Problem 1. (~~7~~ points)

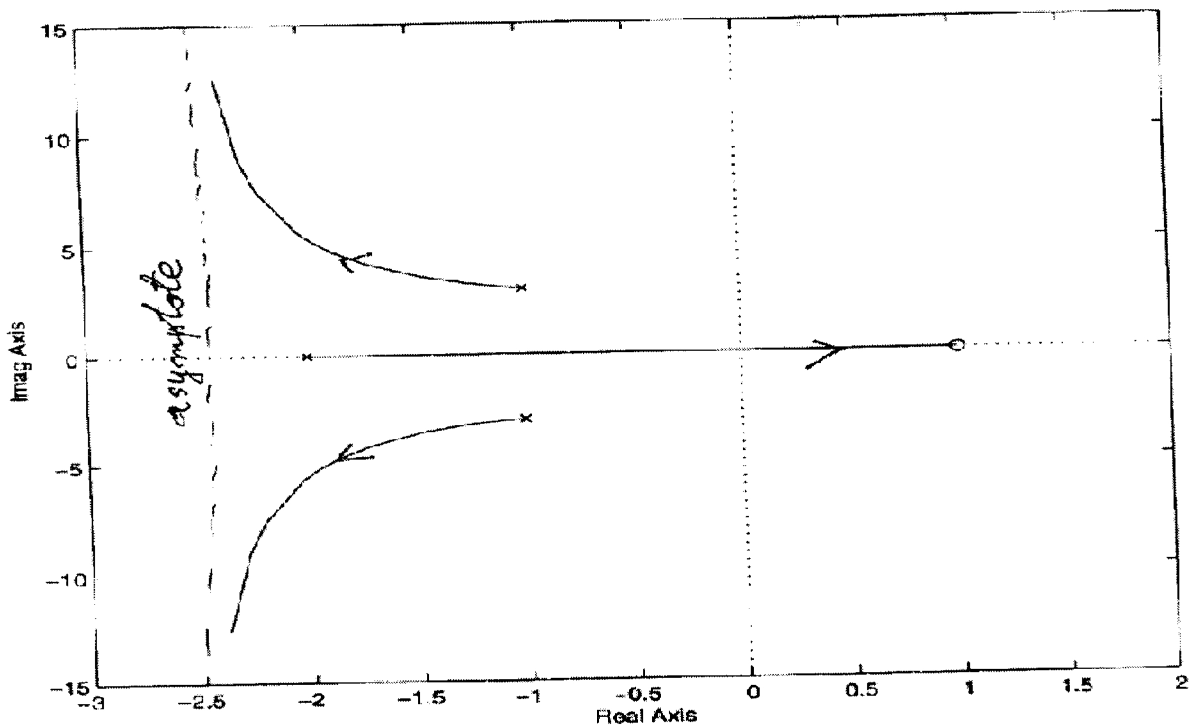
Consider the plant

$$P(s) = \frac{s-1}{(s+2)(s^2+2s+10)}$$

in a feedback loop with a gain $K > 0$. Sketch the root locus.

Rel. deg = 2 $\Longrightarrow$ 2 asymptotes w/ angles $\pm 90°$

Intersect. of asymp. w/ Re axis at $\frac{-2-2-1}{2} = -2.5$



Problem 2. (7 points)

By applying **Routh's criterion** to the system in **Problem 1**, find the range of $K > 0$ such that the system in asymptotically stable.

Closed-loop transfer function:

$$\frac{\frac{s-1}{(s+2)(s^2+2s+10)}}{1+K\frac{s-1}{(s+2)(s^2+2s+10)}}$$

$$= \frac{s-1}{(s+2)(s^2+2s+10)+K(s-1)}$$

$$= \frac{s-1}{s^3+4s^2+14s+20+K(s-1)}$$

$$= \frac{s-1}{s^3+4s^2+(14+K)s+20-K}$$

$$\triangleq \frac{s-1}{s^3+a_2s^2+a_1s+a_0}$$

We derived in class, based on Routh's criterion for a general 3rd order system, that stability is guaranteed if and only if

$$a_0,\ a_1,\ a_2,\ a_2a_1-a_0>0.$$

$a_0 = 20 - K > 0$ for $\underline{K < 20}$

$a_1 = 14 + K > 0$ for all $K > 0$

$a_2 = 4 > 0$

$$a_2 a_1 - a_0 = 4(14 + K) - (20 - K)$$
$$= 36 + 5K > 0 \quad \text{for all } K > 0$$

Conclusion: K must satisfy

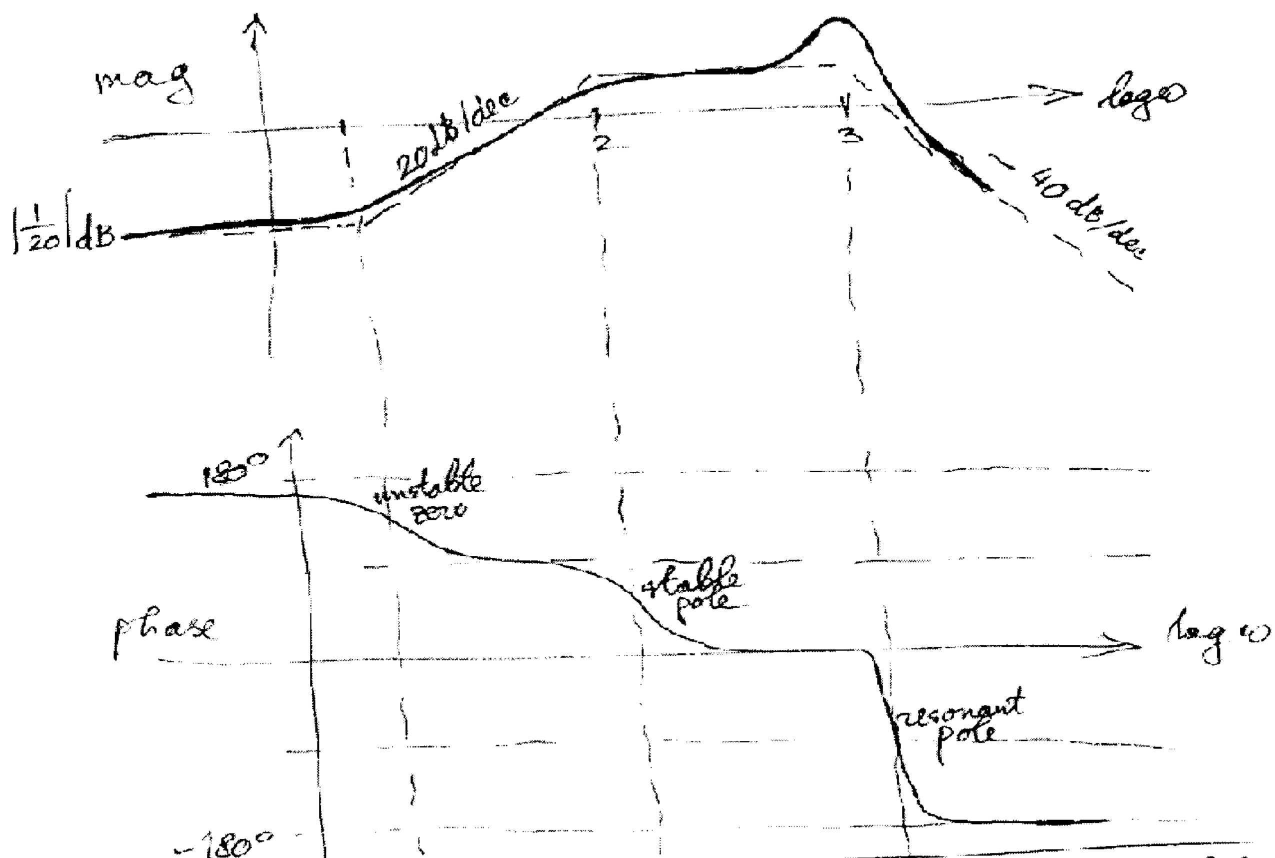
$$\underline{0 < K < 20}$$

Problem 3. (7 points)

Sketch the approximate Bode plot for the plant in Problem 1. (Please exagerate the features so that it is clear if you have understood the procedure.)

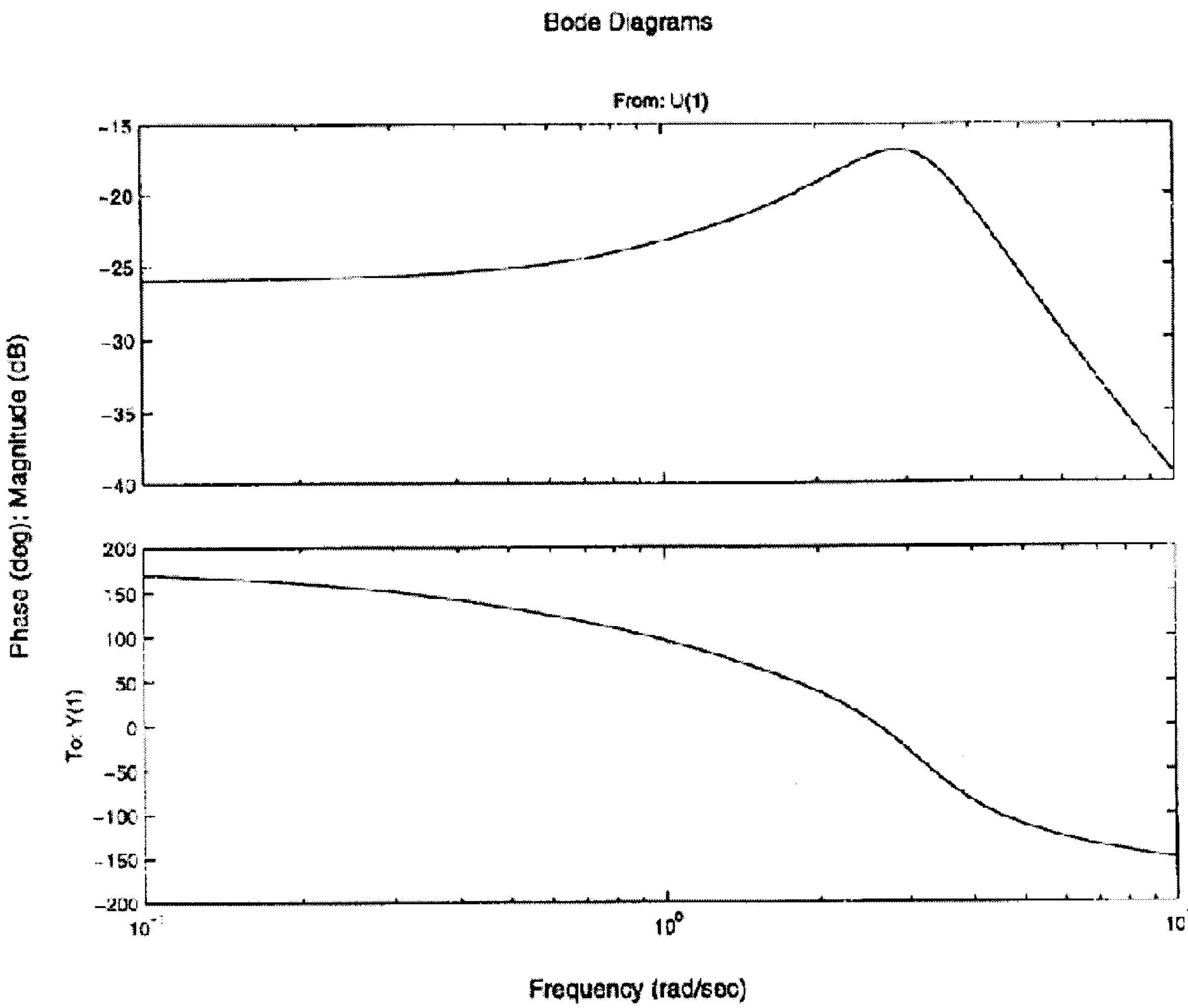
$$P(s) = \frac{s-1}{(s+2)\left[(s+1)^2+3^2\right]}, \quad P(j0) = -\frac{1}{20}$$

DC gain = 1/20, DC phase = 180°



Remember, these are plots with exaggerated features!

Actual Bode plot by Matlab:



Problem 4. (8 points)

Sketch the Nyquist plot for the plant in Problem 1. (Make sure to calculate the intersections with the axes and to indicate clearly the angle of arrival for $\omega \to \infty$.)

$$\omega \to 0 \implies P(j\omega) \to -\frac{1}{20} = -0.05$$

$$\omega \to \infty \implies P(j\omega) \to \frac{1}{(j\omega)^2} = -\frac{1}{\omega^2} = -0$$

To calculate the intersections, let us compute

$$P(j\omega) = \frac{j\omega - 1}{(j\omega)^3 + 4(j\omega)^2 + 14(j\omega) + 20}$$

$$= \frac{j\omega - 1}{20 - 4\omega^2 + j\omega(14 - \omega^2)} \cdot \frac{20 - 4\omega^2 - j\omega(14 - \omega^2)}{20 - 4\omega^2 - j\omega(14 - \omega^2)}$$

$$= \frac{-\omega^4 + 18\omega^2 - 20 + j\omega(34 - 5\omega^2)}{(20 - 4\omega^2)^2 + \omega^2(14 - \omega^2)^2}$$

Intersections with Re axis:

$\omega = 0$ and $34 = 5\omega^2$

$\downarrow$ already done above

$\downarrow$ $\omega = \sqrt{\frac{35}{5}} = 2.6077$

$$P(j2.6077) = \left. \frac{-\omega^4 + 18\omega^2 - 20}{(20 - 4\omega^2)^2 + \omega^2(14 - \omega^2)^2} \right|_{\omega = 2.6077}$$

$$= 0.139$$

Intersections w/ Im axis:

$$\omega^4 - 18\,\omega^2 + 20 = 0$$

$$\rightarrow \omega^2 = 9 \pm \sqrt{61} \rightarrow \omega = \sqrt{9-\sqrt{61}} = 1.0908$$

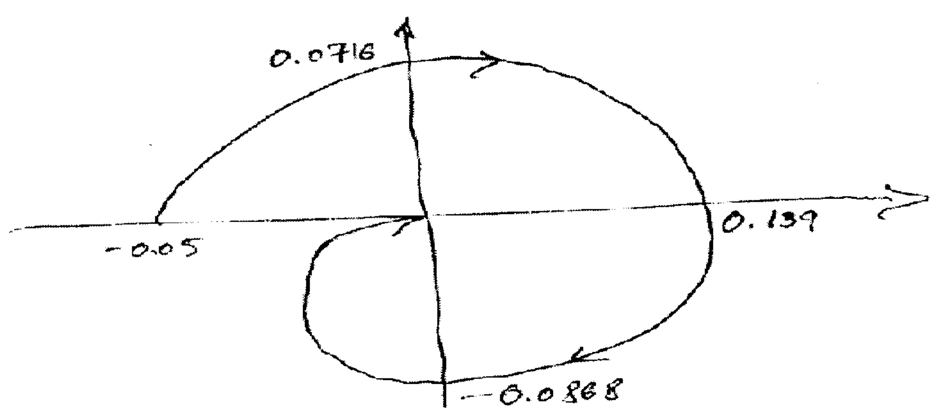
$$\text{and } \omega = \sqrt{9+\sqrt{61}} = 4.1$$

$$P(j1.0908) = j\,\frac{\omega(34-5\omega^2)}{(20-4\omega^2)^2 + \omega^2(14-\omega^2)^2}\Bigg|_{\omega=1.0908}$$

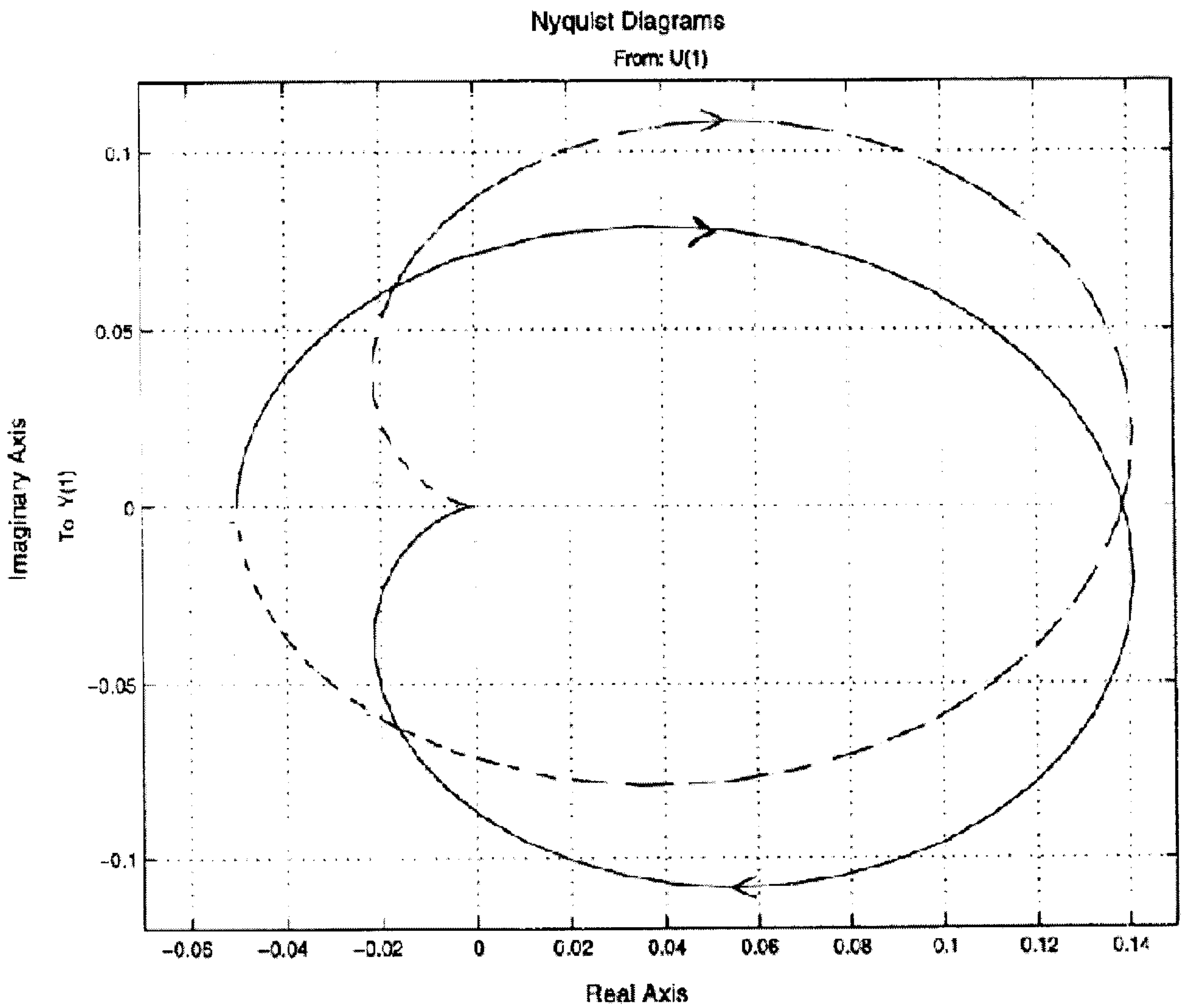
$$= j0.0716$$

$$P(j4.1) = j\,\frac{\omega(34-5\omega^2)}{(20-4\omega^2)^2 + \omega^2(14-\omega^2)^2}\Bigg|_{\omega=4.1}$$

$$= -j\,0.0868$$



Actual Nyquist by Matlab:
(you can ignore the dashed/symmetric part)



$$P(s)\,C(s) = \frac{(s-1)(s+5)}{(s+2)(s+20)(s^2+2s+10)}$$

Problem 5. (7 points)

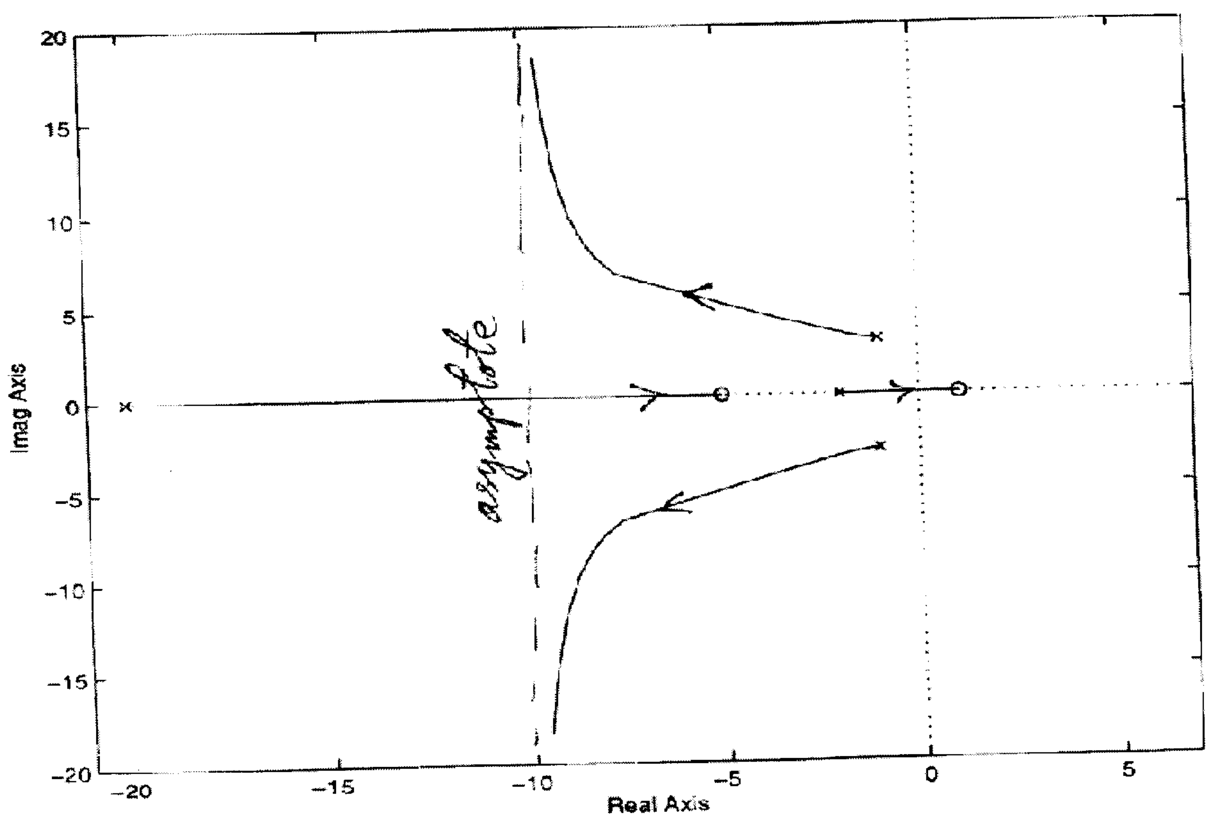
Add the phase-lead compensator

$$C(s) = \frac{s+5}{s+20}$$

to the feedback system in Problem 1. Find the root locus.

Rel deg = 2 $\Longrightarrow$ 2 asymp. w/ angles $\pm 90°$

Intersect. of asymp w/ Re axis at $\frac{-2-2-1-20+5}{2} = -10$



Problem 6. (7 points)

By applying Routh's criterion to the system in Problem 5, find the range of $K > 0$ such that the system in asymptotically stable.

$$P(s)C(s) = \frac{s^2+4s-5}{s^4+24s^3+94s^2+300s+400}$$

Closed-loop:

$$\frac{KP(s)C(s)}{1+KP(s)C(s)} = \frac{s^2+4s-5}{s^4+24s^3+(94+K)s^2+(300+4K)s+400-5K}$$

Routh:

s^4	1	$94+K$	$400-5K$
s^3	24	$300+4K$	
s^2	$\frac{1956+20K}{24}$	$400-5K$	
s^1	$\frac{356400+16704K+80K^2}{1956+20K}$		
s^0	$400-5K$		

For $K>0$ only the last row may become negative. So the range of stabilizing gains is

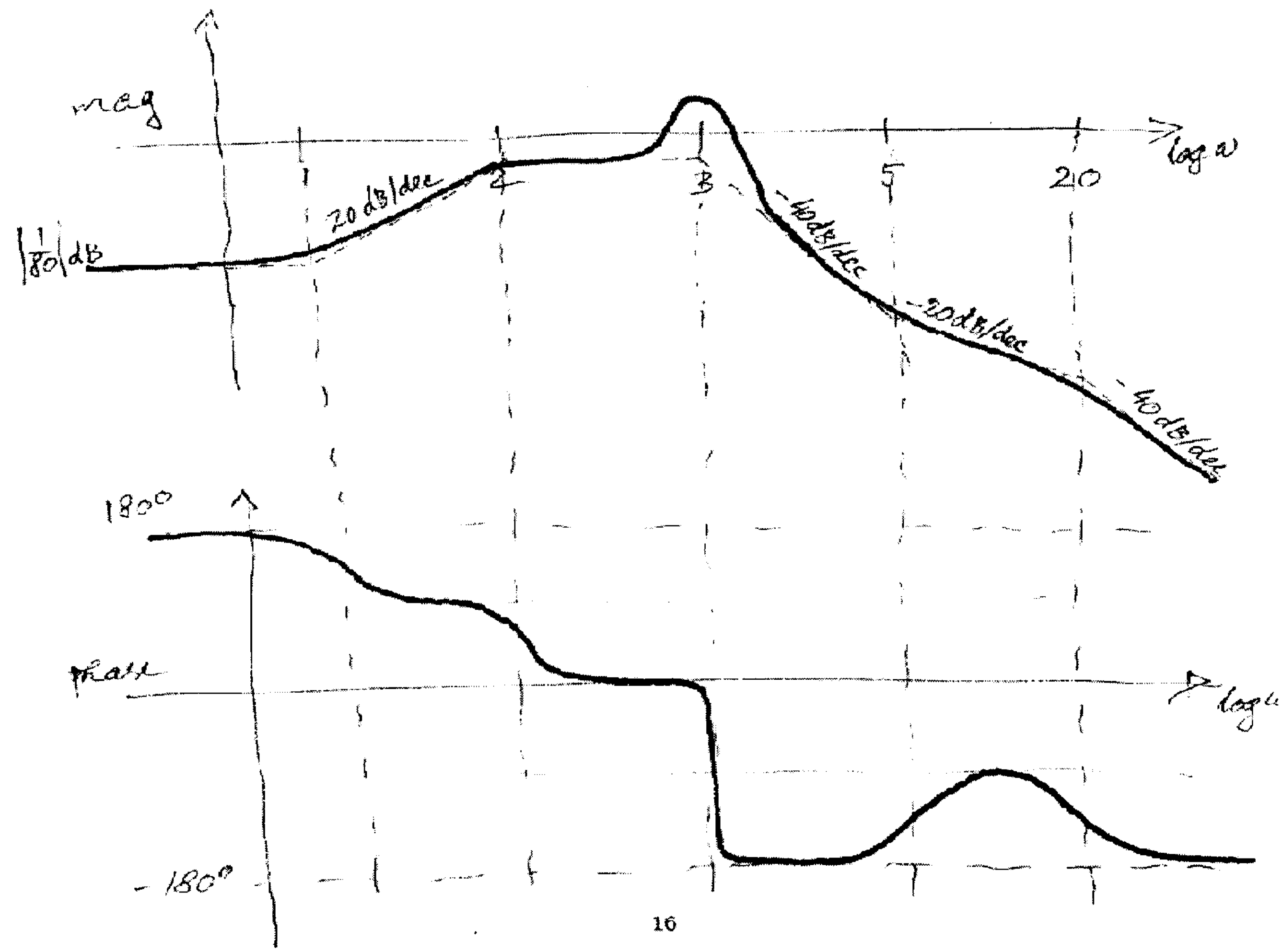
$$\underline{0 < K < 80}$$

Note how the lead compensator has increased the range from problem 2!

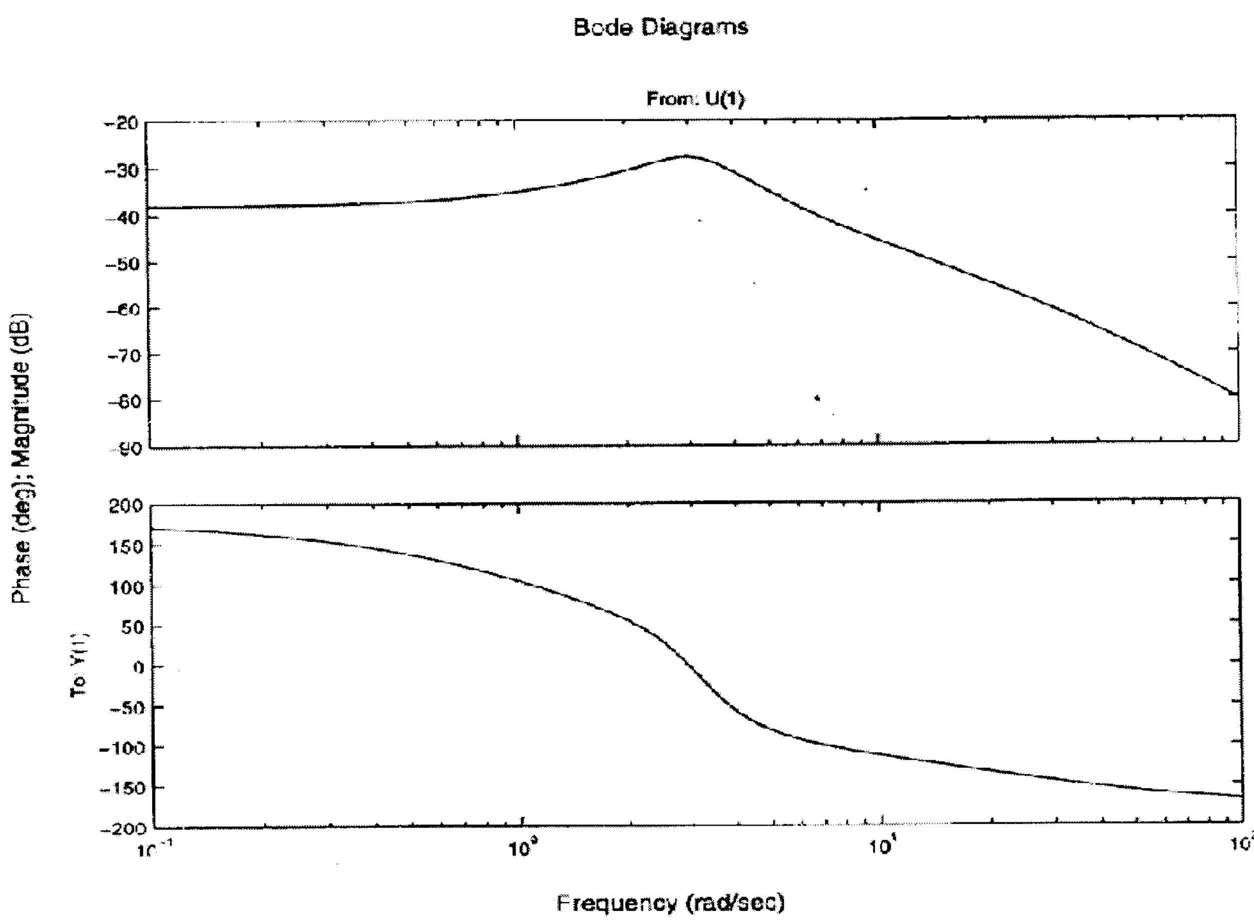
Problem 7. (7 points)

For the system in Problem 6 sketch the Bode plot. Then, based on the Bode plot, sketch the Nyquist plot (no need to calculate the intersection/s with the positive real axis in this problem). Does your Nyquist plot confirm your result for problem 6?

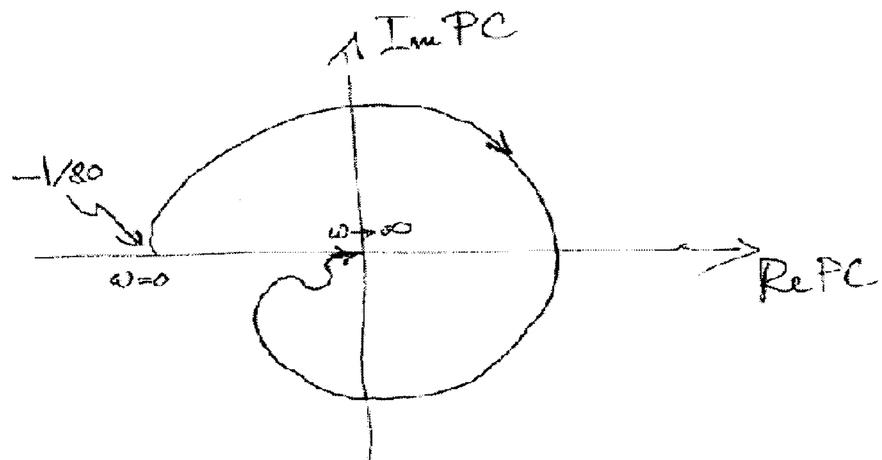
$$P(s)C(s) = \frac{(s-1)(s+5)}{(s+2)(s+20)\left[(s+1)^2+3^2\right]}, \quad P(j0) = -\frac{1}{80}$$



Actual Bode plot by Matlab:



From the Bode plot done by hand (not Matlab) one would expect a Nyquist plot like this:



The " wiggle " at the end of the curve is due to the fact that the phase plot (Bode) has an interval of growth between $\omega = 5$ and $\omega = 20$.

Regarding stability, since the open-loop system is stable, we look for encirclements of the critical point $-1/K$. From the Bode phase plot it is clear that the phase does not cross the value $\pm 180°$ (except for $\omega = 0$). Thus,

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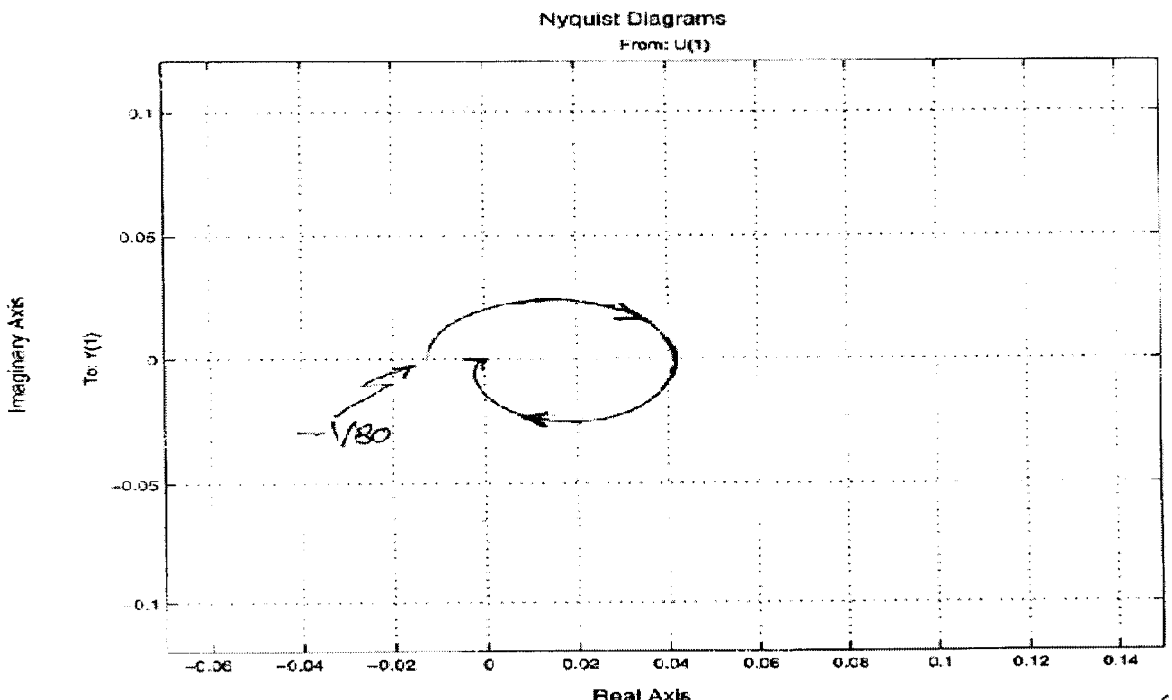
our Nyquist sketch is correct in showing no intersections with the negative real axis (other than $\omega=0$). Hence, encirclement occurs only for $K > 80$. This confirms the result for problem 6 that the system is stable for $K < 80$

The actual Nyquist plot by Matlab is:



(I tried to white out the symmetric / $\omega<0$ part, didn't succeed completely.)

Note the absence of the "wiggle" because the actual phase Bode plot does not have a growth in the [5, 20] frequency range.