfer function.

$$G(s) = \frac{2s+12}{(s+1-2j)(s+1+2j)} = \frac{a}{s+1-2j} + \frac{\bar{a}}{s+1+2j}, \quad (1)$$

where $\bar{a}$ is the conjugate of a.
By the method of residues

$$a = (s+1-2j)G(s)|_{s=-1+2j} = \frac{2-5j}{2} = \frac{1}{2}\sqrt{4+25}e^{-j\tan^{-1}\frac{5}{2}}. \quad (2)$$

From (1) and (2)

$$G(s) = \frac{1}{2}\sqrt{4+25}\left(\frac{e^{-j\tan^{-1}\frac{5}{2}}}{s+1-2j} + \frac{e^{j\tan^{-1}\frac{5}{2}}}{s+1+2j}\right) \quad (3)$$

The inverse Laplace transform of (3) gives

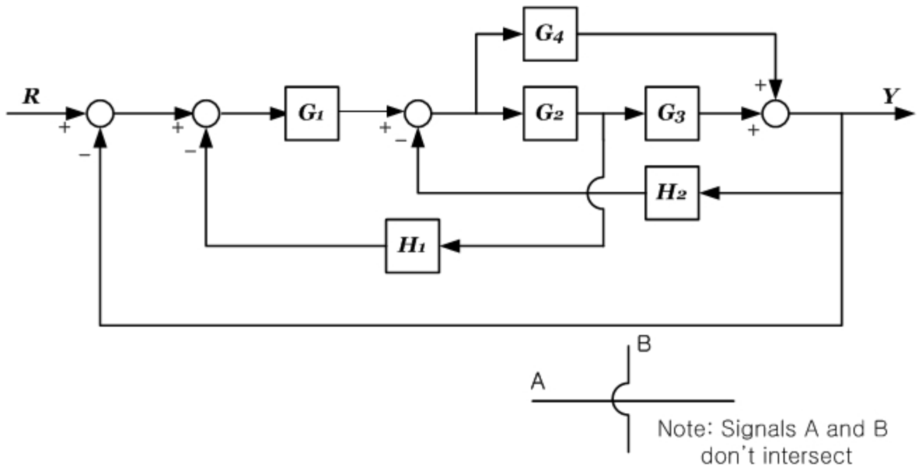
$$g(t) = \frac{1}{2}\sqrt{4+25}e^{-t}\left(e^{-j\tan^{-1}\frac{5}{2}}e^{j2t} + e^{j\tan^{-1}\frac{5}{2}}e^{-j2t}\right).$$

Using the fact $\frac{1}{2}\left(e^{-j\tan^{-1}\frac{5}{2}}e^{j2t} + e^{j\tan^{-1}\frac{5}{2}}e^{-j2t}\right) = \cos(2t - \tan^{-1}\frac{5}{2})$, we have

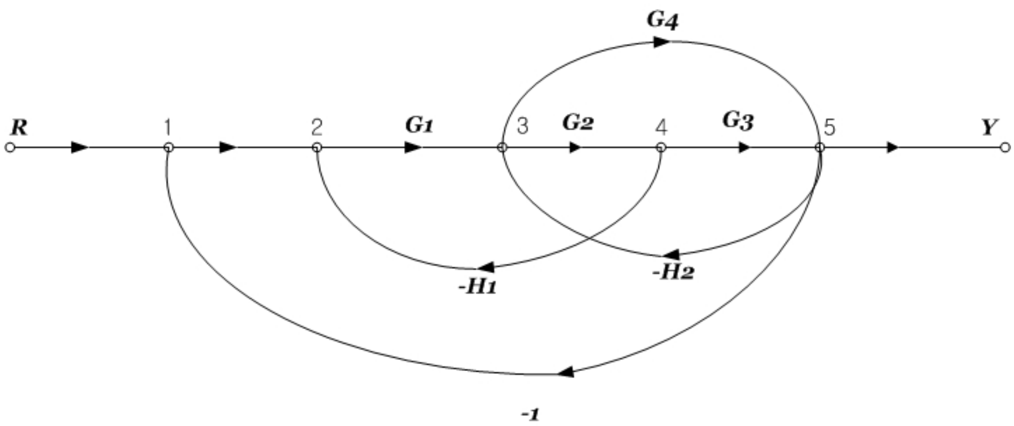
$$g(t) = \sqrt{29}e^{-t}\cos\left(2t - \tan^{-1}\frac{5}{2}\right).$$

Problem 2. (7 points)

Find the closed-loop transfer function $G(s) = \frac{Y(s)}{R(s)}$ for the system



Answer



Forward Path

R1235Y	$T_1 = G_1G_4$
R12345Y	$T_2 = G_1G_2G_3$

Loop Gain Gain

12351	$-G_1G_4$
123451	$-G_1G_2G_3$
2342	$-G_1G_2H_1$
353	$-G_4H_2$
3453	$-G_2G_3H_2$

The system determinant is

$$\Delta = 1 - (-G_1G_4 - G_1G_2G_3 - G_1G_2H_1 - G_4H_2 - G_2G_3H_2).$$

Individual forward paths touch every loop. Hence

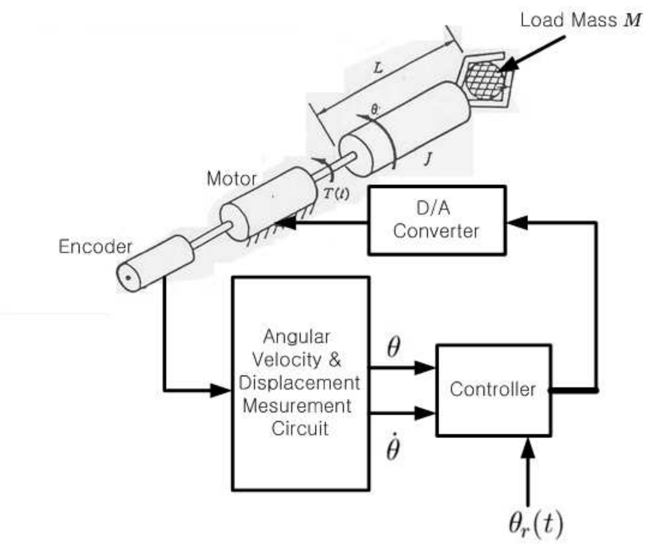
$$\Delta_{T_1} = 1 - 0,$$
$$\Delta_{T_2} = 1 - 0.$$

Therefore the transfer function $G(S)$ is

$$G(s) = \frac{1}{\Delta}(T_1\Delta_{T_1} + T_2\Delta_{T_2})$$
$$= \frac{G_1G_4 + G_1G_2G_3}{1 + G_1G_4 + G_1G_2G_3 + G_1G_2H_1 + G_4H_2 + G_2G_3H_2}.$$

Problem 3. (6 points each)

Consider a robot manipulator described by the following diagram:



If gravity and friction are ignored the system has the dynamics

$$(J + ML^2)\ddot{\theta} = T(t), \tag{4}$$

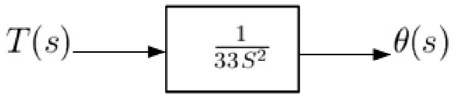
where θ is angular displacement, T is motor torque, $J = 1$, $M = 8$, and $L = 2$.
To improve the performance we consider using the proportional-derivative controller

$$T(s) = K(\theta_r - \theta) - K_v\dot{\theta}(s), \tag{5}$$

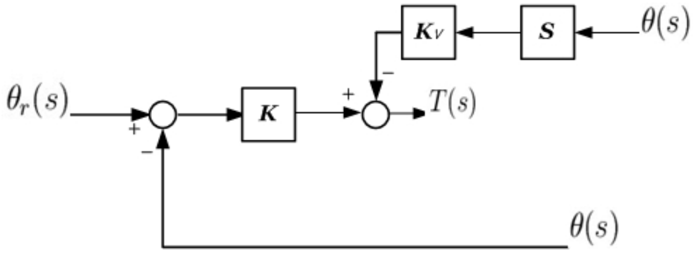
where K and K_v are proportional and derivative controller gains respectively.

(a)(2 points) **Draw the unity feedback** block diagram(Not simplified) described by (4) and (5).

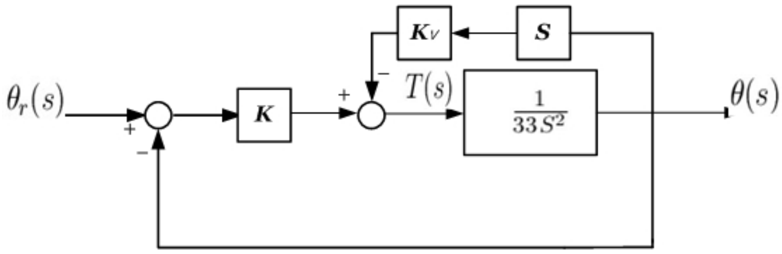
Answer
First, we consider the block diagram of (4):



(5) is described as following:

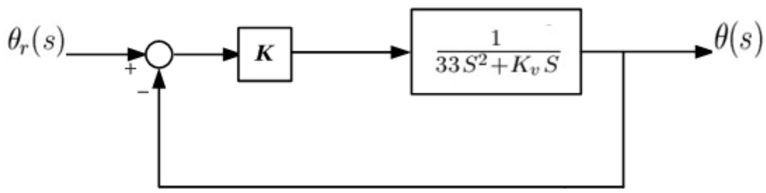


Combining the block diagram of (4) and the description above, we have



(b)(2 points)Use the block diagram to **find** the transfer function $G(s) = \frac{\theta}{\theta_r}$ (You must do **(a)** first).

Answer
The block diagram is reduced to



Hence

$$G(s) = \frac{\theta(s)}{\theta_r(s)} = \frac{\frac{K}{33s^2+K_v s}}{1 + \frac{K}{33s^2+K_v s}} = \frac{K}{33s^2 + K_v s + K}. \qquad (6)$$

(c)(4 points) The system in **(b)** should have Settling time $t_s = 0.49sec$ and Rise time $t_r = 0.12sec$. **Find** the gains K, K_v.

Answer

$$t_s = \frac{4.6}{\zeta\omega_n} = 0.49,$$
$$t_r = \frac{1.8}{\omega_n} = 0.12.$$

Therefore,

$$\omega_n = \frac{1.8}{0.12} = 15rad/sec,$$
$$\zeta = \frac{4.6}{0.49 \times \omega_n} = \frac{4.6}{0.49 \times 15} = 0.63.$$

From (6)

$$G(s) = \frac{\frac{K}{33}}{s^2 + \frac{K_v}{33}s + \frac{K}{33}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{8}$$

(8) implies

$$\begin{aligned} \frac{K}{33} &= \omega_n^2 = 15^2, \\ \frac{K_v}{33} &= 2\zeta\omega_n = 2 \times 0.63 \times 15. \end{aligned} \tag{9}$$

Hence $K = 7425$ and $K_v = 623.7$.

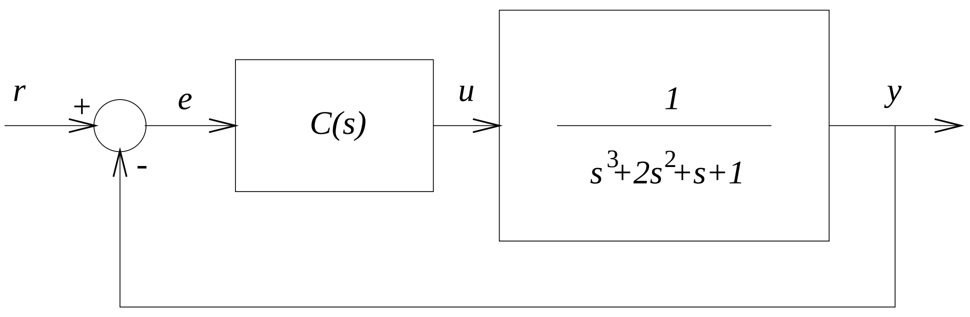
(d)(2 points) Find the Overshoot(M_p) in **(c)**.

Answer

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$
$$= e^{-0.63\pi/\sqrt{1-0.63^2}} = 0.0782 = 7.82\%.$$

Problem 4. (8 points)

Consider the following feedback loop



Find the parameters of a PID controller

$$C(s) = K\left(1 + \frac{1}{T_I s} + T_D s\right)$$

using the Ziegler-Nichols *ultimate sensitivity method* (called Method 2 in the class notes). Instead of performing an impulse response experiment to find K_u and P_u (for which you need Matlab), use the identity

$$\frac{\frac{1}{s^3+2s^2+s+1}}{1+\frac{1}{s^3+2s^2+s+1}} = \frac{1}{(s+2)(s^2+1)}.$$

Answer

Let $C(s) = K_u$ and $r(t) = \delta(t)$. Then

$$Y(s) = \frac{\frac{K_u}{s^3+2s^2+s+1}}{1+\frac{K_u}{s^3+2s^2+s+1}}.$$

For $K_u = 1$, the above identity gives

$$\begin{aligned} Y(s) &= \frac{1}{(s+2)(s^2+1)} = \frac{1}{5(s+2)} + \frac{s-2}{s^2+1} \\ \Longrightarrow y(t) &= \left[\tfrac{1}{5}e^{-2t} + \cos(t) - 2\sin(t)\right] 1(t) \\ &= \left[\tfrac{1}{5}e^{-2t} + \sqrt{5}\sin(t+\theta)\right] 1(t). \end{aligned}$$

Since the first term decays to zero, the long time response is

$$y(t) = \sqrt{5}\sin(t+\theta),$$

with a period of oscillations $P_u = 2\pi$. Hence we have

$$K_u = 1, \qquad P_u = 2\pi.$$

From the Ziegler-Nichols table we get

$$\begin{aligned} K &= 0.6K_u = 0.6, \\ T_I &= \frac{1}{2}P_u = \pi, \\ T_D &= \frac{1}{8}P_u = \frac{\pi}{4}. \end{aligned}$$

Thus

$$C(s) = 0.6\left(1 + \tfrac{1}{\pi s} + \tfrac{\pi s}{4}\right).$$