NAME: Solutions       PID: ____________________________

- One page (front and back) of your own handwritten notes.
- No graphing calculators.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are not ordered by difficulty.
- Total points: 60
- Time: 3 hours.
Problem 1: **Stability** (4 points)

Is the following polynomial stable? If not, how many eigenvalues are in the right-half plane?

\[ p(s) = s^5 + 2s^4 + 3s^3 + 4s^2 + 5s + 6 \]

\[
\begin{array}{c|cccc}
  s^5 & 1 & 3 & 5 \\
  s^4 & 2 & 4 & 6 \\
  s^3 & 1 & 2 & 0 \\
  s^2 & 2 & 6 & 0 \\
  s^1 & 2e-6 & 0 \\
  s^0 & 6 \\
\end{array}
\]

\[
\lim_{\varepsilon \to 0^+} \frac{2\varepsilon - 6}{\varepsilon} = -\infty
\]

\[ \downarrow \]

Unstable, 2 in RHP
Problem 2: Root Locus (12 points)

Sketch the root locus with respect to $K$ for the equation $1 + KG(s) = 0$ for the following:

(a) (4 points)

$$G(s) = \frac{(s + 4)(s^2 + 16)}{s(s^2 + 9)}$$

Describe the nature of the stability of the system for different values of gain $K$.

(b) (4 points)

$$G(s) = \frac{1}{(s + 2)(s^2 + 2s + 2)}$$

Be sure to mark where the Root Locus intersects the imaginary axis. When the Root Locus intersects the imaginary axis, what is the gain $K$?

(c) (4 points)

$$G(s) = \frac{(s + 4)(s + 2)^2}{s(s + 1)^2(s + 6)}$$

Set $K = 1$:

$$s(s^2 + 9) + (s + 4)(s^2 + 16)(1) = 0$$

$$2s^3 + 4s^2 + 25s + 64 = 0$$

Check for stability:

$$s^3 \quad 1 \quad 25/2$$

$$s^2 \quad 2 \quad 32$$

$$s \quad 25 - 32 \quad \text{unstable 2 poles in RHP for } K = 1$$

System is marginally stable for $K = 3$ or $K = 0$

System is unstable for $0 < K < \infty$
\[ \alpha^2 = \frac{-2 - 1 - 1}{3} = \frac{-4}{3} \]

Use Routh's to find \( K \):

\[
\begin{array}{c|ccc}
 s^3 & 4 & -4 + K \\
 s^2 & 1 & 6 \\
 s & 0 & 4 - 4 + K \\
 0 & 24 - 4 + K & 4 \\
\end{array}
\]

Marginally stable when \( K = 20 \)

\[ -jw^3 - 4w^2 + 6jw + 4 + K = 0 \]

\[ (-4w^2 + 4 + K) + (-w^3 + 6w)j = 0 \]

Set real part equal to zero with \( K = 20 \):

\[ -4w^2 + 4 + 20 = 0 \]

\[ w = \frac{-24}{-4} = 6 \]

\[ w = \pm \sqrt{6} \]
Problem 3: Bode Plots (12 points)

Sketch the Bode plots for the following open-loop transfer functions:

(a) (4 points)

\[ G(s) = \frac{.01(s - 10)(s + 100)}{(s + 1)^2} \]

(b) (4 points)

\[ G(s) = \frac{s(s + 100)}{s^2 + .1s + 1} \]

(c) (4 points)

\[ G(s) = \frac{10(s - 1)^2}{s^2(s^2 + 10s + 100)} \]
$G(j\omega) = 0 \quad ; \quad \angle G(j\omega) = 90^\circ$

$\left| G(\omega) \right|$

$\left< G(\omega) \right>$

- Stable complex pole
- Stable zero

$G(j\infty) = 1 = 0 \text{ dB}$
$
\angle G(j\infty) = 0^\circ$

\[
\lim_{\omega \to 0} G(j\omega) = j\omega 100
\]

\[
\left| G(j\omega) \right|_{\omega=10^2} \approx j10^2 100 = 1
\]

$\delta = \frac{\pi}{2}$

$1.1 = 20 \log \left| v \right|$

$= 0 \text{ dB}$
\( |G(j\omega)| = +\infty \quad \angle G(j\omega) = -180^\circ \pm 180^\circ \)

\( |G(j\infty)| = 0 \quad \angle G(j\infty) = -180^\circ \)

\( |G(j1)| = -14\, \text{dB} \)

\[
\lim_{\omega \to 0} G(j\omega) = \frac{10}{(j\omega)^2} = \frac{-1}{10\omega^2}
\]

\[
G(j\omega)|_{\omega=1} = \frac{-1}{10 \cdot 1^2} = -\frac{1}{10}
\]

\[
|1| = 20\log_{10}1 = 0\, \text{dB}
\]

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Graph:

- **\( |G(j\omega)| \)**:
  - Linear scale from -80 dB to 40 dB.
  - Log scale on the x-axis from 10 to 10^3.

- **\( \angle G(j\omega) \)**:
  - Linear scale from 180° to -180°.

- **Unstable double zero**:
  - At \( \omega = 1 \).

- **Stable complex pole**:
  - At \( \omega = 10 \).

- **-40 dB/dec**

Problem 4: Nyquist Plots (12 points)

Sketch the Nyquist plot for each of the transfer functions in problem 4. What does Nyquist's stability criterion tell you about each of the systems? (4 points each part)

(a) \[ G(s) = \frac{0.01(s - 10)(s + 100)}{(s + 1)^2} \]

\[ G(\sigma) = (0.01)(-10)(100) = -10 \]
\[ \angle G(j\sigma) = -180^\circ \]
\[ G(j\omega) = +0.01 \]
\[ \angle G(j\omega) = 0^\circ \]

Find crossings:

\[ G(j\omega) = \frac{0.01(j\omega - 10)(j\omega + 100)}{(j\omega + 1)^2} \]

\[ = 0.01 \times \frac{-\omega^2 - 1000 + 90\omega j}{-\omega^2 + 1 + 2\omega j} \times \frac{-\omega^2 + 1 - 2\omega j}{-\omega^2 + 1 - 2\omega j} \]

\[ = 0.01 \times \frac{\omega^4 + 1179\omega^2 - 1000 + (-88\omega^3 + 2090\omega)}{(-\omega^2 + 1)^2 + (2\omega)^2} \]

Real axis intersection: \( \text{Im}(G(j\omega)) = 0 \)
\[ \omega (88\omega^2 - 2090) = 0 \]
\[ \omega = \{0, 4.8734\} \]
\( G(j4.8734) = 0.45 \)

Imag axis intersection: \( \text{Re}(G(j\omega)) = 0 \)
\[ \omega^4 + 1179\omega^2 - 1000 = 0 \]
\[ \omega^2 = -1179 \pm \sqrt{(1179)^2 + 4(1000)} \]
\[ \omega = 9.206 \]
\( G(j0.9206) = 5.4358 \)
Nyquist says:
stable for \(0 < K < \frac{1}{10}\)
unstable for \(\frac{1}{10} < K < \infty\)

(b) \(G(s) = \frac{s(s+100)}{s^2 + 15s + 1}\)

\(G(0) = 0\), \(\angle G(j0) = 90^\circ\)
\(G(j\infty) = 1\), \(\angle G(j\infty) = 0^\circ\)

Find crossings:
\[G(j\omega) = \frac{-\omega^2 + 100j\omega}{-\omega^2 + 1 + j\omega} \times \frac{-\omega^2 - 1 + j\omega}{-\omega^2 - 1 - j\omega}\]

\[= \frac{\omega^4 + 9\omega^2 + (-99.9\omega^3 + 100\omega)j}{(-\omega^2 + 1)^2 + (1\omega)^2}\]

Real axis intersection:
\[\text{Im}(G(j\omega)) = 0\]
\[\omega(99.9\omega^2 - 100) = 0\]
\[\omega = \{0, 1.001\}\]

\(G(j1.001) = 999.8\)

Imag axis intersection:
\[\text{Re}(G(j\omega)) = 0\]
\[\omega^4 + 9\omega^2 = 0\]
\[\omega = 0\]

Nyquist says stable if \(K\)
\[ G(s) = \frac{10(s-1)^2}{s^2(s^2+100s+1000)} \]

\[ G(0) = \infty, \quad \angle G(0) = 180^\circ \]
\[ G(\infty) = 0, \quad \angle G(\infty) = -180^\circ \]

Find crossings:

\[
G(j\omega) = \frac{-10\omega^2 + 10 - 20\omega j}{\omega^4 - 100\omega^2 - 100\omega^3 j} \times \frac{\omega^4 - 100\omega^2 + 100\omega^3 j}{\omega^4 - 100\omega^2 + 100\omega^3 j}
\]

\[ = \left(\frac{-10\omega^6 + 1210\omega^4 - 1000\omega^2}{(\omega^4 - 100\omega^2)^2 + (10\omega^3)^2}\right) \times \left(\frac{-120\omega^5 + 2100\omega^3}{(\omega^4 - 100\omega^2)^2 + (10\omega^3)^2}\right) \]

Rel axis intersection: \( \text{Im}(G(j\omega)) = 0 \)

\[ \omega^3(120\omega^2 - 2100) = 0 \]

\[ \omega = 0, \quad 4.183 \]

\[ G(j4.18) = 0.1143 \]

Imaginary axis intersection: \( \text{Re}(G(j\omega)) = 0 \)

\[ \omega^2(10\omega^4 - 1210\omega^2 + 1000) = 0 \]

\[ \omega = 0, \quad 0.9122, \quad 10.963 \]

\[ G(j0.9122) = 0.2211 \]
\[ G(j10.96) = 0.0905 \]

\[ \text{unstable} \quad \Rightarrow \quad K \]

\[ \text{Unstable and } K \]

\[ \text{Unstable and } K \]

\[ \text{Unstable and } K \]

\[ \text{Unstable and } K \]
Problem 5: *Bringing it all together (20 points)*

We can learn a lot about the behavior of a system from its Bode plot. Suppose we perform an experiment on a dynamic system, and we obtain the following Bode plot:

![Bode Diagram]

(a) (4 points) Sketch the Nyquist plot.

(b) (2 points) Determine the gain and phase margin(s).

(c) (5 points) Determine the Ziegler-Nichols PID tuning parameters using the ultimate sensitivity method. (Hint: Find $K_u$ and $P_u$ from the Bode plot.) (Use a straight edge to approximate the values the best you can)

(d) (5 points) Determine the transfer function of the open-loop system. (Hint: Use $K_u$ and the corresponding $\omega$ to find $\zeta$.) (Use powers of 10 for your breakpoints)

(e) (4 points) Sketch the Root Locus to confirm that high gain leads to instability.
\( \frac{1}{k_u} \approx -7 \text{ dB} = 10^{-7/20} \) (from graph)

\( k_u = 2.24 \)

\( \omega_u \approx 1.8 \) (from graph)

\( P_u = \frac{2\pi}{\omega} = \frac{2\pi}{1.8} = 3.49 \)

\( k_p = 0.6 \cdot k_u = 1.34 \)

\( k_1 = \frac{1}{2} P_u = 1.745 \)

\( k_0 = \frac{1}{5} P_u = 0.4363 \)

\( k_p = 1.34, \ k_1 = 1.745, \ k_0 = 0.4363 \)
From the bode plot we know the transfer function will be in this form:

\[
\frac{(s - 10)^2}{(s^2 + 235s + 1)(s + 100)} = \frac{1}{Kw}
\]

Use \(Kw\) and \(w\) to determine \(Z\):

\[-\frac{w^2 - 20wj + 100}{(-w^2 + 2w3j + 1)(100 + wj)} = -\frac{1}{Kw}\]

\[
\frac{96.76 - 36j}{(-224 + 3.63j)(100 + 1.8j)} = -\frac{1}{2.24}
\]

\[
\frac{96.76 - 36j}{(-224 - 6.483) + (-4.032 + 36.03j)} \times \frac{(-224 - 6.483) - (-4.032 + 36.03j)}{(-224 - 6.483) - (-4.032 + 36.03j)} = -\frac{1}{2.24}
\]

Reimaginary part:

\[
(96.76)(4.032 - 36.03j) + 36(224 + 6.483) = 0
\]

\[
8454 = 34600j
\]

\[
Z = 0.244
\]

(actually \(Z = .25\), but approximations from Bode alter results)

\[
G(s) = \frac{(s - 10)^2}{(s^2 + .485s + 1)(s + 100)}
\]
Large gain leads to instability