○ *Final Fall 2000 - Solutions*

Problem 1. (12 points)

Consider the plant

$$P(s) = \frac{s^2 - 0.2s + 1.01}{s(s+0.5)(s+5)(s^2+0.4s+4.04)} = \frac{s^2 - 0.2s + 1.01}{s^5 + 5.9s^4 + 8.74s^3 + 23.22s^2 + 10.1s}$$

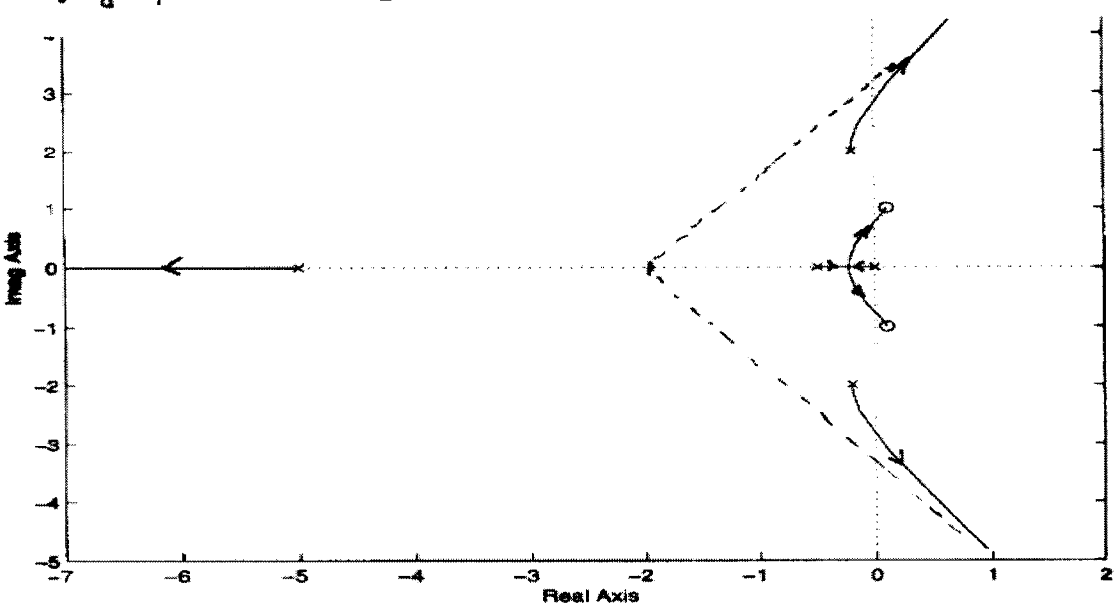
in a feedback loop with a gain $K > 0$. Sketch the root locus.

$$P(s) = \frac{(s-0.1)^2 + 1^2}{s(s+0.5)(s+5)\left((s+0.2)^2+2^2\right)}$$

Relative degree: $r = 3$

Center of asymptotes: $\frac{-0.2 - 5.9}{3} = \frac{-6.1}{3} = -2.033$
≈ -2

Asymptote angles: $\pm 60°$, $180°$



Problem 2. (13 points)

By applying Routh's criterion to the system in Problem 1, find the range of $K > 0$ such that the system in asymptotically stable.

Closed-loop system:

$$\frac{KP(s)}{1+KP(s)} = \frac{s^2 - 0.2s + 1.01}{s^5 + 5.9s^4 + 8.74s^3 + 23.22s^2 + 10.1s \text{ [redacted] } + K(s^2 - 0.2s + 1.01)}$$

$$= \frac{s^2 - 0.2s + 101}{s^5 + 5.9s^4 + 8.74s^3 + (23.22+K)s^2 + (10.1-0.2K)s + 1.01K}$$

Routh table

s^5	1	8.74	$10.1 - 0.2K$
s^4	5.9	$23.22+K$	$1.01K$
s^3	$\frac{5.9 \cdot 8.74 - 23.22 - K}{5.9}$	$\frac{5.9 \cdot 10.1 - 5.9 \cdot 0.2K - 1.01K}{5.9}$	
⋮			

The first column of the Routh's Table is

s^5 1

s^4 5.9

s^3 $4.8044 - 0.1695\,k = \dfrac{k - 28.3460}{-5.9}$ (1)

s^2 $\dfrac{k^2 - 18.047\,k - 306.6131}{k - 28.3460}$ (2)

s^1 $\dfrac{k^3 - 35.4615\,k^2 - 345.4175\,k + 15484}{-5\left(k^2 - 18.0470\,k - 306.6131\right)}$ (3)

s^0 $1.01\,k$ (4)

SOLUTION. $0 < k < 28.3460$

FULL CREDIT

PART TO BE GRADED !!

From (4) $\boxed{k > 0}$

From (1) $\boxed{k < 28.3460}$

From (2) $\boxed{-10.6751 < k < 28.3460} \vee \boxed{k > 28.7221}$

From (3) $\boxed{k < -20.0896} \vee \boxed{-10.6751 < k < 26.7998}$

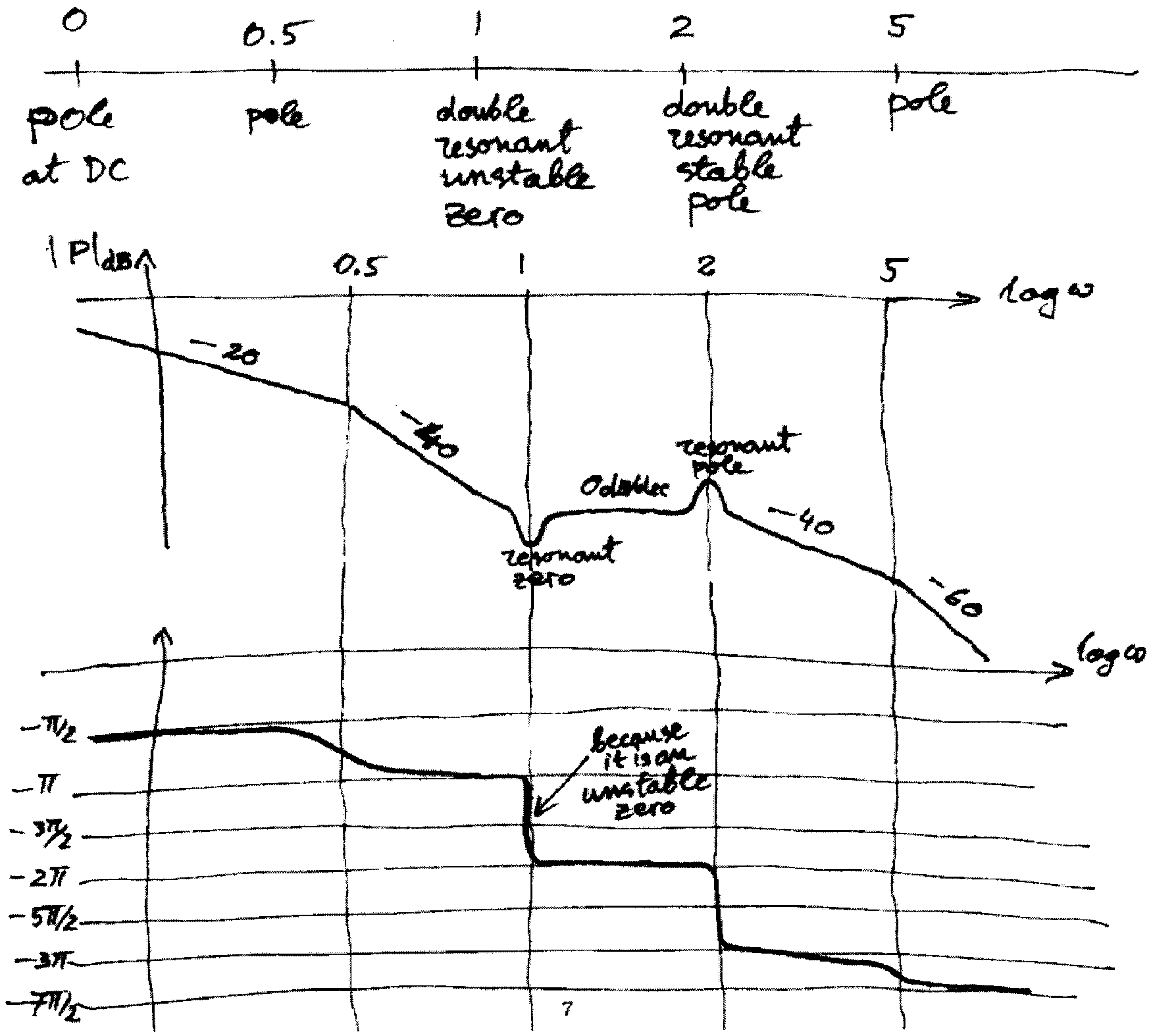
$\vee \boxed{28.7221 < k < 28.7592}$

Finally: $\boxed{0 < k < 26.7999}$ CORRECT FINAL SOLUTION

NOT EXPECTED

Problem 3. (12 points)

Sketch the approximate Bode plot for the plant in Problem 1. (Please exagerate the features so that it is clear if you have understood the procedure.)



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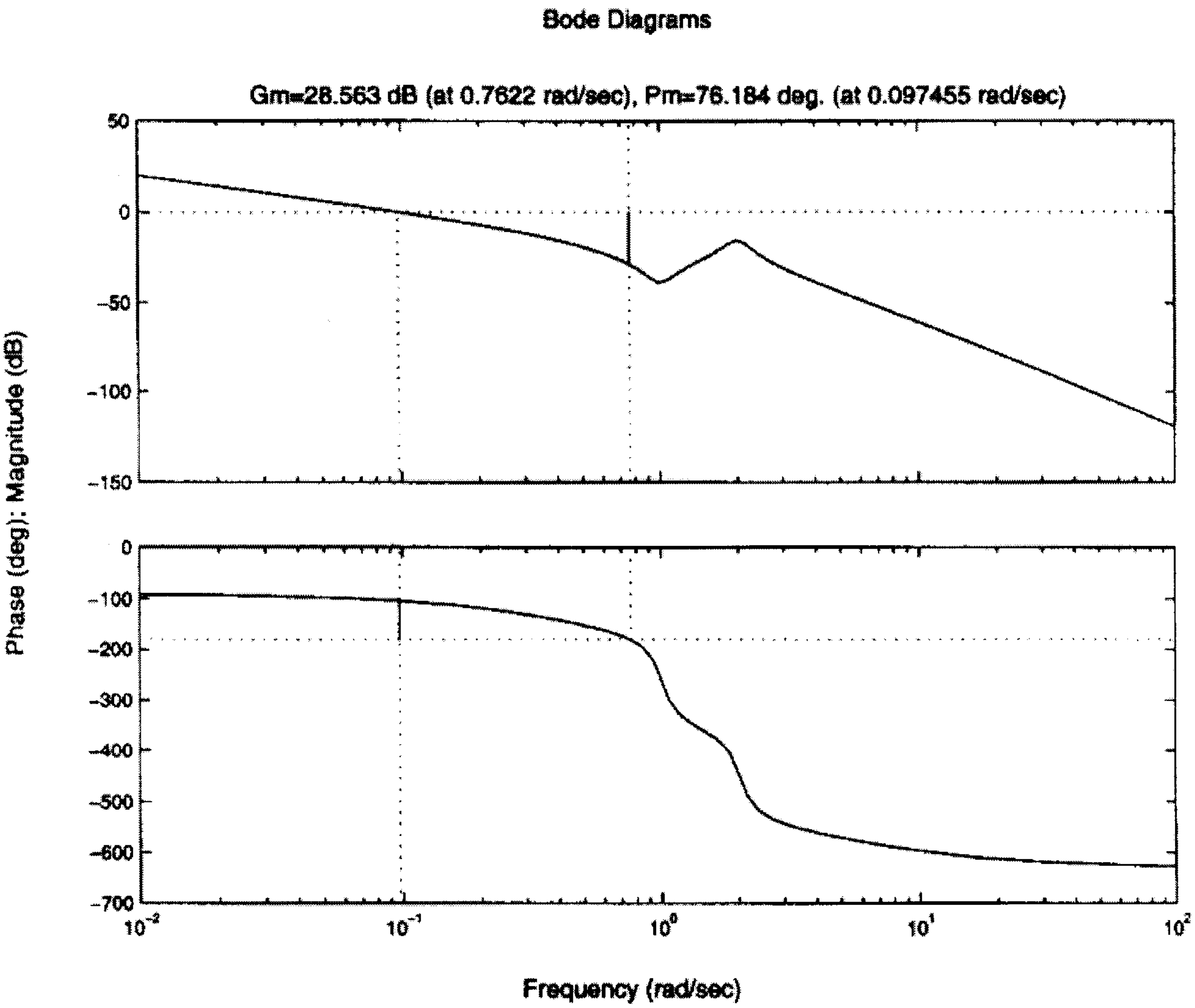

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$$P(s) = \frac{s^2 - 0.2s + 1.01}{s(s+0.5)(s+5)(s^2+0.4s+4.04)}$$

$$= \frac{1.01\left[\left(\frac{s}{\sqrt{1.01}}\right)^2 - \frac{0.2}{1.01}s + 1\right]}{s\;0.5\left(\frac{s}{0.5}+1\right)\;5\left(\frac{s}{5}+1\right)\;4.04\left[\left(\frac{s}{\sqrt{4.04}}\right)^2 + \frac{0.4}{4.04}s + 1\right]}$$

$$= \frac{0.1\left[\left(\frac{s}{\sqrt{1.01}}\right)^2 + \frac{0.2}{1.01}s + 1\right]}{s\left(\frac{s}{0.5}+1\right)\left(\frac{s}{5}+1\right)\left[\left(\frac{s}{\sqrt{4.04}}\right)^2 + \frac{0.4}{4.04} + 1\right]}$$

$s^2 - 0.2s + 1.01 = 0 \iff s = 0.1 \pm i$

$s^2 - 0.2s + 1.01 = s^2 - 2\zeta\omega_n s + \omega_n^2$ with $\omega_n = \sqrt{1.01} \cong 1.005$

$\zeta \cong 0.0995$

$s^2 + 0.4s + 4.04 = 0 \iff s = -0.2 \pm 2i$

$s^2 + 0.4s + 4.04 = s^2 + 2\zeta\omega_n s + \omega_n^2$ with $\omega_n = \sqrt{4.04} \cong 2.01$

$\zeta = 0.0995$

$$\left|P(j\omega)\right|_{dB} = \underset{(1)}{20\log 0.1} + \underset{(2)}{20\log\sqrt{\left[1-\left(\frac{\omega}{\sqrt{1.01}}\right)^2\right]^2 + \left[\frac{0.2}{1.01}\omega\right]^2}} -$$

$$- \underset{(3)}{20\log\omega} - \underset{(4)}{20\log\sqrt{1+\left(\frac{\omega}{0.5}\right)^2}} - \underset{(5)}{20\log\sqrt{1+\left(\frac{\omega}{5}\right)^2}} - \underset{(6)}{20\log\sqrt{\left[1-\left(\frac{\omega}{\sqrt{4.04}}\right)^2\right]^2 + \left[\frac{0.4\omega}{4.04}\right]^2}}$$

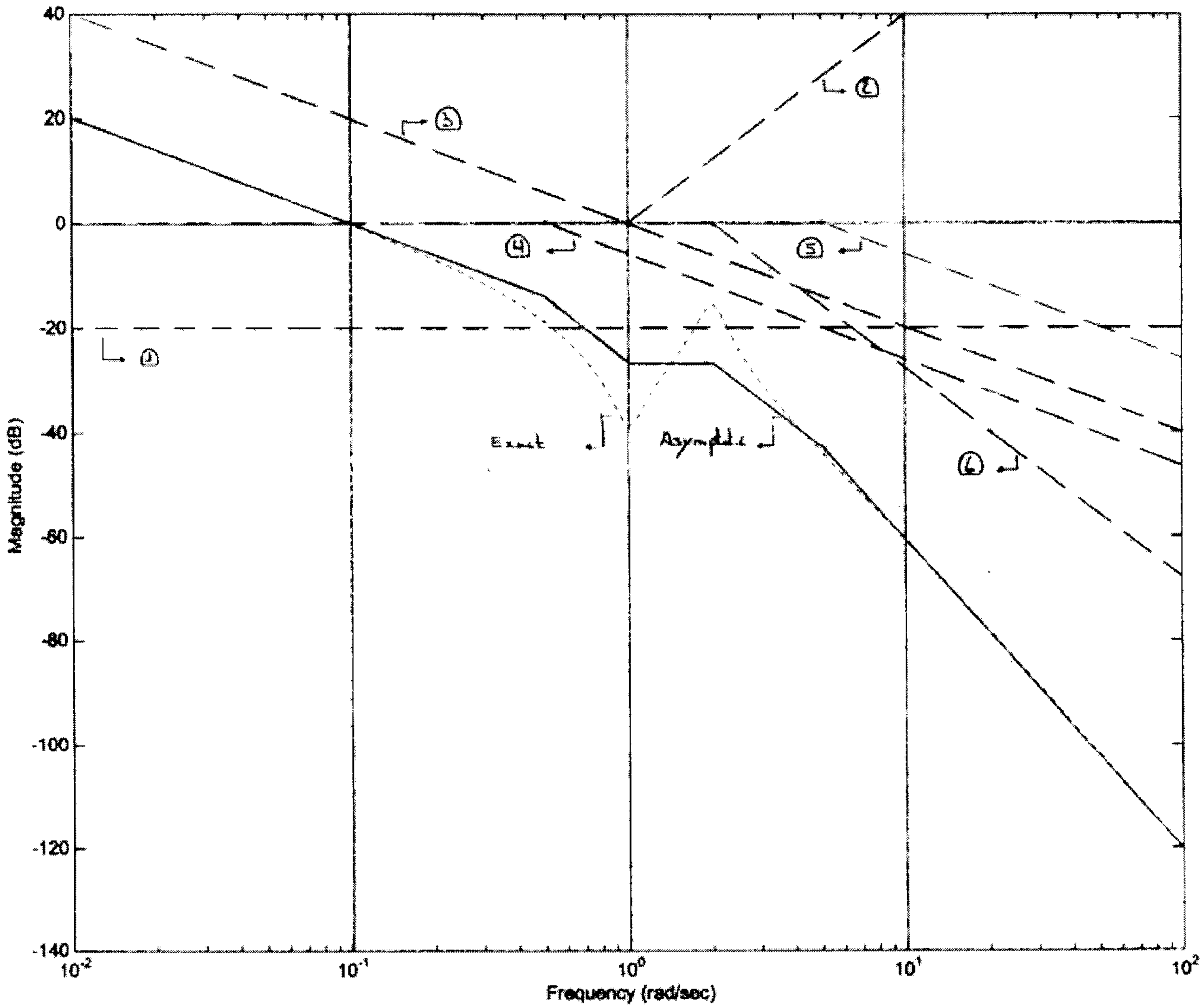
overpeak of the unstable resonant zeros : $20\log\sqrt{\left[\frac{0.2}{1.01}\sqrt{1.01}\right]^2} \cong -14$ dB at $\omega = \omega_n = \sqrt{1.01}$

overpeak of the stable resonant poles : $-20\log\sqrt{\left[\frac{0.4}{4.04}\sqrt{4.04}\right]^2} \cong 14$ dB at $\omega = \omega_n = \sqrt{4.04}$

$$\angle P(j\omega) = \angle 0.1 + \angle\left(1 - \frac{\omega^2}{1.01}\right) - i\frac{0.2}{1.01}\omega - \angle j\omega - \angle\; 1 + i\frac{\omega}{0.5} - \angle\; 1 + i\frac{\omega}{5}$$

$$- \angle\left(1 - \frac{\omega^2}{4.04}\right) + i\frac{0.4}{4.04}\omega$$

$$= \underset{(1)}{0^\circ} + \underset{(2)}{\angle\left[\left(1 - \frac{\omega^2}{1.01}\right) - i\frac{0.2}{1.01}\omega\right]} - \underset{(3)}{\frac{\pi}{2}} - \underset{(4)}{\operatorname{atan}\frac{\omega}{0.5}} - \underset{(5)}{\operatorname{atan}\frac{\omega}{5}} - \underset{(6)}{\angle\left[\left(1-\frac{\omega^2}{4.04}\right) + i\frac{0.4\omega}{4.04}\right]}$$



40
20
0
-20
-40
-60
-80
-100
-120
-140
Magnitude (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
1
2
3
4
5
6
Exact
Asymptotic

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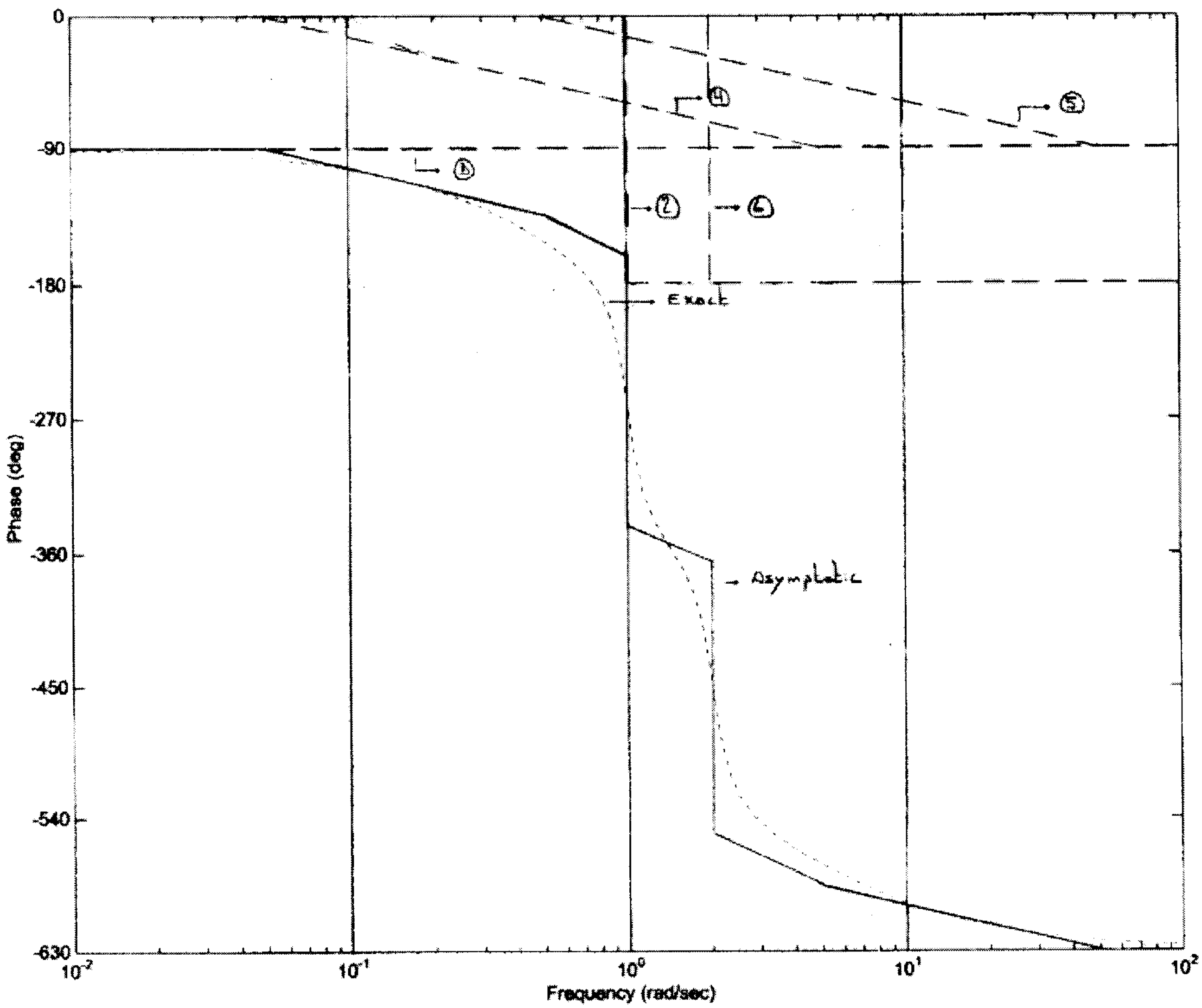
0
-90
-180
-270
-360
-450
-540
-630
Phase (deg)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
1
2
4
5
6
Exact
Asymptotic

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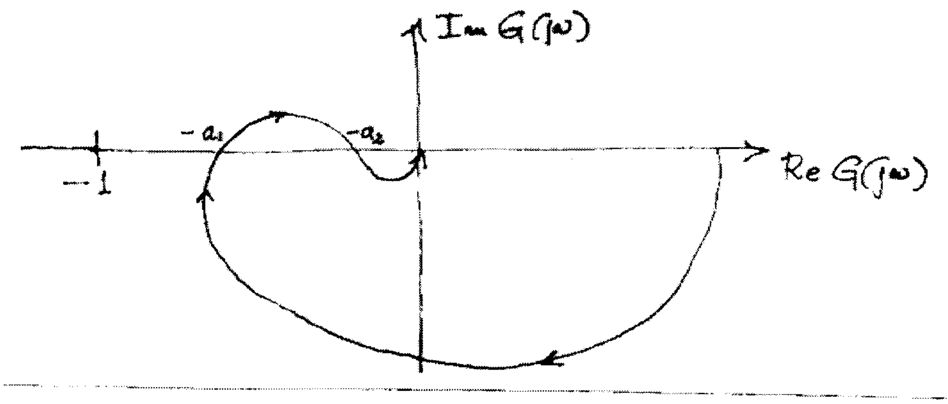
Problem 4. (13 points)

Consider the Nyquist plot



This is the Nyquist plot of the plant

$$G(s) = \frac{p^{2+r}(s+z)^2}{z^2(s+p)^{2+r}}, \qquad p, z > 0$$

but with the features of the plot exagerated (i.e., the scaling of the axes is not linear).

Answer the following questions (answers without justification will not receive credit even if they happen to be correct):

(a) If this plant is in a feedback loop with a gain K, what is the set of K's for which the system is asymptotically stable?

(b) What is the gain margin of the system?

(c) Based on the plot, which of the choices is correct:

- $r = 0$
- $r = 1$
- $r = 2$
- $r = 3$?

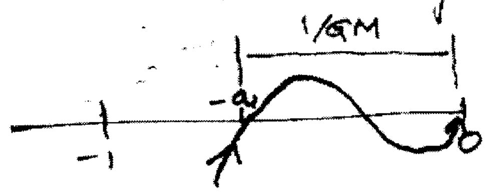
(d) Based on the plot, which of the choices is correct:

- $z > p$
- $z >> p$
- $p > z$
- $p >> z$?

(a) Since $G(s)$ is stable, we apply the Nyquist corrolary. The feedback system is stable if $(-1/K, j0)$ is not encircled. An encirclement occurs only if $-1/K \in [-a_1, -a_2]$. Hence, for stability we need that $1/K > a_1$ or $1/K < a_2$, that is,

$$\boxed{0 < K < 1/a_1 \quad \text{or} \quad K > 1/a_2}$$

(b) Since GM is defined as



we have $\frac{1}{GM} =$, which gives

$$\boxed{GM = \frac{1}{a_1}}$$

(c) Since the angle of arrival at the origin is $-90°$, the relative degree is $\boxed{r = 1}$

(d) Since the phase of the system first drops and then rises, the pole comes before the zero, i.e., $z > p$.
However, if the zero was close to the pole the phase drop due to the pole would not be able to develop as much as the plot shows – the plot shows a phase drop well below $-180°$.
Thus, $\boxed{z \gg p}$