MAE 143B
LINEAR CONTROL

Prof. M. Krstic

FINAL

June 13, 2007

NAME: Solutions PID: ____________________

- One page (front and back) of your own handwritten notes.
- No graphing calculators.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 60
- Time: 3 hours.

Problem 1: *Stability* (4 points)

Is the following polynomial stable? If not, how many eigenvalues are in the right-half plane?

$$p(s) = s^5 + 2s^4 + 3s^3 + 4s^2 + 5s + 6$$

$$\begin{array}{c|ccc}
s^5 & 1 & 3 & 5 \\
s^4 & 2 & 4 & 6 \\
\hline
s^3 & 1 & 2 & 0 \\
s^2 & \epsilon & 6 & 0 \\
s^1 & \frac{2\epsilon - 6}{\epsilon} & 0 & \\
s^0 & 6 & &
\end{array}$$

$$\lim_{\epsilon \to 0^+} \frac{2\epsilon - 6}{\epsilon} = -\infty$$

$$\Downarrow$$

Unstable, 2 in RHP

Problem 2: *Root Locus* (12 points)

Sketch the root locus with respect to K for the equation $1 + KG(s) = 0$ for the following:

(a) (4 points)

$$G(s) = \frac{(s+4)(s^2+16)}{s(s^2+9)}$$

Describe the nature of the stability of the system for different values of gain K.

(b) (4 points)

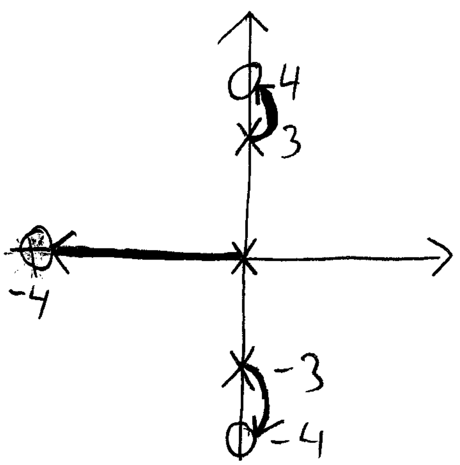
$$G(s) = \frac{1}{(s+2)(s^2+2s+2)}$$

Be sure to mark where the Root Locus intersects the imaginary axis. When the Root Locus intersects the imaginary axis, what is the gain K?

(c) (4 points)

$$G(s) = \frac{(s+4)(s+2)^2}{s(s+1)^2(s+6)}$$

a



Set $K=1$: $s(s^2+9) + (s+4)(s^2+16)(1) = 0$

$$2s^3 + 4s^2 + 25s + 64 = 0$$

Check for stability:

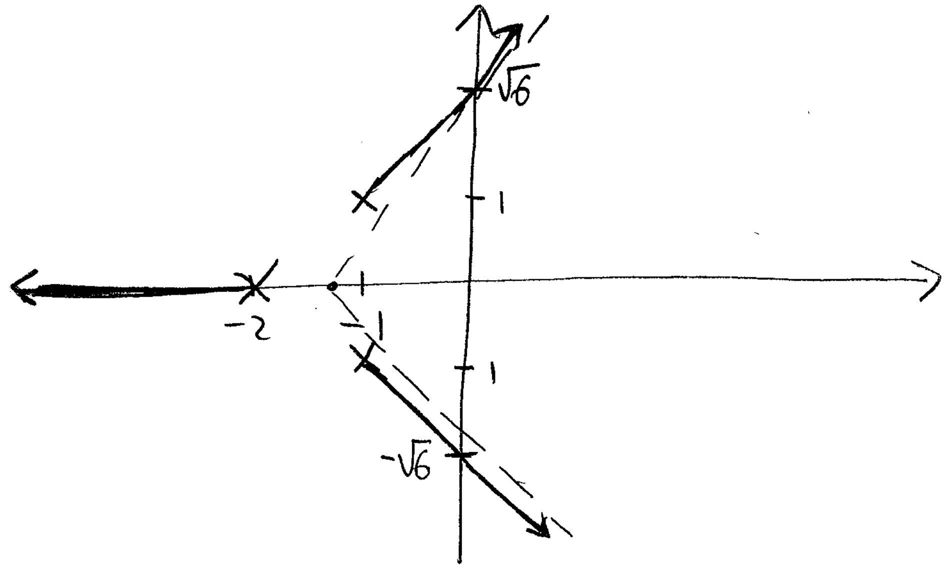
s^3	1	25/2
s^2	2	32
s^1	$\frac{25-32}{2}$	
s^0	32	

$\Rightarrow$ unstable 2 poles in RHP for $K=1$

System is marginally stable for $K=0$ or $K=\infty$

System is unstable for $0 < K < \infty$

(b)



$$\alpha = \frac{-2-1-1}{3} = -\frac{4}{3}$$

Use Routh's to find K:

$$(s+2)(s^2+2s+2)+K=0$$

$$s^3+4s^2+6s+4+K=0$$

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 4 & 4+K \\ \hline s^1 & \frac{24-4-K}{4} & \\ s^0 & 4+K & \end{array}$$

Marginally stable when $K=20$

$$-j\omega^3 - 4\omega^2 + 6j\omega + 4 + K = 0$$

$$(-4\omega^2+4+K)+(-\omega^3+6\omega)j=0$$

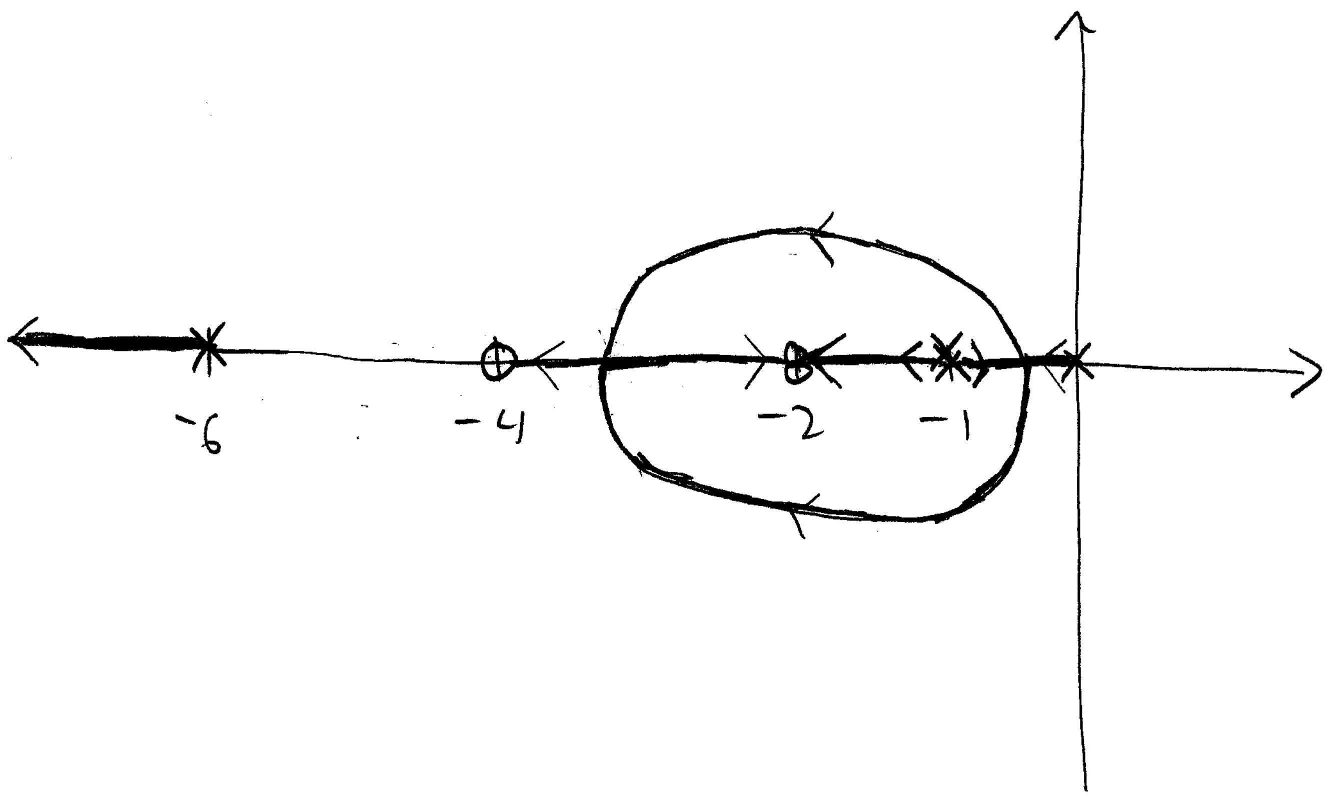
Set real part equal to zero with $K=20$:

$$-4\omega^2+4+20=0$$

$$\omega = \frac{-24}{-4}$$

$$\omega = \pm\sqrt{6}$$

(c)



Problem 3: *Bode Plots* (12 points)

Sketch the Bode plots for the following open-loop transfer functions:

(a) (4 points)

$$G(s) = \frac{.01(s-10)(s+100)}{(s+1)^2}$$

(b) (4 points)

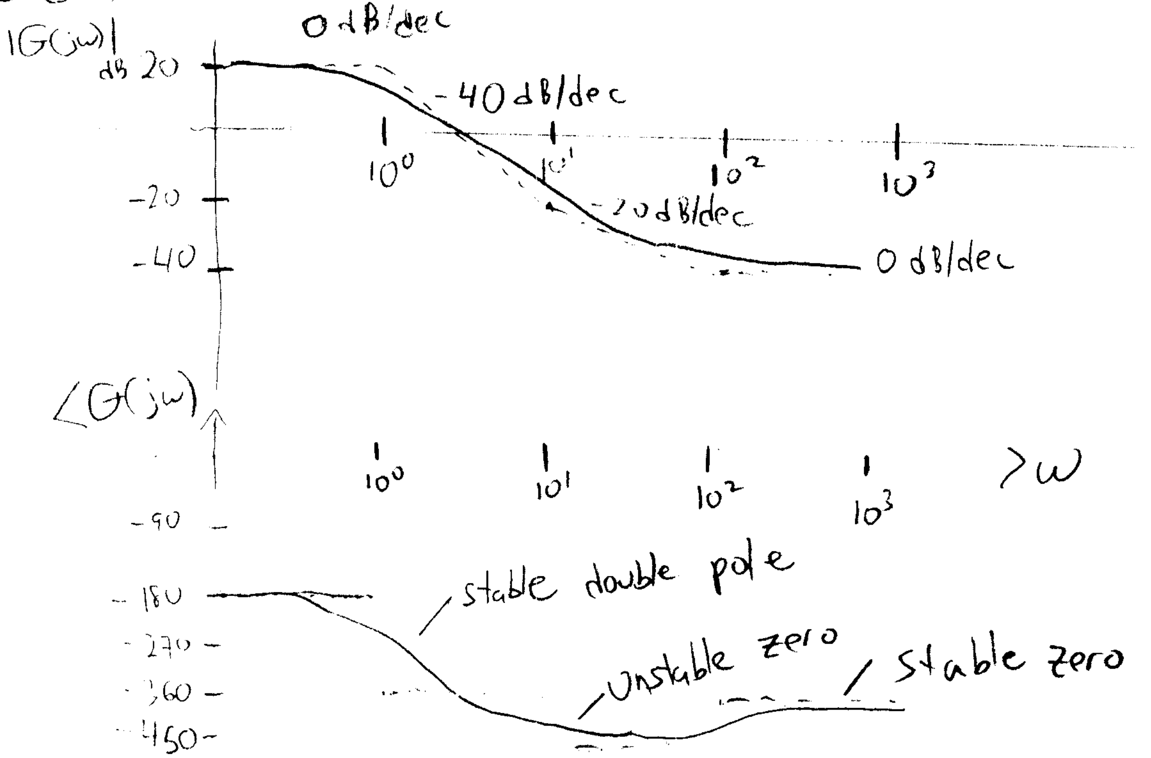
$$G(s) = \frac{s(s+100)}{s^2+.1s+1}$$

(c) (4 points)

$$G(s) = \frac{10(s-1)^2}{s^2(s^2+10s+100)}$$

a) $|G(j0)| = |.01(-10)(100)| = |-10| = 10 \Rightarrow 20\text{dB}$

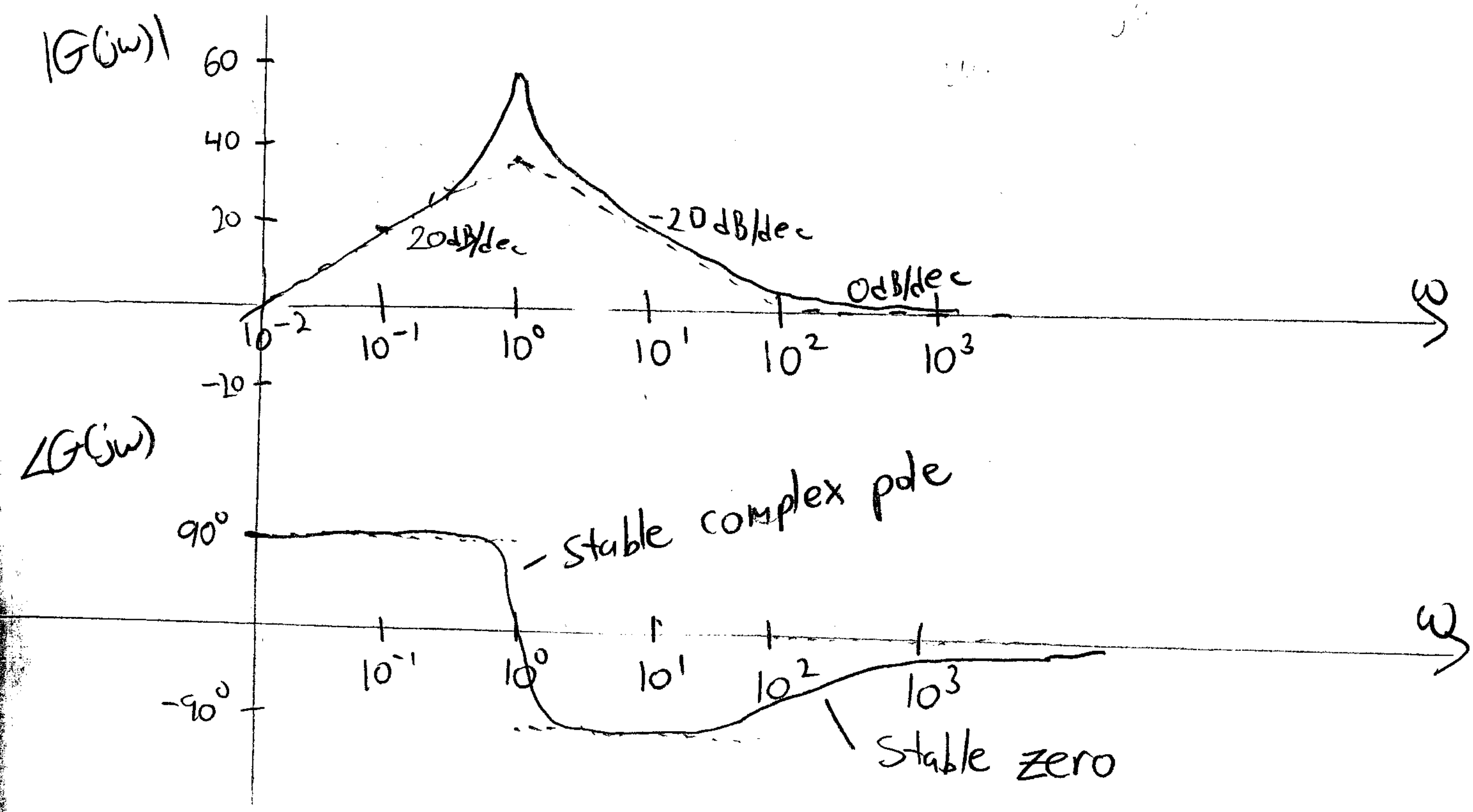
$\angle G(j0) = 180^\circ = -180^\circ = \pi = -\pi$



$|G(j\infty)| = |.01| = -40\text{ dB}$

$\angle G(j\infty) = 0^\circ = \pm 360^\circ = \pm 2\pi$

(b) $G(j0) = 0 \;;\; \angle G(j0) = 90^\circ$



$G(j\infty) = 1 = 0\,dB$

$\angle G(j\infty) = 0^\circ$

$$\lim_{\omega \to 0} G(j\omega) = j\omega 100 \qquad G(j\omega)\Big|_{\omega = 10^{-2}} \approx j10^{-2}100 = j$$

$$\measuredangle = \pi/2$$

$$|\cdot| = 20\log 1$$

$$= 0\,dB$$

(c) $|G(j0)| = +\infty$; $\angle G(j0) = -180^\circ \hat{=} +180^\circ$

$|G(j\infty)| = 0$; $\angle G(j\infty) = -180^\circ$

$|G(j1)| = -14\text{dB}$

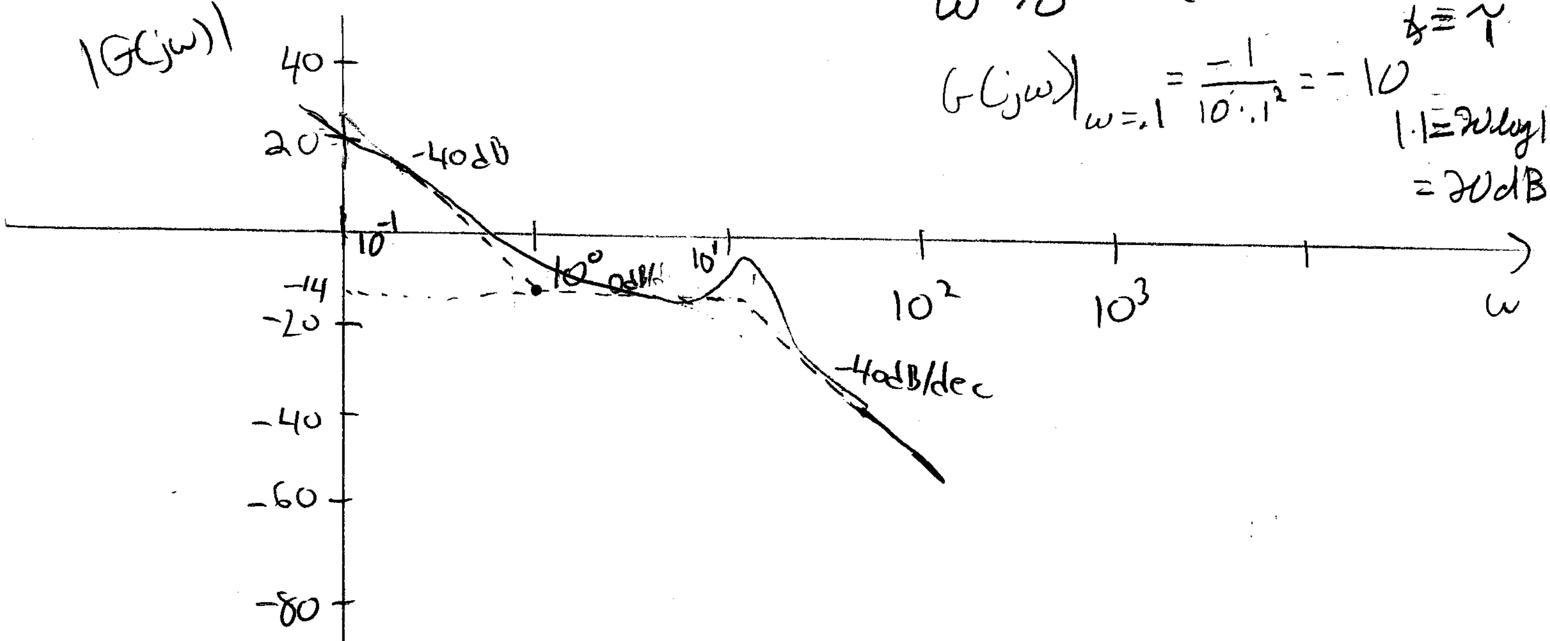
$\lim_{\omega \to 0} G(j\omega) = \frac{10}{(j\omega)^2 100} = \frac{-1}{10\omega^2}$

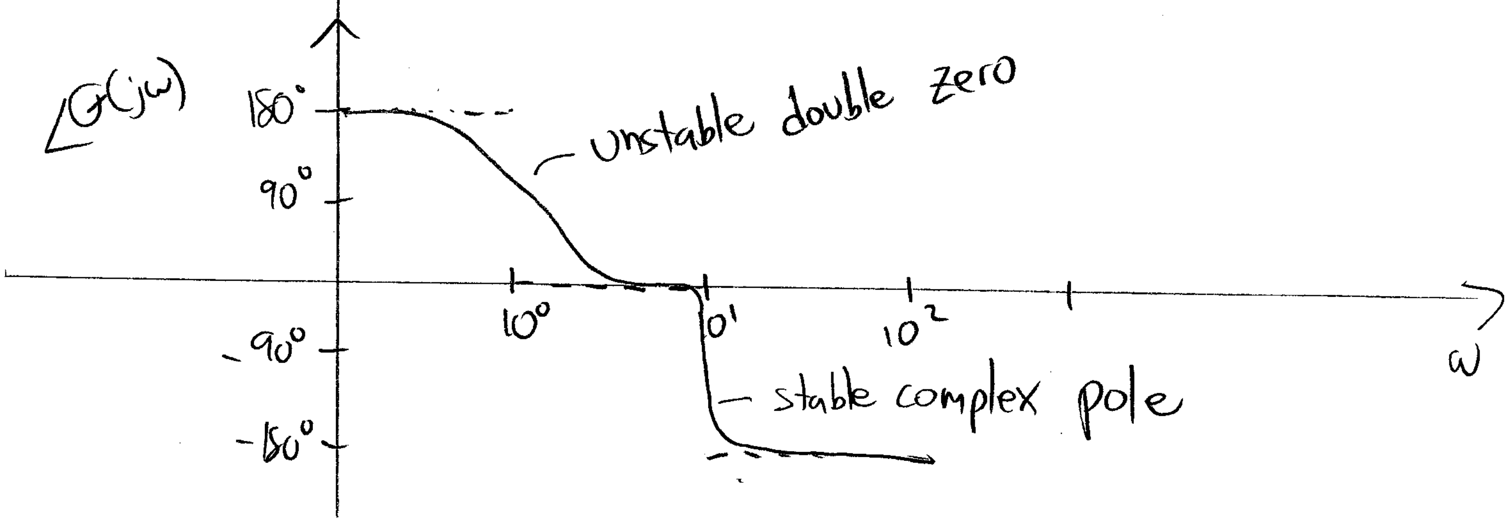
$\measuredangle = \pi$

$G(j\omega)\big|_{\omega=.1} = \frac{-1}{10 \cdot .1^2} = -10$

$|.1| = 20\log|$

$= 20\text{dB}$





Problem 4: *Nyquist Plots* (12 points)

Sketch the Nyquist plot for each of the transfer functions in problem 4. What does Nyquist's stability criterion tell you about each of the systems? (4 points each part)

(a) $G(s) = .01\dfrac{(s-10)(s+100)}{(s+1)^2}$

$G(j0) = (.01)(-10)(100) = -10$, $\angle G(j0) = -180°$

$G(j\infty) = +.01$, $\angle G(j\infty) = 0°$

Find crossings:

$$G(j\omega) = .01\frac{(j\omega-10)(j\omega+100)}{(j\omega+1)^2}$$

$$= .01 \times \frac{-\omega^2 - 1000 + 90\omega j}{-\omega^2+1+2\omega j} \times \frac{-\omega^2+1-2\omega j}{-\omega^2+1-2\omega j}$$

$$= .01 \times \frac{\omega^4 + 1179\omega^2 - 1000 + (-88\omega^3 + 2090\omega)j}{(-\omega^2+1)^2 + (2\omega)^2}$$

Real axis intersection: $\mathrm{Im}(G(j\omega)) = 0$

$$\omega(88\omega^2 - 2090) = 0$$

$$\omega = \{0, 4.8734\}$$

$G(j4.8734) = 0.45$

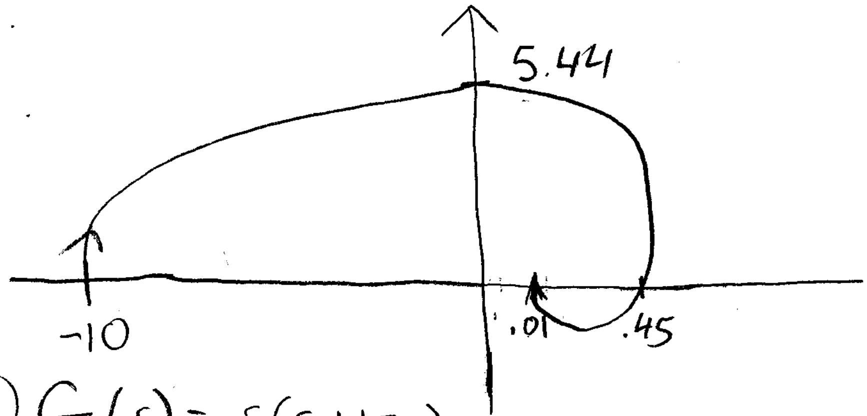
Imag axis intersection: $\mathrm{Re}(G(j\omega)) = 0$

$$\omega^4 + 1179\omega^2 - 1000$$

$$\omega^2 = \frac{-1179 \pm \sqrt{(1179)^2 + 4(1000)}}{2}$$

$\omega = .9206$

$G(j0.9206) = 5.4358$



Nyquist says:
stable for $0 < K < \frac{1}{10}$
unstable for $\frac{1}{10} < K < \infty$

(b) $G(s) = \dfrac{s(s+100)}{s^2 + .1s + 1}$

$G(j0) = 0$, $\angle G(j0) = 90°$

$G(j\infty) = 1$, $\angle G(j\infty) = 0°$

Find crossings:

$$G(j\omega) = \frac{-\omega^2 + 100\omega j}{-\omega^2 + 1 + .1\omega j} \times \frac{-\omega^2 + 1 - .1\omega j}{-\omega^2 + 1 - .1\omega j}$$

$$= \frac{\omega^4 + 9\omega^2 + (-99.9\omega^3 + 100\omega)j}{(-\omega^2+1)^2 + (.1\omega)^2}$$

Real axis intersection: $Im(G(j\omega)) = 0$

$\omega(99.9\omega^2 - 100) = 0$

$\omega = \{0, 1.001\}$

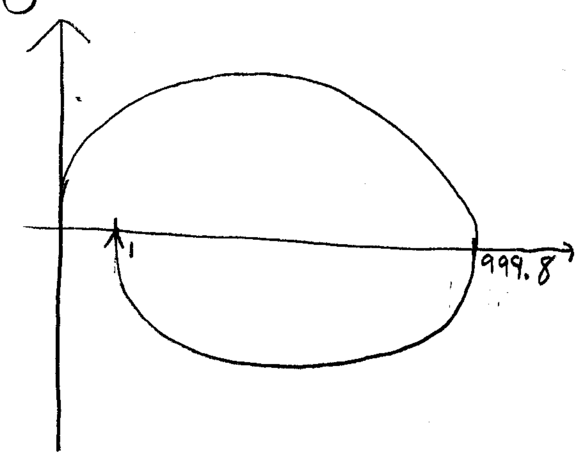
$G(j1.001) = 999.8$

Imag axis intersection: $Re(G(j\omega)) = 0$

$\omega^4 + 9\omega^2 = 0$

$\omega = 0$

Nyquist says
stable $\forall K$



(c) $G(s) = \dfrac{10(s-1)^2}{s^2(s^2+10s+100)}$

$G(j0) = \infty \quad \angle G(j0) = 180^\circ$

$G(j\infty) = 0, \quad \angle G(j\infty) = -180^\circ$

Find crossings:

$$G(j\omega) = \frac{-10\omega^2+10-20\omega j}{\omega^4-100\omega^2-10\omega^3 j} \times \frac{\omega^4-100\omega^2+10\omega^3 j}{\omega^4-100\omega^2+10\omega^3 j}$$

$$= \frac{(-10\omega^6+1210\omega^4-1000\omega^2)+(-120\omega^5+2100\omega^3)j}{(\omega^4-100\omega^2)^2+(10\omega^3)^2}$$

Real axis intersection: $Im(G(j\omega)) = 0$

$$\omega^3(120\omega^2-2100) = 0 \quad \omega = \{0, 4.18\}$$

$G(j4.18) = .1143$

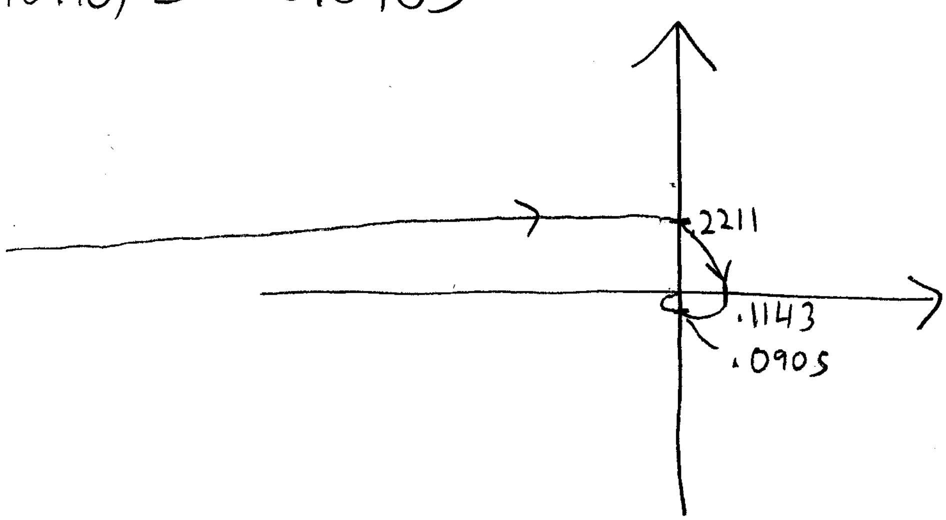
Imaginary axis intersection: $Re(G(j\omega)) = 0$

$$\omega^2(10\omega^4-1210\omega^2+1000) = 0$$

$$\omega = \{0, .9122, 10.96\}$$

$G(j.9122) = .2211$

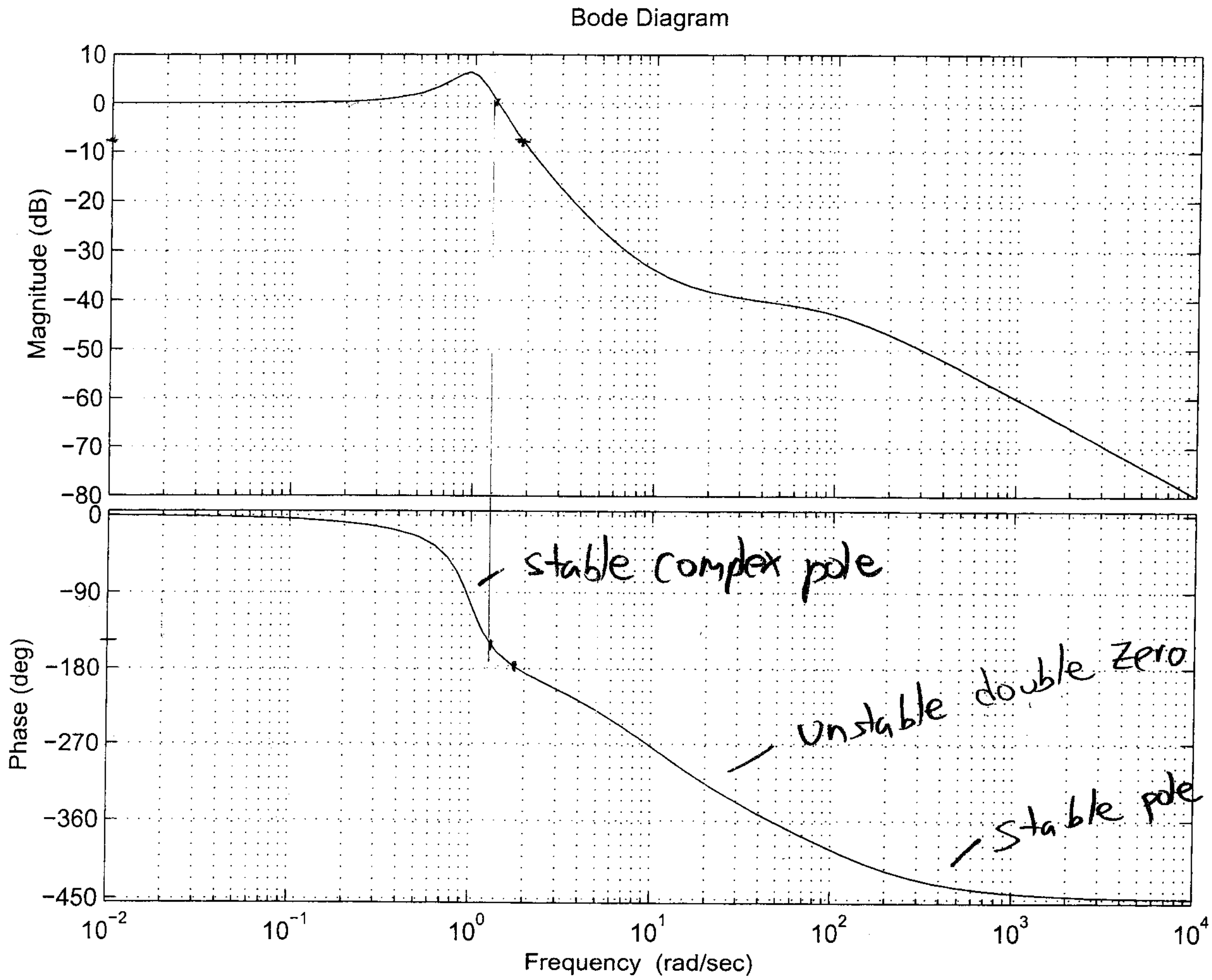
$G(j10.96) = -0.0905$



unstable $\forall K$

Problem 5: *Bringing it all together* (20 points)

We can learn a lot about the behavior of a system from its Bode plot. Suppose we perform an experiment on a dynamic system, and we obtain the following Bode plot:



(a) (4 points) Sketch the Nyquist plot.

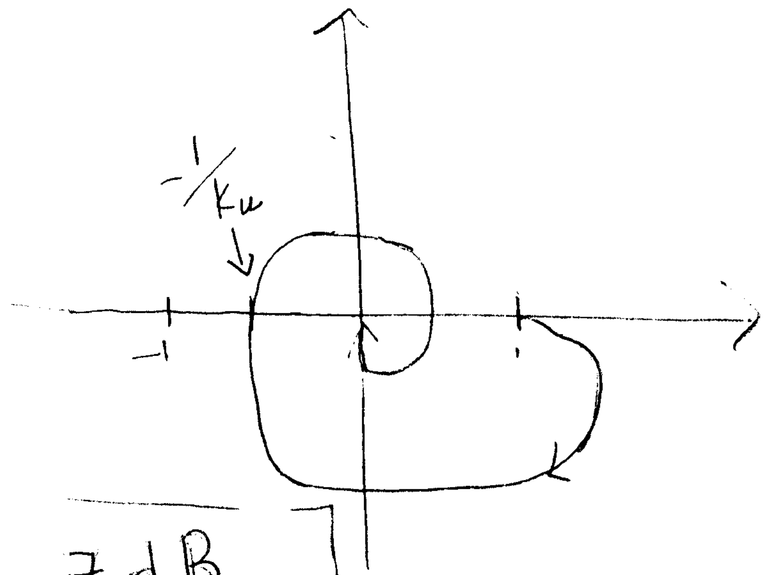
(b) (2 points) Determine the gain and phase margin(s).

(c) (5 points) Determine the Ziegler-Nichols PID tuning parameters using the ultimate sensitivity method. (Hint: Find K_u and P_u from the Bode plot.) (Use a straight edge to approximate the values the best you can)

(d) (5 points) Determine the transfer function of the open-loop system. (Hint: Use K_u and the cooresponding ω to find ζ.) (Use powers of 10 for your breakpoints)

(e) (4 points) Sketch the Root Locus to confirm that high gain leads to instability.

(a)



G.M.	7 dB	
P.M.	32°	$\rightarrow$ 0dB @ $\angle -148^\circ$ $\quad 148^\circ - 180^\circ = -32^\circ = 32^\circ$

$$\frac{1}{K_u} \approx -7 \text{ dB} = 10^{-7/20} \text{ (from graph)}$$

$$K_u = 2.24$$

$$\omega_u \approx 1.8 \text{ (from graph)}$$

$$P_u = \frac{2\pi}{\omega} = \frac{2\pi}{1.8} = 3.49$$

$$K_p = 0.6 K_u = 1.34$$

$$K_I = \frac{1}{2} P_u = 1.745$$

$$K_D = \frac{1}{8} P_u = .4363$$

$$\boxed{K_p = 1.34,\ K_I = 1.745,\ K_D = .4363}$$

(d) From the bode plot we know the transfer function will be in this form:

$$\left.\frac{(s-10)^2}{(s^2+2\zeta s+1)(s+100)}\right|_{s=j\omega} = -\frac{1}{K_u}$$

Use K_u and ω to determine ζ:

$$\frac{-\omega^2-20\omega j+100}{(-\omega^2+2\omega\zeta j+1)(100+\omega j)} = -\frac{1}{K_u}$$

$$\frac{96.76-36j}{(-2.24+3.6\zeta j)(100+1.8j)} = -\frac{1}{2.24}$$

$$\frac{96.76-36j}{(-224-6.48\zeta)+(-4.032+360\zeta)j} \times \frac{(-224-6.48\zeta)-(-4.032+360\zeta)j}{(-224-6.48\zeta)-(-4.032+360\zeta)j} = \frac{-1}{2.24}$$

Imaginary part:

$$(96.76)(4.032-360\zeta)+36(224+6.48\zeta)=0$$

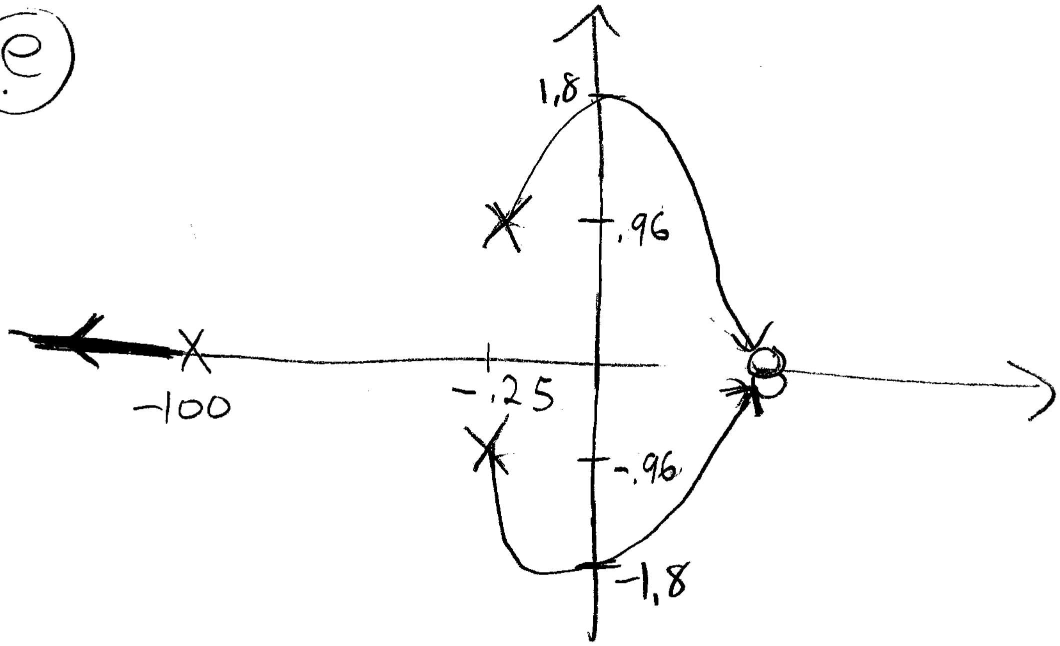
$$8454 = 34600\zeta$$

$$\boxed{\zeta = .244}$$

(actually $\zeta = .25$, but approximations from Bode alter results)

$$\boxed{G(s) = \frac{(s-10)^2}{(s^2+.48s+1)(s+100)}}$$

(e)



Large gain leads to instability