MAE 143B

LINEAR CONTROL — Prof. M. Krstic

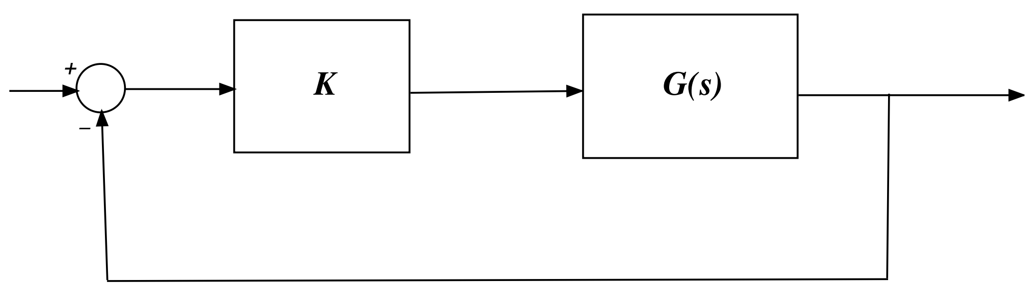
FINAL

September 7, 2007

- One page (front and back) of your own handwritten notes.
- No graphing calculators.
- Present your reasoning and calculations clearly. Inconsistent etchings will not be graded.
- Write answers only in the blue book.
- Total points: 65. Time: 2.5 hours.

Problem 1. (9 points)

Consider the feedback system



What type of feedback gains K would you use for these plants, large or small? why?

(a) (4 points) $G_1(s) = \dfrac{s^2 - 4s + 10}{s(s+2)(s+8)}$

(b) (5 points) $G_2(s) = \dfrac{s+15}{(s+6)(s+3)(s^2+s+3)}$

(Sketch the root locus for G_1 and G_2.)

Problem 2. (9 points)

Consider again the feedback system of Problem 1 (root locus continued).

Suppose that as a feedback designer you are allowed to use only either very large K or unity K. Which would you choose for each of the following two plants? (If you opt for unity K you need to justify your answer by Routh's criterion.)

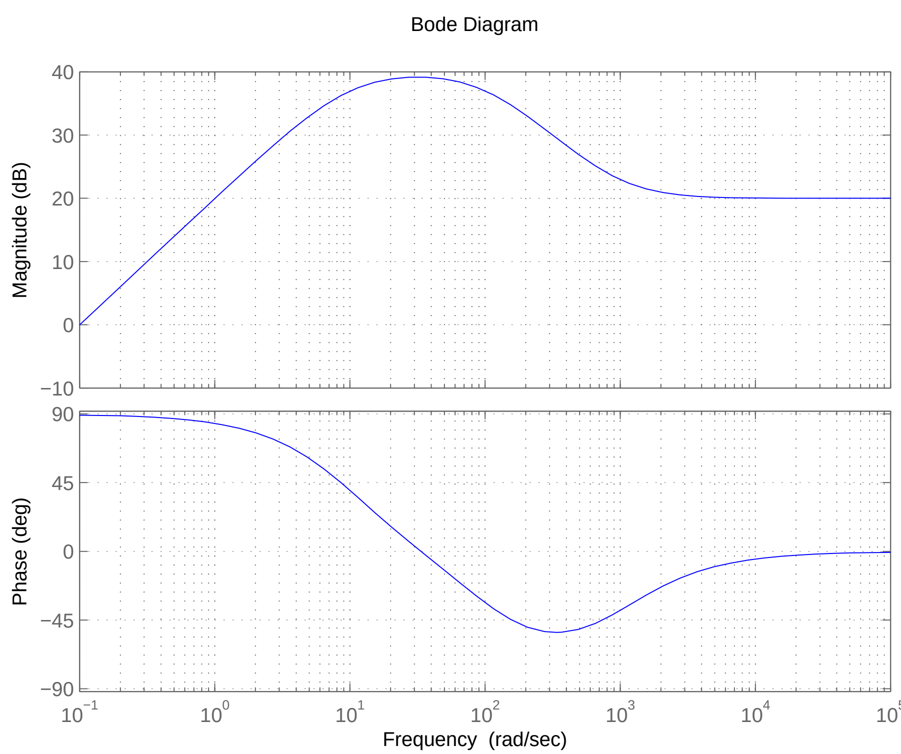
(a) (4 points) $G_3(s) = \dfrac{2(s+10)}{(s+5)(s^2+2s-3)}$

(b) (5 points) $G_4(s) = \dfrac{s^2+2s+4}{(s^2-16)(s+2)(s+6)}$

(Sketch the root locus for both G_3 and G_4.)

Problem 3. (9 points)

The Bode plots of a system $G(s)$ are



Justifying your answer, write an approximate expression for $G(s)$ (5 points). Then sketch a Nyquist diagram based on the Bode plot (4 points).

Problem 4. (9 points)

Sketch the Bode plots for the system

$$P(s) = \frac{s^2 + 3s + 9}{(s + 0.05)(s^2 + 2s + 100)}$$

Problem 5. (10 points)

Sketch the Nyquist diagram for the system from Problem 4.

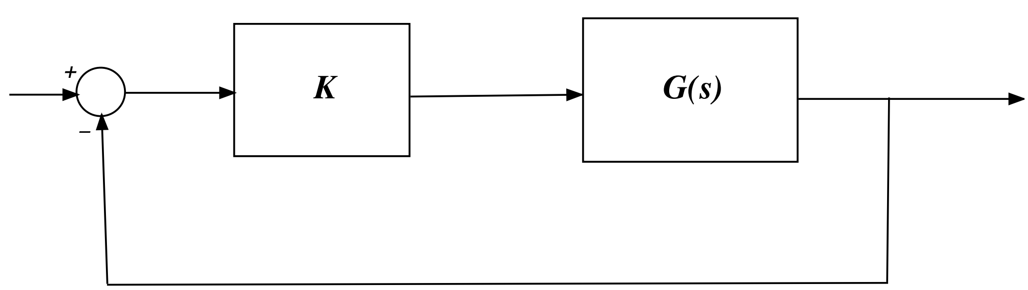
Problem 6. (9 points)

Sketch the Bode plots for the system

$$G(s) = \frac{(s + 20)^2}{(s - 5)^2(s + 50)}$$

Problem 7. (10 points)

Sketch the Nyquist diagram for the system from Problem 6. Then, consider the problem of stability for this system in feedback with a gain K, as in the figure. It turns out that, as K is varied from 0 to $+\infty$, stability character changes at some points. For which gain range is the loop stable? (Very large gain, very small gain, or a gain in an interval $[K_1, K_2]$ for some positive values K_1 and K_2?)

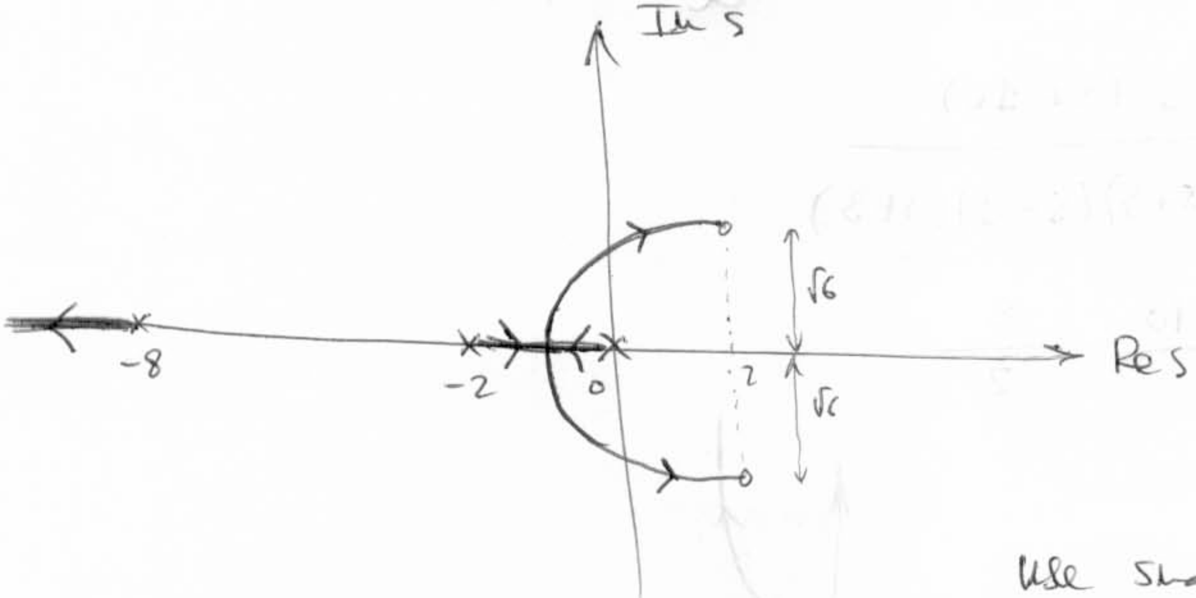


Problem 1

a) $$G_1(s) = \frac{s^2 - 4s + 10}{s(s+2)(s+8)}$$

factorize the numerator: $s^2 - 4s + 10 = 0 \Leftrightarrow s = \frac{4 \pm \sqrt{16-40}}{2} = 2 \pm j\sqrt{6}$

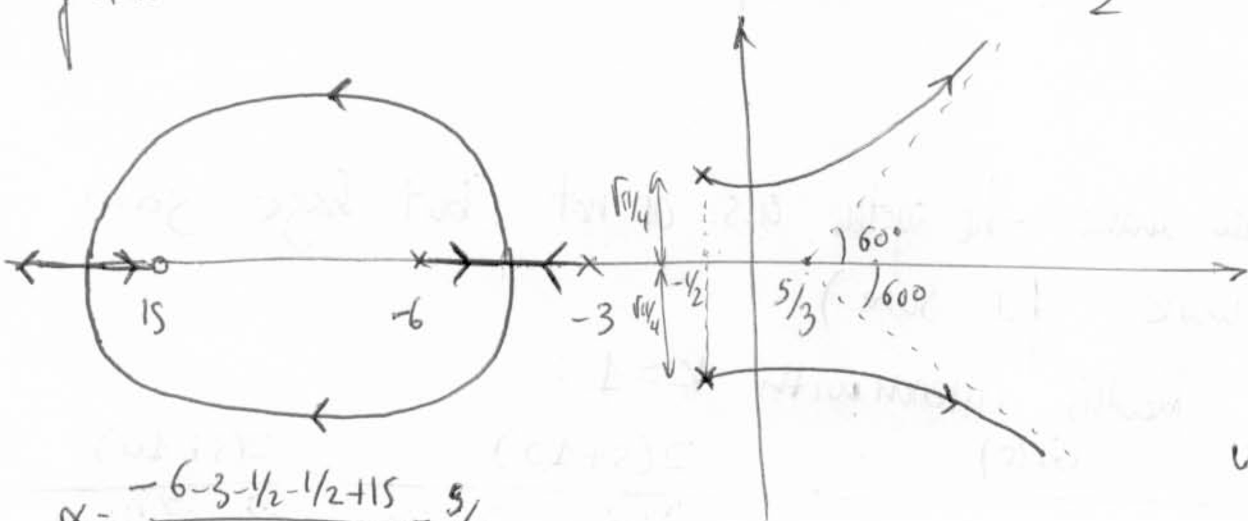
$$G_1(s) = \frac{(s-2-j\sqrt{6})(s-2+j\sqrt{6})}{s(s+2)(s+8)}$$



use small k!

b) $$G_2(s) = \frac{s+15}{(s+6)(s+3)(s^2+s+3)}$$

factorize the denominator: $s^2 + s + 3 = 0 \Leftrightarrow s = \frac{-1 \pm \sqrt{1-12}}{2} = -\frac{1}{2} \pm j\sqrt{11/4}$



use small k!

$$\alpha = \frac{-6-3-1/2-1/2+15}{3} = 5/3$$

Problem 2

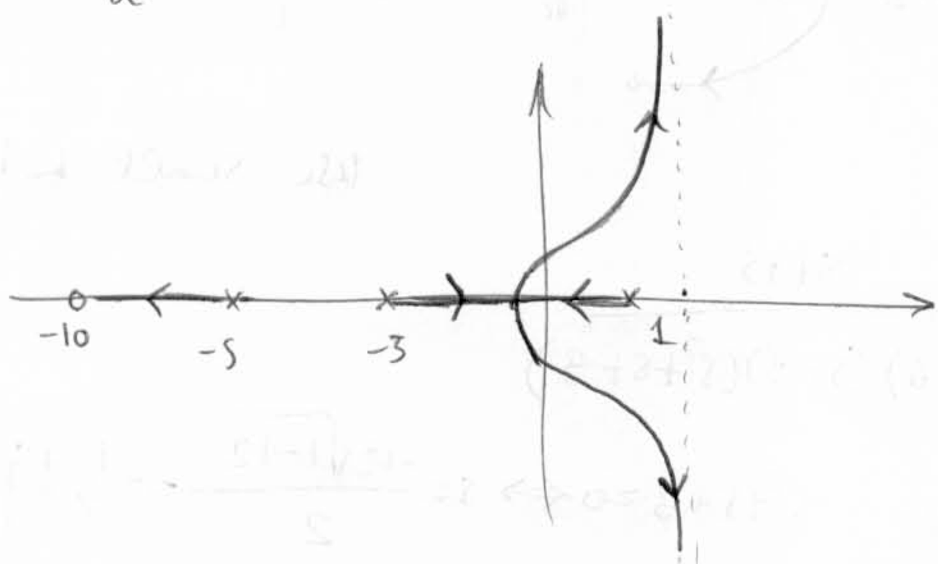
(a) $G_3(s) = \dfrac{2(s+10)}{(s+5)(s^2+2s-3)} =$

factorize the denominator:

$$s^2+2s-3=0 \Rightarrow s = \frac{-2\pm\sqrt{4+12}}{2} \begin{cases} 1 \\ -3 \end{cases}$$

$$G_3(s) = \frac{2(s+10)}{(s+5)(s-1)(s+3)}$$

$$\alpha = \frac{-5+1-3+10}{2} = \frac{3}{2}$$



unity fbk can make the system a.s. or not (but large gain does not work for sure)

Check with Routh's criterion with $K=1$:

$$T(s) = \frac{G_3(s)}{1+G_3(s)} = \frac{2(s+10)}{(s+5)(s^2+2s-3)+2(s+10)} = \frac{2(s+10)}{s^3+7s^2+9s+5}$$

s^3	1	9
s^2	7	5
s^1	$\frac{63-5}{7} > 0$	0
1	5	

$K=1$ works!

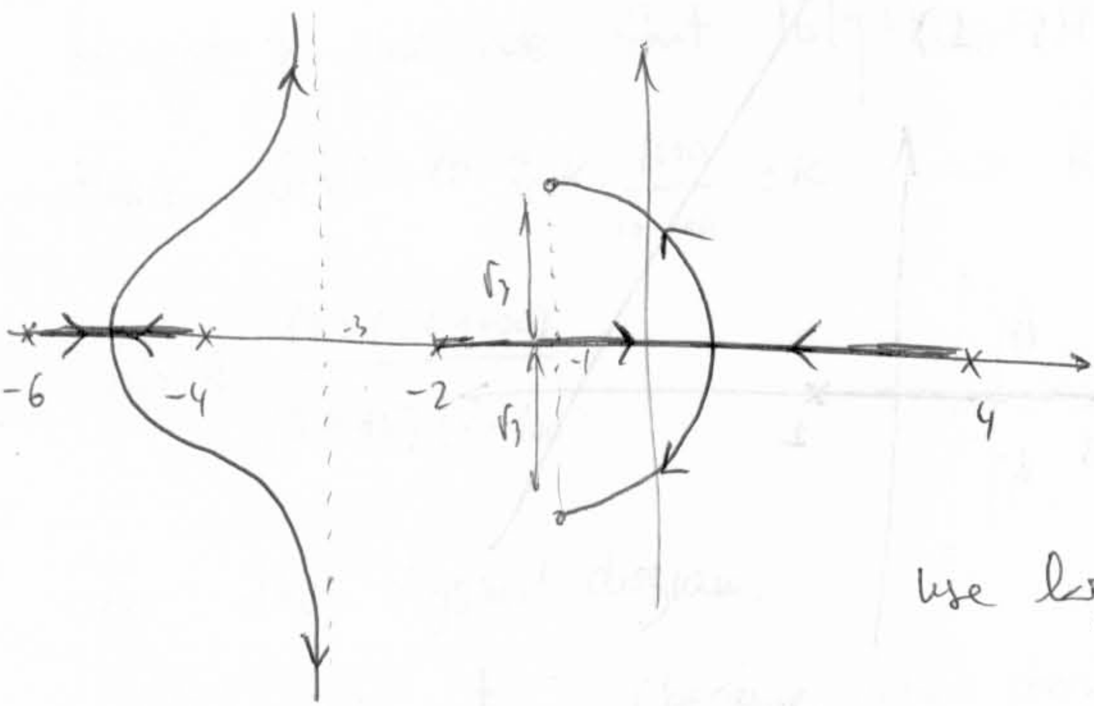
b)

$$G_4(s) = \frac{s^2+2s+4}{(s^2-16)(s+2)(s+6)}$$

$$s^2+2s+4=0 \Leftrightarrow s = \frac{-2\pm\sqrt{4-16}}{2} = -1 \pm j\sqrt{3}$$

$$G_4(s) = \frac{(s+1-j\sqrt{3})(s+1+j\sqrt{3})}{(s+4)(s-4)(s+2)(s+6)}$$

$$\alpha = \frac{4-4-2-6+1+1}{2} = -3$$



use large gain!

Problem 3

From the Magnitude plot:

- starts at a slope of +20dB/dec → zero at the origin
- at $\omega \approx 10^1$ it goes to 0 dB/dec → pole at $s = 10$
- at $\omega \simeq 10^2$ it goes to -20dB/dec → pole at $s = 100$
- at $\omega \approx 10^3$ it goes to +0 dB/dec → zero at $s = 1000$

The phase plot verifies the above zeros and poles, moreover they are stable.

Hence $G(s) = K\, s \dfrac{(s+1000)}{(s+10)(s+100)}$

to get K we note that $|G(j\cdot 1)|_{dB} = 20$ dB

Hence $G(j1) = 10 \approx K \cdot \dfrac{1000}{10 \cdot 100} = K \implies K \approx 10$

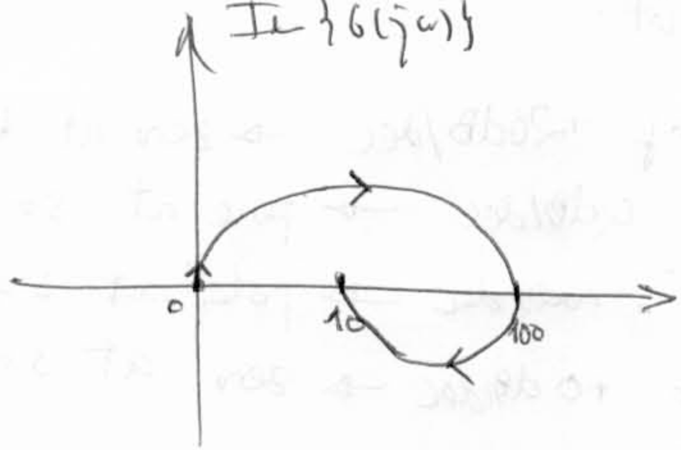
$$G(s) \approx \frac{10\, s\,(s+1000)}{(s+10)(s+100)}$$

For the Nyquist diagram:

$G(j0) = j\cdot 0$ ↑ (because initial phase is 90°)

$G(j\infty) = 20\text{dB} = 10$

There is one cross from the 1st quadrant into the second at $\omega \approx 30$ with $|G(j\cdot 30)|_{dB} \approx 40dB \rightarrow$ cross at ≈ 100 (not exactly)



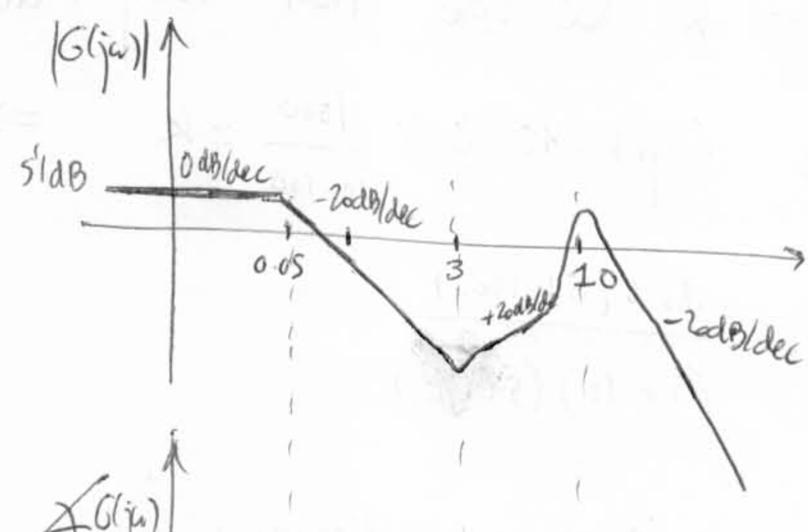
Problem 4.

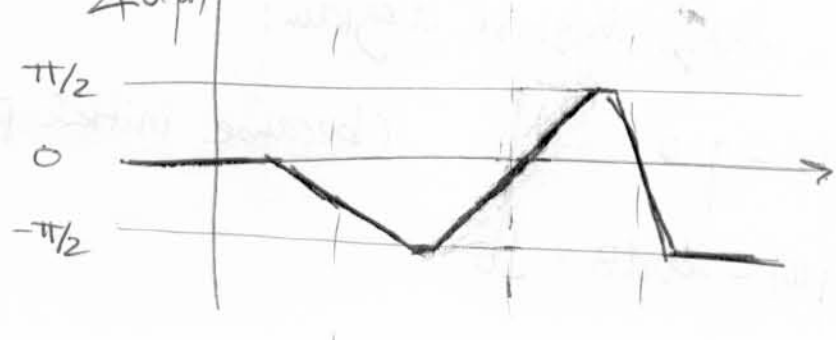
$$P(s) = \frac{s^2+3s+9}{(s+0.05)(s^2+2s+100)}$$

resonant zero: $\omega_n = 3$, $\zeta = 0.05$

resonant pole: $\omega_n = 10$, $\zeta = 0.1$

$|P(j\cdot 0)|_{dB} = |9/5|_{dB} = 5.1 dB$





Problem 5

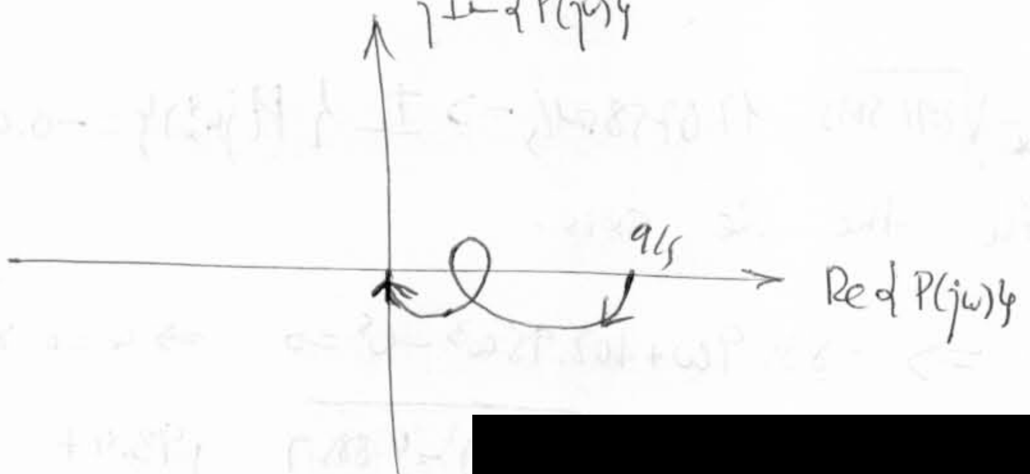
From the previous problem:

$$P(j\cdot 0) = 9/5$$

$$P(j\cdot \infty) = -j\cdot 0$$

There are two intersections when crossing from the 2nd quadrant into the 1st and back.

Hence:



[redacted]

One can find the crossing points:

$$P(j\omega) = \frac{9-\omega^2+3j\omega}{(j\omega+0.05)(100-\omega^2+2j\omega)}$$

Call $\Delta = (0.05^2-\omega^2)\left((100-\omega^2)^2+4\omega^2\right)$

then $$P(j\omega) = \frac{\left((9-\omega^2)+3j\omega\right)(-j\omega+0.05)(100-\omega^2-2j\omega)}{\Delta}$$

$$= \frac{\left[\left(0.05(9-\omega^2)+3\omega^2\right)+j\left(3\omega\cdot 0.05-\omega(9-\omega^2)\right)\right]\left((100-\omega^2)-2j\omega\right)}{\Delta}$$

Finally

$$P(j\omega) = \frac{45+276.85\omega^2-0.95\omega^4}{\Delta} + j\,\frac{-885.9\omega+102.95\omega^3-\omega^5}{\Delta}$$

Crosses with the Im axis:

$$Re\{P(j\omega)\}=0 \Rightarrow 45+276.85\omega^2-0.95\omega^4=0$$

$$\Rightarrow \omega^2 = \frac{-276.85 \pm \sqrt{(276.85)^2+0.95\cdot 4\cdot 45}}{-2\cdot 0.95}$$

$$= \begin{cases} 291.5835 \\ -0.1625 \quad \times \end{cases}$$

Crosses at $\omega_* = \sqrt{291.5835} = 17.0758\ rd/s \Rightarrow Im\{P(j\omega_*)\} = -0.0864$

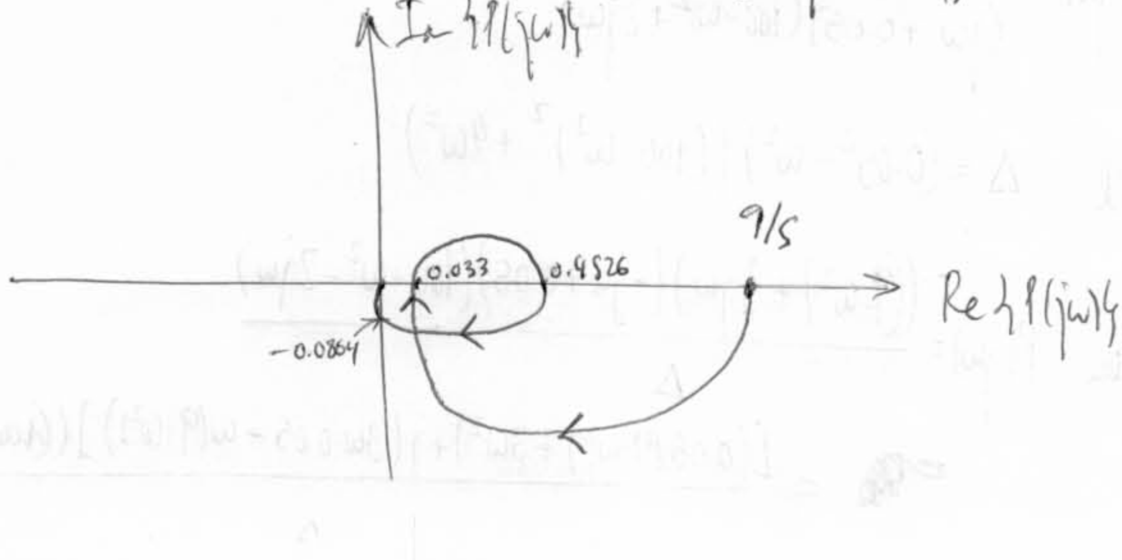
Crosses with the Re axis:

$$Im\{P(j\omega)\}=0 \Rightarrow -885.9\omega+102.95\omega^3-\omega^5=0 \quad \Rightarrow \omega=0 \quad \times$$

$$\omega^2 = \frac{-102.95 \pm \sqrt{(102.95)^2-4\cdot 885.9}}{-2} = \begin{cases} 93.47 \\ 9.47 \end{cases}$$

Crosses at $\omega_* = \begin{cases} \sqrt{93.47} = 9.6681\ rd/s \\ \sqrt{9.47} = 3.0786\ rd/s \end{cases} \Rightarrow \begin{cases} Re\{P(j\cdot 9.6681)\} = 0.4526 \\ Re\{P(j\cdot 3.0786)\} = 0.033 \end{cases}$

more exact:

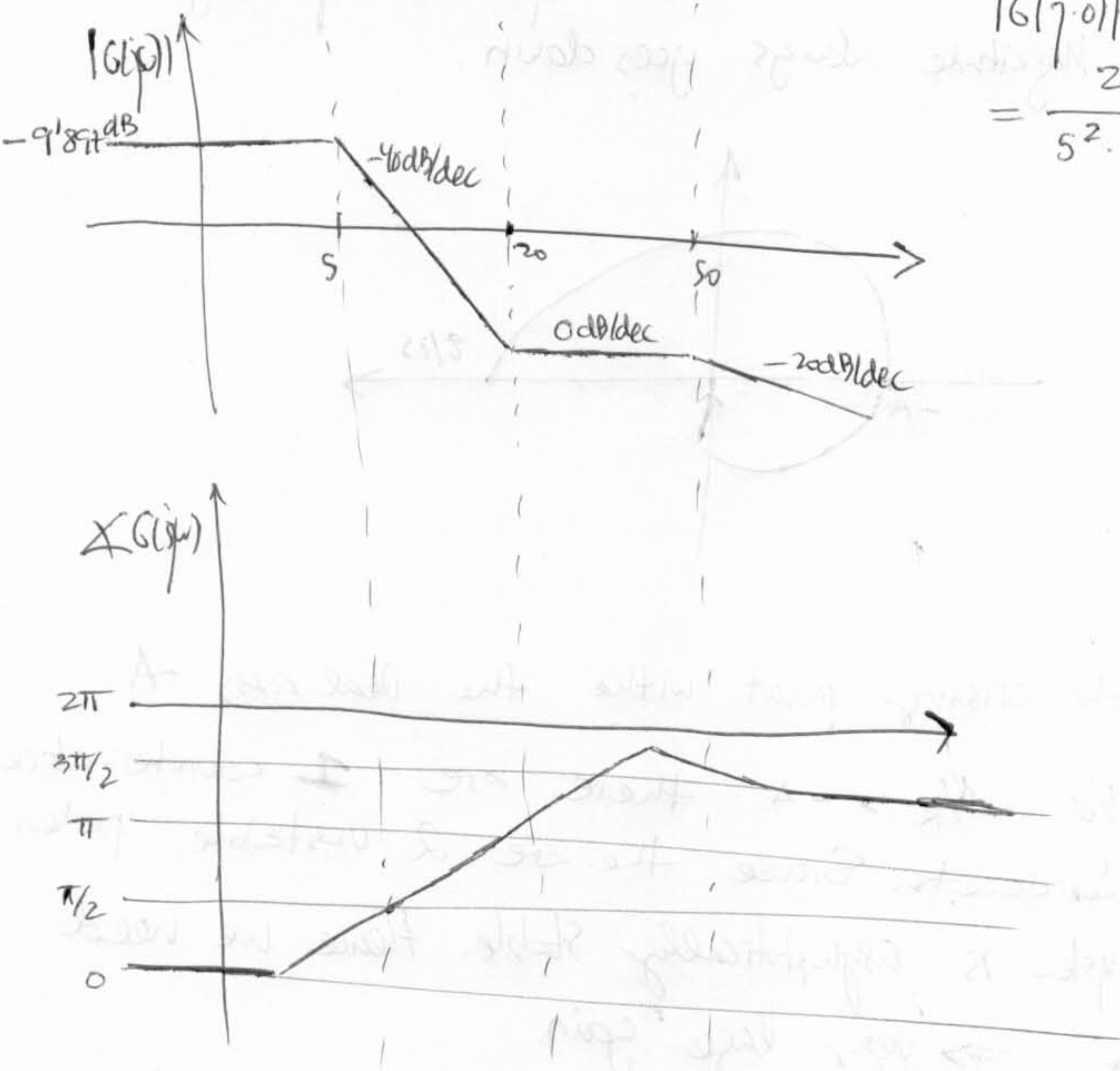


Problem 6

$$G(s) = \frac{(s+20)^2}{(s-5)^2(s+50)}$$

note there are unstable poles! they raise the phase instead of lowering it.

$$|G(j\cdot 0)| = \frac{20^2}{5^2 \cdot 50} = \frac{8}{25} = -9.897\,dB$$



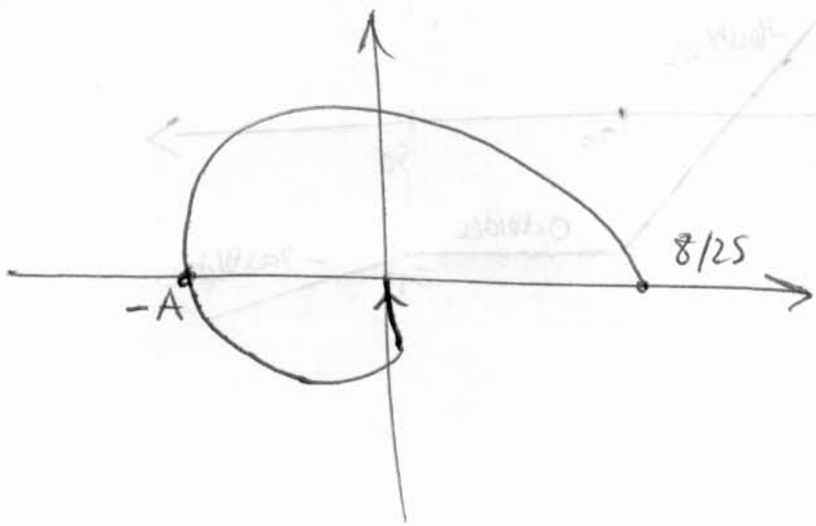
Problem 7

From the Bode:

$G(j\cdot 0) = 8/25$

$G(j\cdot\infty) = -j\cdot 0$

Goes from the 1st to the 4th quadrant finishing at the $-j$ axis. Magnitude always goes down.



Call the crossing point withe the Real axis $-A$.

- Then for $-1/K > -A$ there are 1 counter-clock-wise encirclements. Since there are 2 unstable poles $\Rightarrow$ system is asymptotically stable. Hence we need $K > A$ $\Rightarrow$ very large gain
- if $-1/K < -A$ $(K < A)$ no encirclements $\Rightarrow$ system is unstable.

[redacted]

We can solve problem 7 using the transfer function.

$$G(j\omega) = \frac{(j\omega+20)^2}{(j\omega-5)^2(j\omega+50)}$$

Call $\Delta = (\omega^2+25)^2(\omega^2+50^2)$

Then $$G(j\omega) = \frac{(j\omega+20)^2(-j\omega-5)^2(-j\omega+50)}{\Delta}$$

$$= \frac{500.000 - 36250\omega^2}{\Delta} + j\,\frac{240.000\omega - 1675\omega^3 - \omega^5}{\Delta}$$

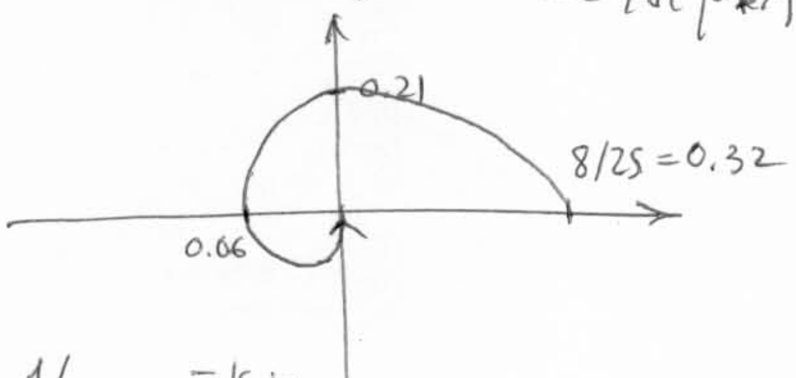
$Re\{G(j\omega_*)\} = 0 \Rightarrow \omega_* = \sqrt{\frac{500.000}{3625}} = 3.7139 \text{ rad/s} \Rightarrow Im\{G(j\omega_*)\} = 0.2127$

$Im\{G(j\omega_*)\} = 0 \Rightarrow \omega = 0$ ✗

or $\omega^4 + 1675\omega^2 - 240.000 = 0$

$$\Rightarrow \omega^2 = \frac{-1675 \pm \sqrt{(1675)^2 + 4\cdot 240.000}}{2} = \begin{cases} -1807.8 \text{ ✗} \\ 132.76 \end{cases}$$

$\omega_* = \sqrt{132.76} = 11.522 \text{ rad/s} \Rightarrow Re\{G(j\omega_*)\} = -0.0658$



Nyquist criterion: $\underline{K > 1/0.0658 = 15.19}$