MAE 143B

LINEAR CONTROL — Prof. M. Krstic

FINAL EXAM

June 9, 2005

NAME: Solution

- Open books and notes.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 60.
- Time: 3:00–6:00 (3 minutes/point)

Problem 1. (8 points)

Consider the plant

$$P(s) = \frac{s-2}{(s+3)(s^2+2s+17)}$$

in a feedback loop with a gain $K > 0$. Sketch the root locus.

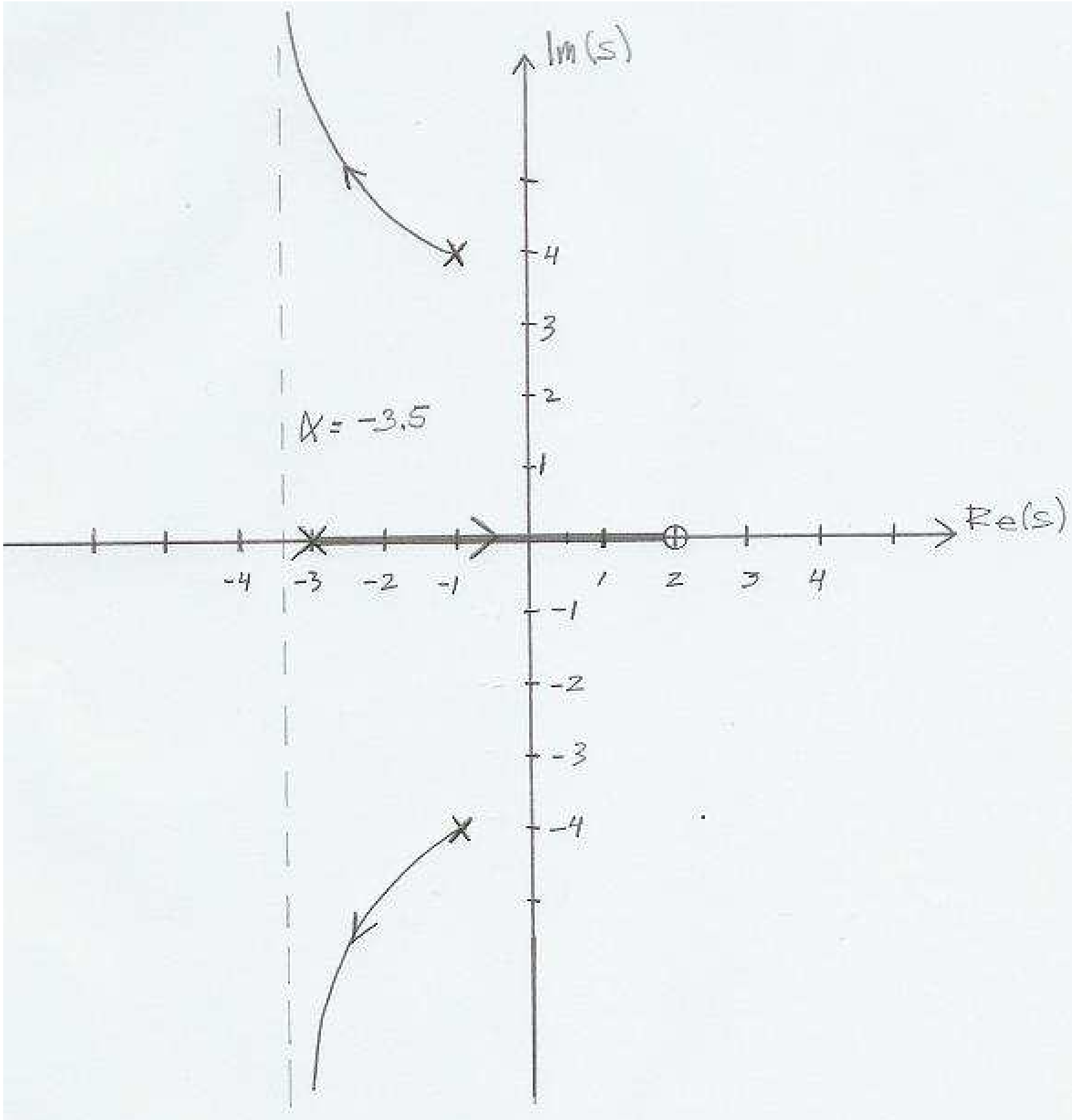
Solution: The poles and zeros of $P(s)$ are:

$$\begin{aligned} p_1 &= -3 \\ p_{2,3} &= -1 \pm j4 \\ z_1 &= 2 \end{aligned}$$

The relative degree is $n - m = 2$. Therefore there exist two asymptotes at $\pm 90°$, located at

$$\alpha = \frac{(-3-1-1)-(2)}{(3-1)} = -3.5.$$

The Root Locus is:



Sketch of Root Locus

Problem 2. (8 points)

By applying Routh's criterion to the system in Problem 1, find the range of $K > 0$ such that the system is asymptotically stable.

Solution: The characteristic equation is (the denominator of the Closed Loop transfer function):

$$\begin{aligned} 1 + KP(s) &= 0 \\ (s+3)(s^2+2s+17) + K(s-2) &= 0 \\ s^3 + 5s^2 + (23+K)s + (51-2K) &= 0 \end{aligned}$$

We have already derived the necessary and sufficient conditions for the stability of a 3^{rd} order system of the form: $s^3 + a_2 s^2 + a_1 s + a_0$. Based on Routh's criterion, stability is guaranteed if and only if

$$a_0,\ a_1,\ a_2,\ a_2 a_1 - a_0 > 0.$$

For this problem:

$$\begin{aligned} a_0 &= 51 - 2K \Rightarrow K < 25.5 \\ a_1 &= 23 + K > 0 \quad \text{for } K > 0 \\ a_1 a_2 - a_0 &= 3K + 64 > 0 \quad \text{for } K > 0 \end{aligned}$$

Therefore:

$$0 < K < 25.5$$

Problem 3. (8 points)

Sketch the approximate Bode plot for the plant in Problem 1. (Please exaggerate the features so that it is clear if you have understood the procedure.)

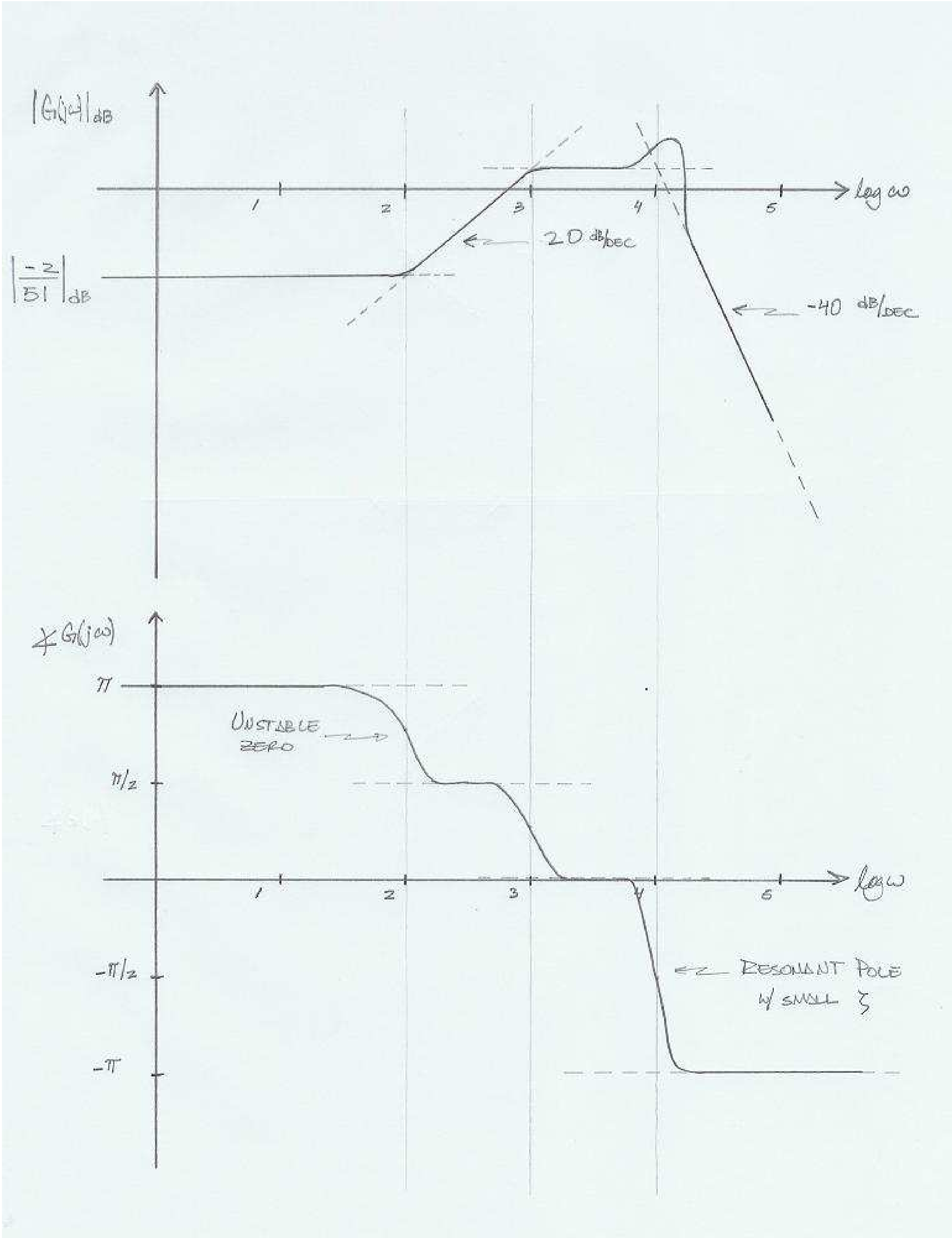
Solution: Rewriting $P(s)$ in *Bode form*

$$\begin{aligned} P(s) &= \frac{s-2}{(s+3)(s^2+2s+17)} \\ &= \frac{2}{51}\left(\frac{s}{2}-1\right)^1\left(\frac{s}{3}+1\right)^{-1}\left[\left(\frac{s}{\sqrt{17}}\right)^2+\frac{2}{17}s+1\right]^{-1} \end{aligned}$$

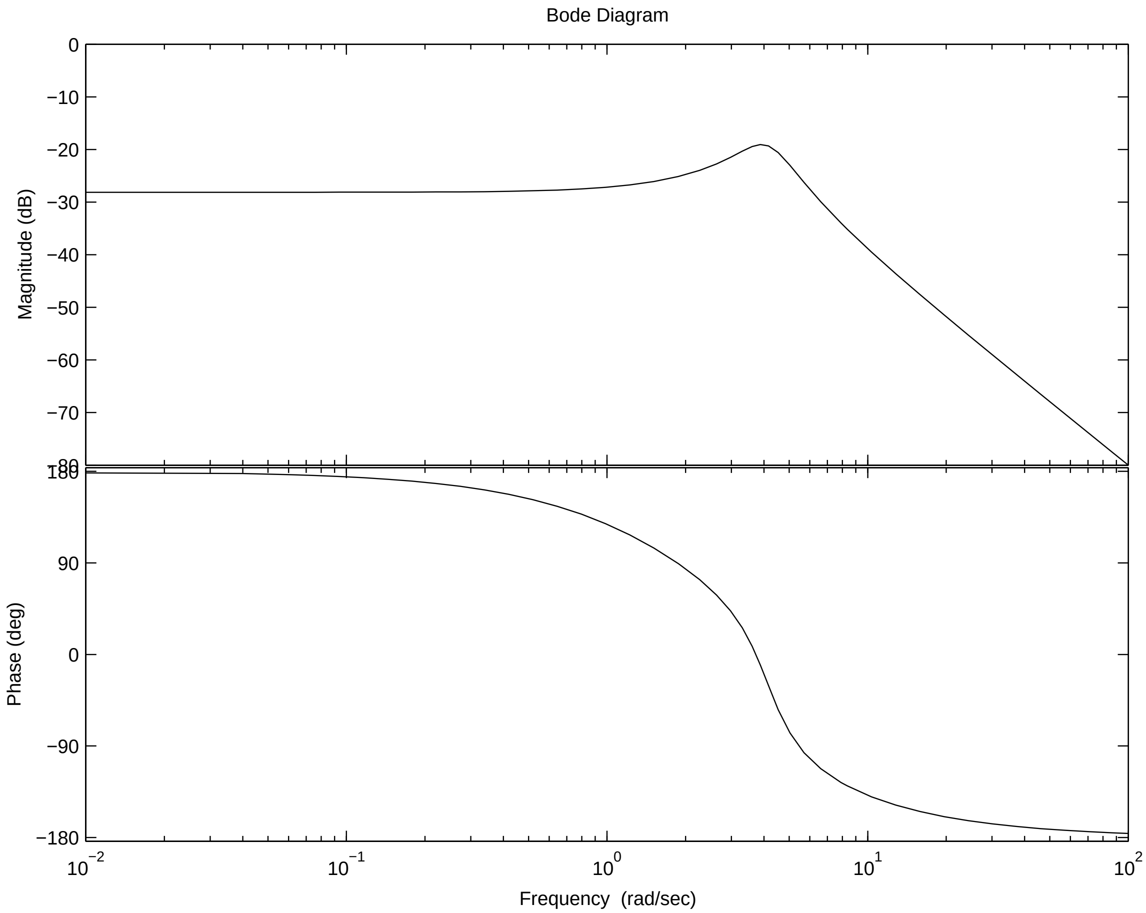
$$\text{DC Gain: } P(j0) = -\frac{2}{51}$$

$$\begin{aligned} 0\frac{rad}{s} < \omega < 2\frac{rad}{s} &: \ |P(j0)|_{dB} \approx -28dB, \quad \angle P(j0) = \pi \\ 2\frac{rad}{s} < \omega < 3\frac{rad}{s} &: \ m = 20\frac{dB}{dec}, \quad \Delta\phi = -\pi/2 \text{ (RHP zero)} \\ 3\frac{rad}{s} < \omega < \sqrt{17}\frac{rad}{s} &: \ m = 0\frac{dB}{dec}, \quad \Delta\phi = -\pi/2 \\ \sqrt{17}\frac{rad}{s} < \omega < \infty\frac{rad}{s} &: \ m = -40\frac{dB}{dec}, \quad \Delta\phi = -\pi \end{aligned}$$

The Bode plot is:



Sketch of Bode plot. *Note: this figure has been exaggerated, it is not to scale.*



Bode done in MATLAB

Problem 4. (11 points)

Sketch the Nyquist plot for the plant in Problem 1. (Make sure to calculate the intersections with the axes and to indicate clearly the angle of arrival for $\omega \to \infty$.)

Solution:

$$\begin{aligned}\omega &\to 0 \Rightarrow P(j\omega) \to -\frac{2}{51} \approx -0.04 \\ \omega &\to \infty \Rightarrow P(j\infty) \to -0 \text{ along } -\pi \text{ (from Bode diagram)}\end{aligned}$$

$$\begin{aligned}P(j\omega) &= \frac{j\omega - 2}{(j\omega)^3 + 5(j\omega)^2 + 23(j\omega) + 51} \\ &= \frac{-2 + j\omega}{(51 - 5\omega^2) + j(23\omega - \omega^3)} \, \frac{(51 - 5\omega^2) - j(23\omega - \omega^3)}{(51 - 5\omega^2) - j(23\omega - \omega^3)} \\ &= \frac{(-\omega^4 + 33\omega^2 - 102) + j(97\omega - 7\omega^2)}{(51 - 5\omega^2)^2 + (23\omega - \omega^3)^2}\end{aligned}$$

- Find intersection(s) with $Re - axis$, $(Im\{P(j\omega)\} = 0)$:

$$97\omega - 7\omega^3 = 0$$

$$\Rightarrow \omega = \{0, 3.72\}$$

$$\begin{aligned}P(j0) &= \left.\frac{-\omega^4 + 33\omega^2 - 102}{(51 - 5\omega^2)^2 + (23\omega - \omega^3)^2}\right|_{\omega=0} = -0.04 \\ P(j3.72) &= \left.\frac{-\omega^4 + 33\omega^2 - 102}{(51 - 5\omega^2)^2 + (23\omega - \omega^3)^2}\right|_{\omega=3.72} = 0.109\end{aligned}$$

Therefore the intersections occur at: $(-0.04, 0)$, $(0.109, 0)$

- Find intersection(s) with $Im - axis$ ($Re\{P(j\omega)\} = 0$):

$$\omega^4 - 33\omega^2 = 0$$

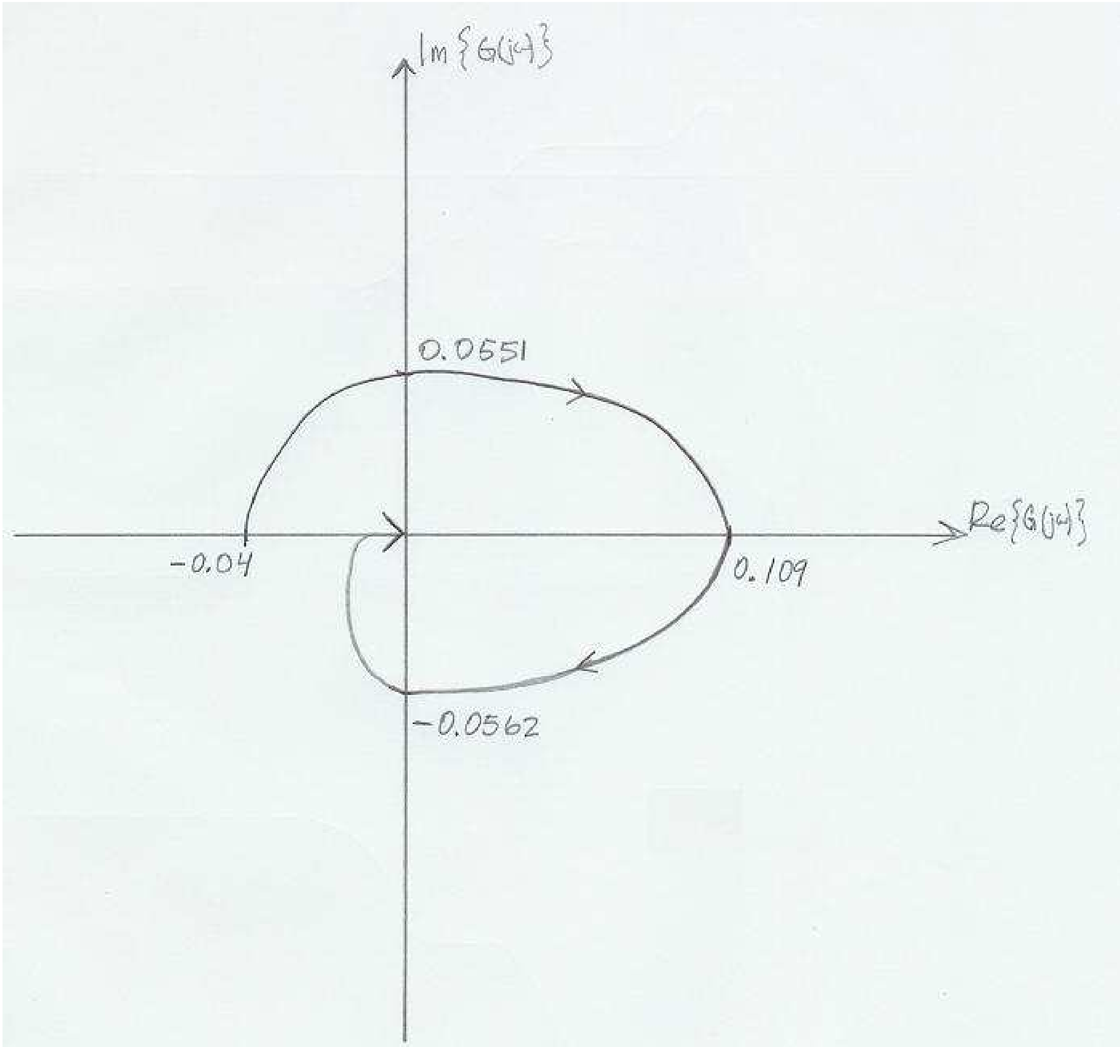
$$\begin{aligned}
\omega^2 &= \frac{33 \pm \sqrt{33^2 - 4(102)}}{2} \\
\omega_{1,2} &= \sqrt{\frac{33 \pm \sqrt{33^2 - 4(102)}}{2}}
\end{aligned}$$

$$\Rightarrow \omega_{1,2} = \{1.858, 5.436\}$$

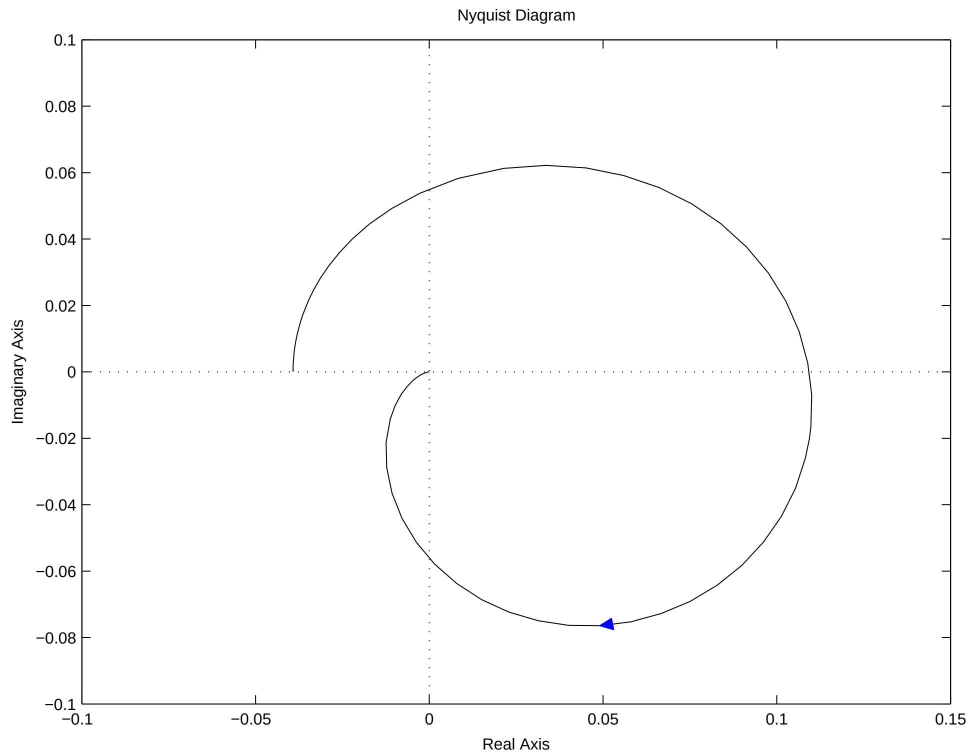
$$\begin{aligned}
P(j1.858) &= \left.\frac{97\omega - 7\omega^2}{(51 - 5\omega^2)^2 + (23\omega - \omega^3)^2}\right|_{\omega=1.858} = 0.0551 \\
P(j5.436) &= \left.\frac{97\omega - 7\omega^2}{(51 - 5\omega^2)^2 + (23\omega - \omega^3)^2}\right|_{\omega=5.436} = -0.0562
\end{aligned}$$

Therefore the intersections occur at: $(0, -0.0562)$, $(0, 0.0551)$

The Nyquist diagram is:



Sketch of Nyquist diagram



Nyquist done in MATLAB

Problem 5. (8 points)

Add the phase-lead compensator

$$C(s) = \frac{s+6}{s+20}$$

to the feedback system in Problem 1. Find the root locus.

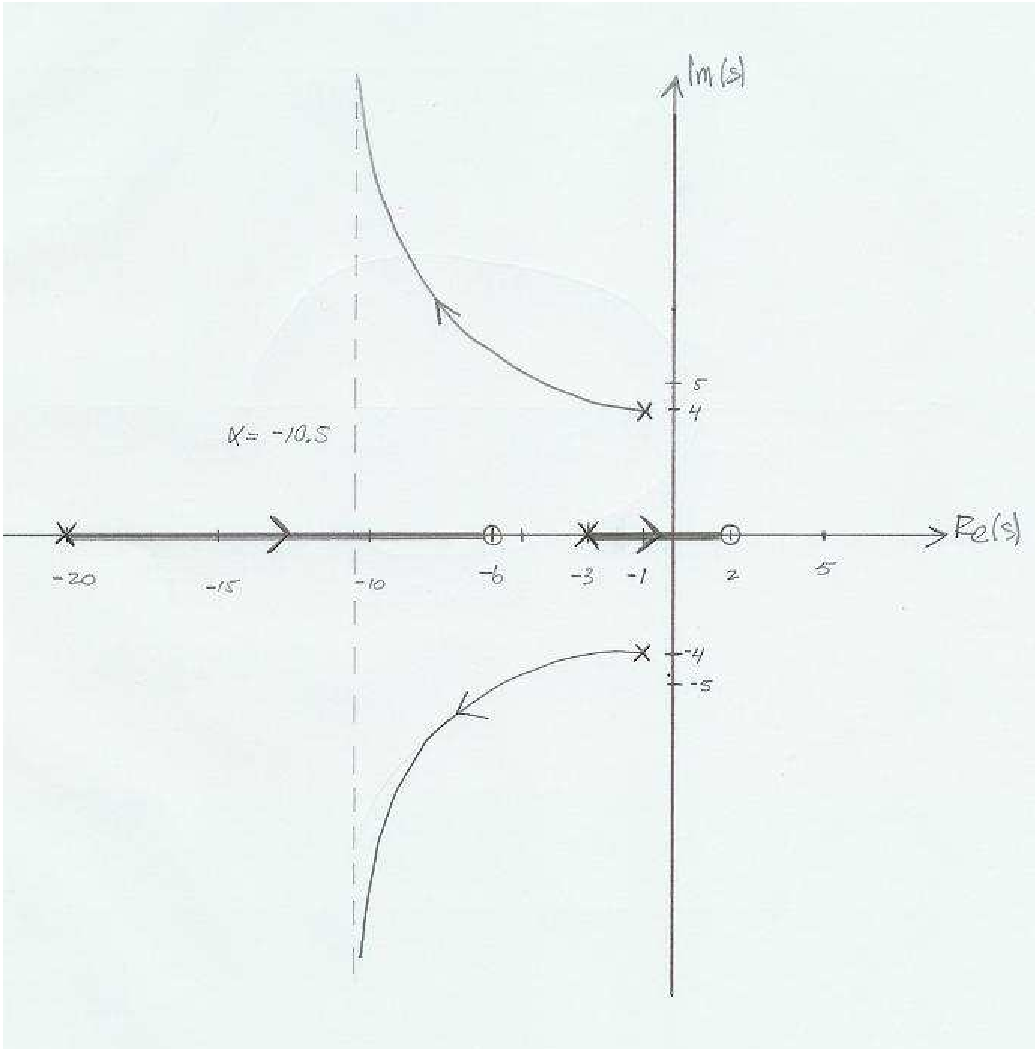
Solution: The poles and zeros of $C(s)P(s)$ are:

$$\begin{aligned} p_1 &= -3 \\ p_{2,3} &= -1 \pm j4 \\ p_4 &= -20 \\ z_1 &= 2 \\ z_2 &= -6 \end{aligned}$$

The relative degree is $n - m = 2$. Therefore there exist two asymptotes at $\pm 90°$, located at

$$\alpha = \frac{(-3-1-1-20)-(2-6)}{(3-1)} = -10.5.$$

The Root Locus is:



Sketch of Root Locus

Problem 6. (9 points)

By applying Routh's criterion to the system in Problem 5, find the range of $K > 0$ such that the system is asymptotically stable.

Solution: The characteristic equation is:

$$\begin{aligned} 1 + KC(s)P(s) &= 0 \\ (s+20)(s+3)(s^2+2s+17) + K(s+6)(s-2) &= 0 \\ s^4 + 25s^3 + (123+K)s^2 + (511+4K)s + (1020-12K) &= 0 \end{aligned}$$

$$\begin{array}{c|cccc} s^4 & 1 & (123+K) & (1020-12K) & 0 \\ s^3 & 25 & (511+4K) & 0 & \\ s^2 & b_1 & (1020-12K) & 0 & \\ s^1 & c_1 & 0 & & \\ s^0 & (1020-12K) & & & \end{array}$$

Where:

$$\begin{aligned} b_1 &= \frac{25(123+K)-(511+4K)}{25} = \frac{21K+2564}{25} \\ c_1 &= \frac{b_1(511+4K)-25(1020-12K)}{b_1} = \frac{84K^2+28,487K+685,204}{21K+2564} \end{aligned}$$

From first column of Routh array:

$$\begin{aligned} 21K+2564 &> 0 \quad \text{for all} K > 0 \\ \frac{84K^2+28,487K+685,204}{21K+2564} &> 0 \quad \text{for all} K > 0 \\ 1020-12K &> 0 \Rightarrow K < 85 \end{aligned}$$

Therefore:

$$0 < K < 85$$

note: the lead compensator has increased the range of K from Problem 2!

Problem 7. (8 points)

For the system in Problem 5 sketch the Bode plot. Then, based on the Bode plot, sketch the Nyquist plot (no need to calculate the intersection[s] with the positive real axis in this problem). Does your Nyquist plot confirm your result for problem 6?

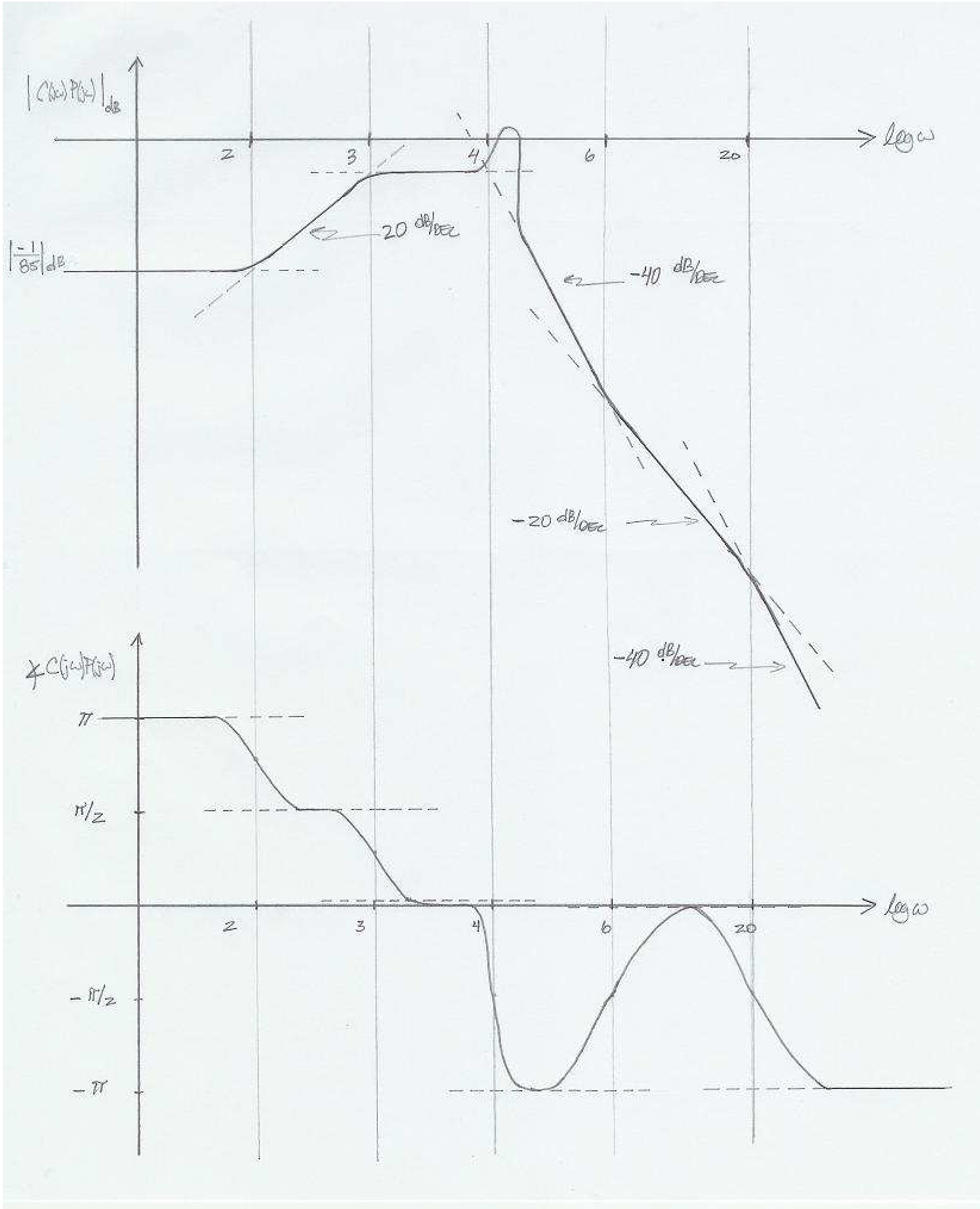
Solution: Rewriting $C(s)P(s)$ in *Bode form*

$$
\begin{aligned}
C(s)P(s) &= \frac{(s+6)(s-2)}{(s+20)(s+3)(s^2+2s+17)} \\
&= \frac{1}{85}\left(\frac{s}{2}-1\right)^1\left(\frac{s}{6}+1\right)^1\left(\frac{s}{3}+1\right)^{-1}\left[\left(\frac{s}{\sqrt{17}}\right)^2+\frac{2}{17}s+1\right]^{-1}\left(\frac{s}{20}+1\right)^{-1}
\end{aligned}
$$

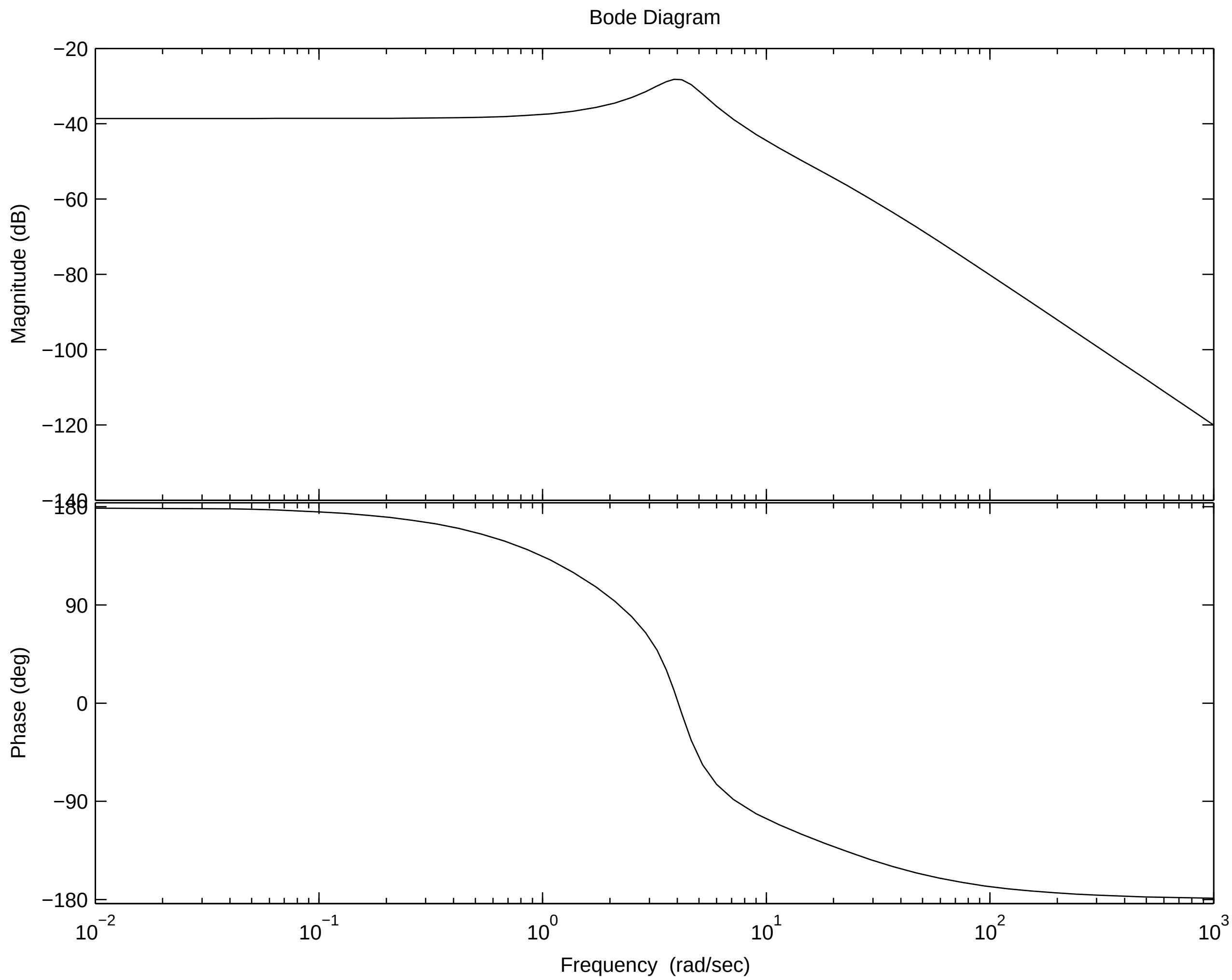
$$\text{DC Gain: } P(j0) = -\frac{1}{85}$$

$$
\begin{aligned}
0\frac{rad}{s} < \omega < 2\frac{rad}{s} &: \ | P(j0) |_{dB} \approx -38dB, \quad \angle P(j0) = \pi \\
2\frac{rad}{s} < \omega < 3\frac{rad}{s} &: \ m = 20\frac{dB}{dec}, \quad \triangle\phi = -\pi/2 \text{ (RHP zero)} \\
3\frac{rad}{s} < \omega < \sqrt{17}\frac{rad}{s} &: \ m = 0\frac{dB}{dec}, \quad \triangle\phi = -\pi/2 \\
\sqrt{17}\frac{rad}{s} < \omega < 6\frac{rad}{s} &: \ m = -40\frac{dB}{dec}, \quad \triangle\phi = -\pi \\
6\frac{rad}{s} < \omega < 20\frac{rad}{s} &: \ m = -20\frac{dB}{dec}, \quad \triangle\phi = \pi/2 \\
20\frac{rad}{s} < \omega < \infty\frac{rad}{s} &: \ m = -40\frac{dB}{dec}, \quad \triangle\phi = -\pi/2
\end{aligned}
$$

The Bode plot is:

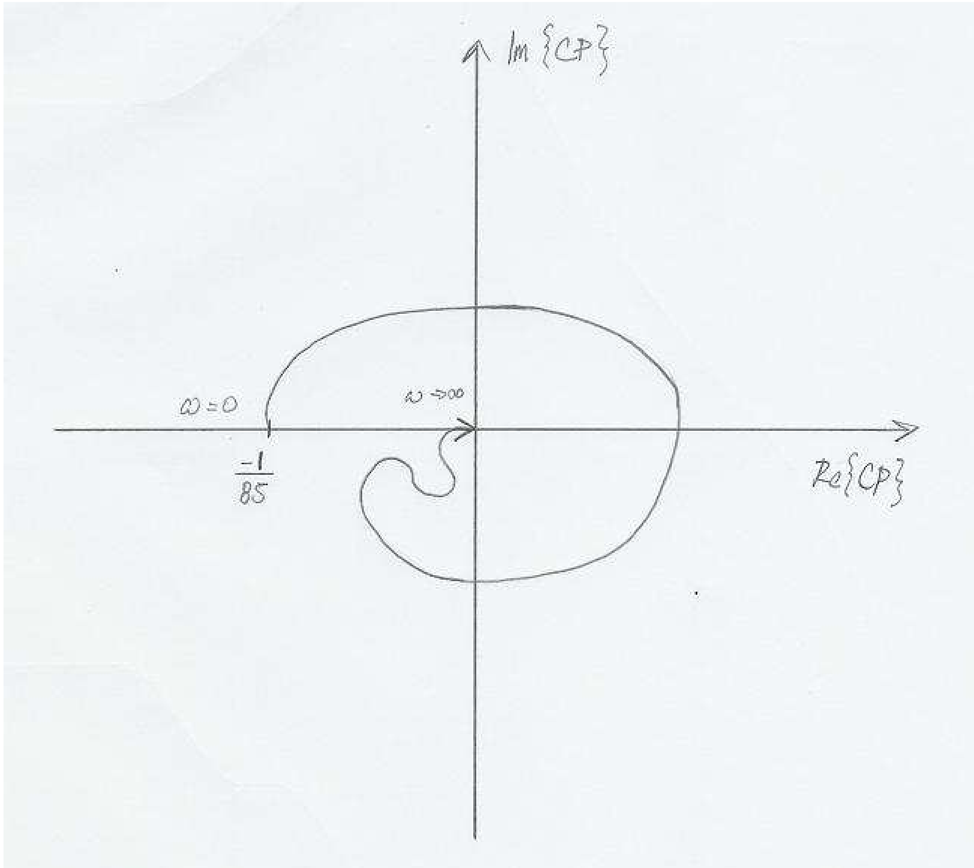


Sketch of Bode Plot. *Note: figure has been exaggerated, not drawn to scale.*



Bode done in MATLAB

From the Bode plot done by hand, one would expect a Nyquist plot like this:

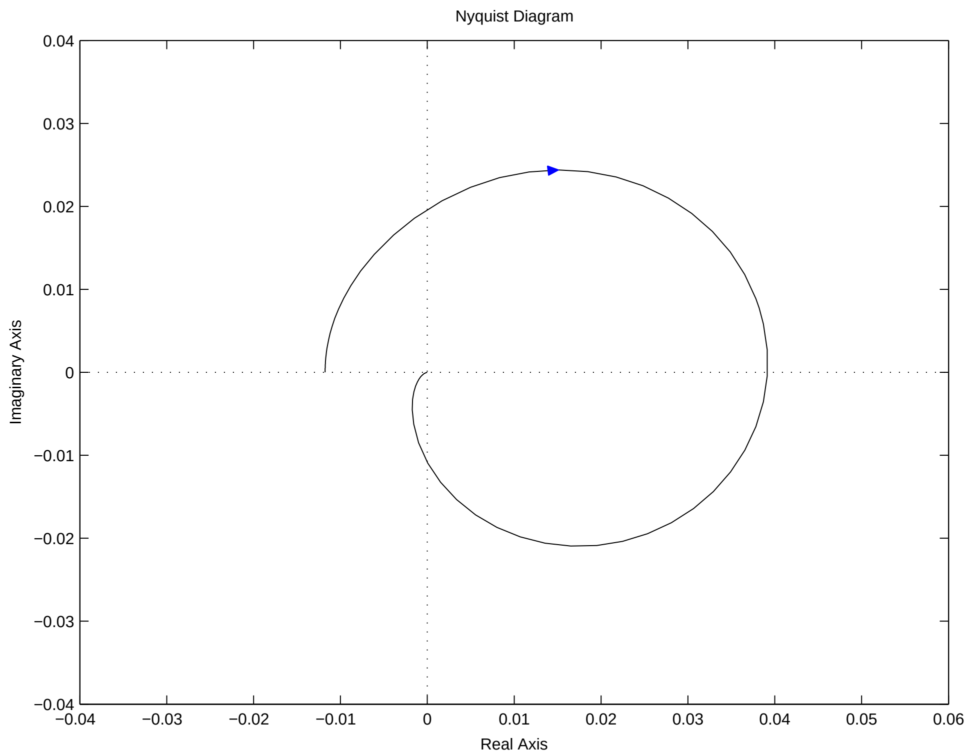


Sketch of Nyquist diagram

The “wiggle” at the end of the curve is due the the fact that the phase plot (Bode) has an interval of growth between $\omega = 6$ and $\omega = 20$.

Regarding stability, since the open-loop system is stable, we look for encirclements of the critical point $-1/K$. From the Bode phase plot is it clear that the phase does not cross the value $\pm\pi$ (except for $\omega = 0$). Thus, the Nyquist sketch is correct in showing no intersections with the negative real axis (other then $\omega = 0$). Hence, encirclement occurs only for $K > 85$. This confirms the result for Problem 6 that the system is stable for $K < 85$.

The actual Nyquist plot by MATLAB is:



Nyquist done in MATLAB

Note the absence of the “wiggle” because the actual phase Bode plot does not exhibit a growth in the [6,20] frequency range.