MAE 143B
LINEAR CONTROL

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FINAL Solutions

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Problem 1. (14 points) Sketch the Nyquist diagrams for the following systems.

a) (2 points) $G(s) = \dfrac{2s+1}{s^2(s+1)}$

b) (2 points) $G(s) = \dfrac{s-5}{(s+1)^2}$

c) (2 points) $G(s) = \dfrac{1}{s^2(s^2+3s+1)}$

d) (3 points) $G(s) = \dfrac{s^2+s+1}{(s-1)^2(s+1)}$

e) (5 points) $G(s) = \dfrac{2s-3}{(s+8)(s^2-2s+3)}$

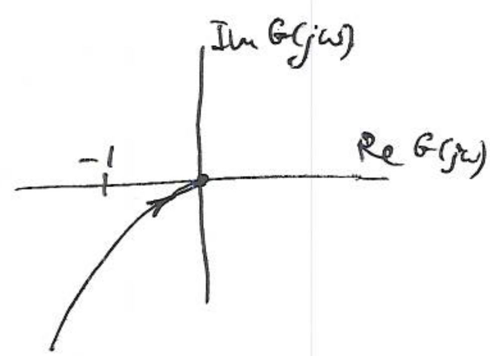
In each case, using Nyquist stability criterion, determine *exactly* the interval for the gain K such that the negative feedback loop of $G(s)$ and K is asymptotically stable.

(a) $G(j\omega) = \dfrac{2j\omega+1}{-\omega^2(j\omega+1)} = \dfrac{-1-2\omega^2-j\omega}{\omega^2(\omega^2+1)}$

$\mathrm{Re}\, G(j0) = -\infty$; $\mathrm{Im}\, G(j0) = -\infty$

$\mathrm{Re}\, G(j\infty) = -0$; $\mathrm{Im}\, G(j\infty) = -0$

No intersections with axes.



Stable for all $K > 0$

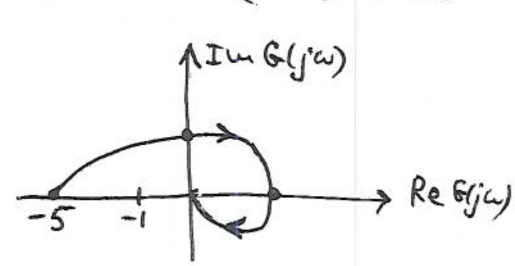
(b) $G(j\omega) = \dfrac{j\omega-5}{(j\omega+1)^2} = \dfrac{j\omega-5}{1-\omega^2+2j\omega} = \dfrac{(j\omega-5)(1-\omega^2-2j\omega)}{(1-\omega^2)^2+4\omega^2} = \dfrac{7\omega^2-5+j\omega(11-\omega^2)}{(1-\omega^2)^2+4\omega^2}$

$\mathrm{Re}\, G(j0) = -5$, $\mathrm{Im}\, G(j0) = 0$

$\mathrm{Re}\, G(j\infty) = +0$; $\mathrm{Im}\, G(j\infty) = -0$

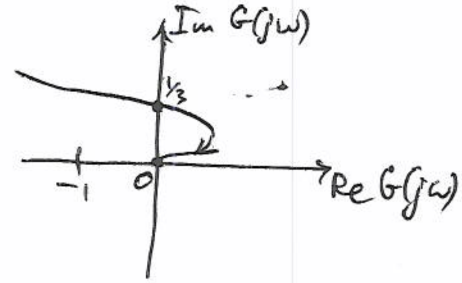
$\mathrm{Re}\, G(j\omega_1) = 0 \Rightarrow \omega_1 = \sqrt{\tfrac{5}{7}} \Rightarrow \mathrm{Im}\, G(j\omega_1) = 2.96$

$\mathrm{Im}\, G(j\omega_2) = 0 \Rightarrow \omega_2 = \sqrt{11} \Rightarrow \mathrm{Re}\, G(j\omega_2) = \tfrac{1}{2}$



Stable for $K < \frac{1}{5}$

1(c) $G(j\omega) = \dfrac{-1}{\omega^2(1-\omega^2+3j\omega)} = \dfrac{\omega^2-1+3j\omega}{\omega^2((1-\omega^2)^2+9\omega^2)}$



$\mathrm{Re}\,G(j0) = -\infty$; $\mathrm{Im}\,G(j0) = +\infty$

$\mathrm{Re}\,G(j\infty) = +0$; $\mathrm{Im}\,G(j\infty) = +0$

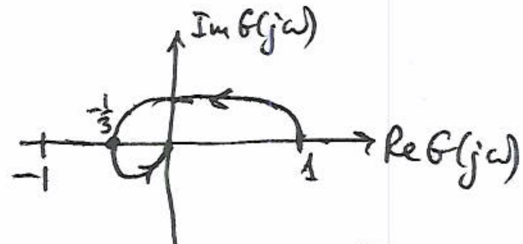
Intersection: $\mathrm{Re}\,G(j\omega_1) = 0 \Rightarrow \omega_1 = 1$

$\Rightarrow \mathrm{Im}\,G(j\omega_1) = \frac{1}{3}$

Unstable for all K

1(d) $G(j\omega) = \dfrac{1-\omega^2+j\omega}{(j\omega+1)(j\omega-1)^2} = \dfrac{1-\omega^2+j\omega}{1+\omega^2-j\omega(1+\omega^2)} = \dfrac{(1-\omega^2+j\omega)(1+j\omega)}{(1+\omega^2)(1+\omega^2)}$

$= \dfrac{1-2\omega^2+j\omega(2-\omega^2)}{(1+\omega^2)^2}$



$\mathrm{Re}\,G(j0) = 1$; $\mathrm{Im}\,G(j0) = 0$

$\mathrm{Re}\,G(j\infty) = -0$; $\mathrm{Im}\,G(j\infty) = -0$

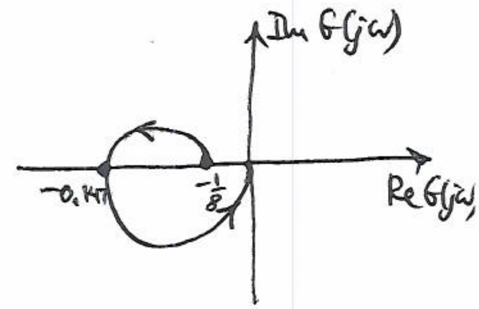
Intersections: $\mathrm{Re}\,G(j\omega_1) = 0 \Rightarrow \omega_1 = \frac{1}{\sqrt{2}} \Rightarrow \mathrm{Im}\,G(j\omega_1) = \frac{\sqrt{2}}{3}$

$\mathrm{Im}\,G(j\omega_2) = 0 \Rightarrow \omega_2 = \sqrt{2} \Rightarrow \mathrm{Re}\,G(j\omega_2) = -\frac{1}{3}$

Stable for $K > 3$ (2 unstable open-loop poles, 1 ccw encirclement for $K>3$)

1(e) $G(j\omega) = \dfrac{2j\omega-3}{(j\omega+8)(3-\omega^2-2j\omega)} =$

$= \dfrac{-2\omega^4-8\omega^2-72+j\omega(9-15\omega^2)}{36(\omega^2-4)^2+\omega^2(\omega^2+13)^2}$



$\mathrm{Re}\,G(j0) = -\frac{1}{8}$; $\mathrm{Im}\,G(j0) = 0$

$\mathrm{Re}\,G(j\infty) = -0$; $\mathrm{Im}\,G(j\infty) = -0$

$\mathrm{Re}\,G(j\omega) = 0 \Rightarrow$ no solutions because $\omega^4+4\omega^2+36 = 0 \Rightarrow \omega^2 = -2 \pm \sqrt{-32}$ not physical

$\mathrm{Im}\,G(j\omega_1) = 0 \Rightarrow \omega_1 = \frac{3}{\sqrt{15}}$ (besides $\omega = 0$) $\Rightarrow \mathrm{Re}\,G(j\omega_1) = -0.147$

Two open-loop unstable poles: $s = 1 \pm j\sqrt{2}$

One ccw encirclement: when -1 is between

$-K \cdot 0.147 < -1 < -K \cdot \frac{1}{8} \Rightarrow$

$\boxed{\text{Stable for } 6.8 < K < 8}$

Problem 2. (8 points)

Consider the system

$$G(s) = \frac{K}{s(s+1)(s+3)}$$

in the negative unity feedback loop.

Using the Nyquist diagram,

(a) (4 points) Find K for which the gain margin (GM) is exactly 2.

(b) (4 points) Find K for which the phase margin (PM) is exactly 45 degrees.

(a) $G(j\omega) = \dfrac{k}{j\omega(j\omega+1)(j\omega+3)} = \dfrac{-Kj}{\omega(3-\omega^2+4j\omega)}$

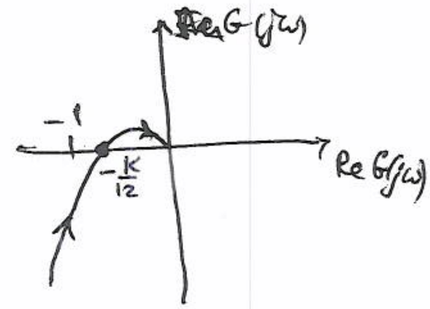
$= \dfrac{-Kj(3-\omega^2-4j\omega)}{\omega((3-\omega^2)^2+16\omega^2)} = \dfrac{-4K\omega + jK(\omega^2-3)}{\omega((3-\omega^2)^2+16\omega^2)}$

$\operatorname{Re} G(j0) = -\frac{4}{9}K$; $\operatorname{Im} G(j0) = -\infty$

$\operatorname{Re} G(j\infty) = -0$; $\operatorname{Im} G(j\infty) = +0$

Intersection: $\operatorname{Im} G(j\omega_1) = 0 \Rightarrow \omega_1 = \sqrt{3} \Rightarrow \operatorname{Re}(j\omega_1) = -\frac{K}{12}$

$-\frac{1}{GM} = -\frac{K}{12} \Rightarrow K = \frac{12}{2} = 6.$



(b) $\tan(\pi + PM) = \dfrac{\omega_0^2-3}{-4\omega_0} = \dfrac{3-\omega_0^2}{4\omega_0}$

$\tan(\pi + \frac{\pi}{4}) = 1 \Rightarrow \omega_0^2 + 4\omega_0 - 3 = 0$

$\omega_0 = -2 \pm \sqrt{7} \Rightarrow \omega_0 = \sqrt{7} - 2 \approx 0.646$

$|G(j\omega_0)|^2 = 1 \Rightarrow \dfrac{K^2(16\omega_0^2 + (\omega_0^2-3)^2)}{\omega_0^2((3-\omega_0^2)^2+16\omega_0^2)^2} = 1$

$\Rightarrow K = \omega_0\sqrt{16\omega_0^2 + (\omega_0^2-3)^2} \approx 2.359$

Problem 3. (8 points) The general form of the lead compensator is

$$G(s) = K\frac{1+T_1 s}{1+T_2 s},$$

where $T_1 > T_2$.

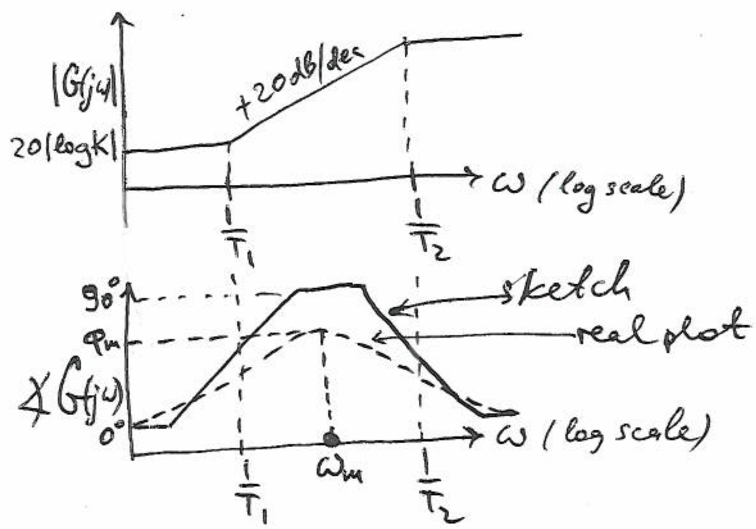
(a) (2 points) Sketch the Bode plots of $G(s)$.

(b) (5 points) From the Bode plots you can see that there is a maximum phase advance at the particular frequency. Find this frequency and the maximum phase advance.

Hint: First derive the equation $\tan(\phi(\omega)) = f(\omega)$, ϕ being the phase (don't use the sketch obtained in (a), derive the exact expression from $G(s)$); since $\tan(\phi)$ is monotonically increasing function of ϕ, the same frequency maximizes $f(\omega)$ and $\phi(\omega)$, so it is sufficient to find the maximum of $f(\omega)$ by differentiating $f(\omega)$ w.r.t. ω and setting the result to zero; find ω from this equation and calculate the corresponding phase.

(c) (1 point) What is the maximum phase advance if T_1 and T_2 are separated by a decade, i.e. $T_1 = 10T_2$?

(a)



(b) $G(j\omega) = K\dfrac{1+T_1 j\omega}{1+T_2 j\omega} = K\dfrac{(1+T_1 j\omega)(1-T_2 j\omega)}{1+T_2^2\omega^2} = \dfrac{K}{1+T_2^2\omega^2}\left(1+T_1T_2\omega^2 + j\omega(T_1-T_2)\right)$

$$\tan\phi = \frac{\omega(T_1-T_2)}{1+T_1T_2\omega^2} = f(\omega).$$

$$f'(\omega) = \frac{(T_1-T_2)(1+T_1T_2\omega^2) - 2T_1T_2\omega^2(T_1-T_2)}{(1+T_1T_2\omega^2)^2}; \quad f'(\omega_m) = 0$$

$$\Rightarrow 1+T_1T_2\omega_m^2 = 2T_1T_2\omega_m^2 \Rightarrow \omega_m = \frac{1}{\sqrt{T_1T_2}} \Rightarrow \tan\phi_m = \frac{T_1-T_2}{2\sqrt{T_1T_2}}$$

$$\boxed{\phi_m = \tan^{-1}\left[\frac{T_1-T_2}{2\sqrt{T_1T_2}}\right]}$$

(c) $\phi_m = \tan^{-1}\left[\dfrac{10T_2-T_2}{2\sqrt{10T_2^2}}\right] = \tan^{-1}\left(\dfrac{9}{2\sqrt{10}}\right) = 0.958 \Rightarrow \boxed{\phi_m = 54.9^\circ}$