MAE 143B Linear Control | Prof. M. Krstic

FINAL EXAM | June 12, 2008

Closed book, one sheet of notes allowed. Duration: 1 hr 20 mins. Total points: 30.

Problem 1. (7 points) Plot the Nyquist diagram of the system $G(s) = \dfrac{1}{s(s+2)^2}$ and determine *exactly* the gain margin, GM (not in decibels but in regular, `linear' units).

Problem 2. (9 points) Plot the Nyquist diagram of the system $G(s) = \dfrac{10}{s^2+s+9}$ and determine *exactly* the phase margin, PM. If your (non-graphing) calculator does not have the arc tangent function, present the result in the form PM = arctan(*number*).

Problem 3. (14 points) Plot the Nyquist diagram of the system $G(s) = \dfrac{(s^2+1)(s+5)}{s(s+2)(s^2-3s+4)}$ and, using the Nyquist stability criterion, determine *exactly* the interval (or intervals) for the gain K such that the negative feedback loop of $G(s)$ and K is asymptotically stable.

(In all three problems, if you are not able to complete the calculations needed to obtain the GM/PM and to apply the Nyquist stability criterion *exactly*, sketch the Bode plots and use them to sketch the Nyquist plots to receive some modest partial credit – provided your plots are correct.)

Problem 1

$$G(j\omega) = \frac{1}{j\omega(j\omega+2)^2}$$

A similar problem was considered in class

$$G(j\omega) = \frac{1}{j\omega(j\omega+2)^2}\frac{(2-j\omega)^2}{(2-j\omega)^2} = -\frac{j(2-j\omega)^2}{\omega(\omega^2+4)^2}$$

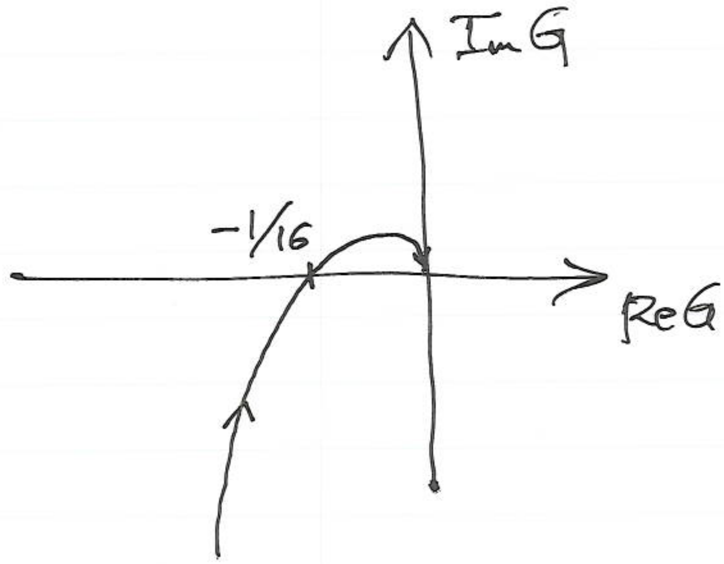
$$= -\frac{j(4-\omega^2-j4\omega)}{\omega(\omega^2+4)^2}$$

$$= \frac{-4\omega+j(\omega^2-4)}{\omega(\omega^2+4)^2}$$

$$\operatorname{Re} G(j\omega) = -\frac{4}{(\omega^2+4)^2}, \quad \operatorname{Im} G(j\omega) = \frac{\omega^2-4}{\omega(\omega^2+4)^2}$$

$\operatorname{Im} G(j\omega^*) = 0$ for $\omega^* = 2$

$\operatorname{Re} G(j\omega^*) = -\frac{1}{16}$

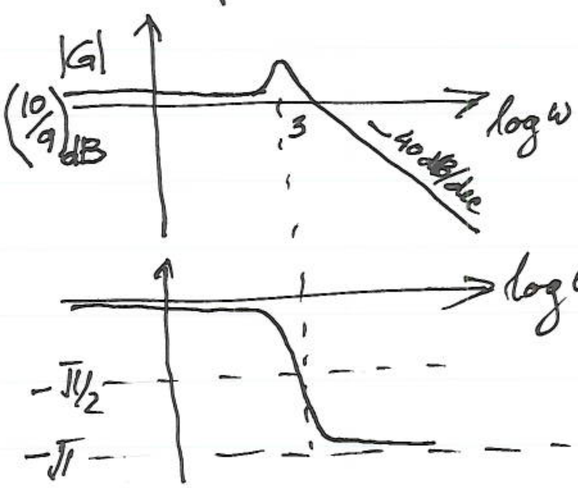


$\underline{GM = 16}$

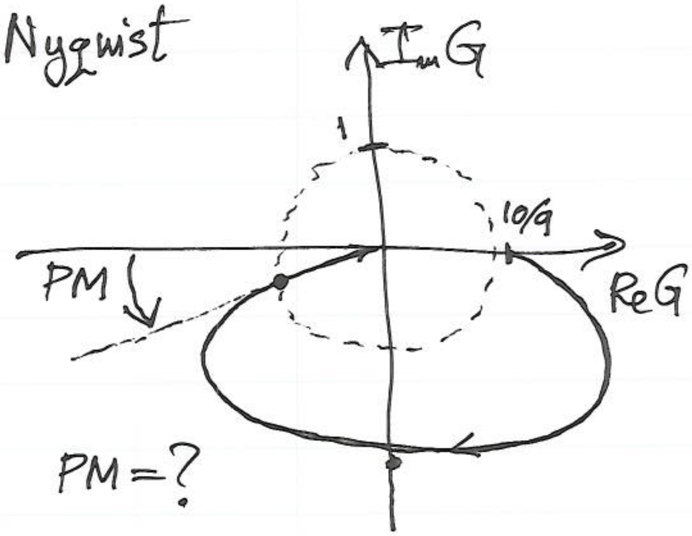
Problem 2

$$G(j\omega) = \frac{10}{(j\omega)^2 + j\omega + 9}$$

Bode plots



Nyquist



$$|G(j\omega)|^2 = \frac{100}{(9-\omega^2)^2 + \omega^2} = 1 \longrightarrow \omega^* = ?$$

$$(9-\omega^2)^2 + \omega^2 = 100$$

$$\omega^4 - 17\omega^2 - 19 = 0$$

$$(\omega^2)_{1,2} = \frac{17 \pm \sqrt{17^2 + 4 \cdot 19}}{2} = \frac{17 \pm 19.1}{2}$$

$$(\omega^2)_1 = 18.05, \quad (\omega^2)_2 = -1.05 \quad \text{discard}$$

So, $\underline{\omega^* = \sqrt{18.05} = 4.25}$

$$G(j\omega) = \frac{10}{9-\omega^2+j\omega} = \frac{10}{9-\omega^2+j\omega}\,\frac{9-\omega^2-j\omega}{9-\omega^2-j\omega}$$

$$= \frac{10}{(9-\omega^2)^2+\omega^2}\left(9-\omega^2-j\omega\right)$$

Since $\dfrac{100}{(9-\omega^2)^2+\omega^2} = 1$, we get

$$G(j\omega) = \frac{1}{10}\left(9-\omega^2-j\omega\right)$$

Then $G(j\omega^*) = -\left(0.905 + j0.425\right)$

$$\measuredangle G(j\omega^*) = -\pi + \arctan\frac{0.425}{0.905}$$

$$= -\pi + \arctan 0.47$$

$$= -\pi + \underbrace{0.44}_{\text{PM in radians}}$$

So

$$\underline{PM = 0.44 \text{ rad} = 25^\circ}$$

Problem 3 $\quad G(s) = \dfrac{(s^2+1)(s+5)}{s(s+2)(s^2-3s+4)}$

$$G(j\omega) = \frac{(1-\omega^2)(5+j\omega)}{j\omega(2+j\omega)(4-\omega^2-j3\omega)}$$

$$= \frac{(1-\omega^2)(\omega-j5)}{\omega(2+j\omega)(4-\omega^2-j3\omega)} \quad \frac{(2-j\omega)(4-\omega^2+j3\omega)}{(2-j\omega)(4-\omega^2+j3\omega)}$$

$$= \frac{(1-\omega^2)(\omega-j5)(2-j\omega)(4-\omega^2+j3\omega)}{\omega(\omega^2+4)\left[(4-\omega^2)^2+9\omega^2\right]}$$

$$= \frac{(1-\omega^2)(2\omega-5\omega-j10-j\omega^2)(4-\omega^2+j3\omega)}{\omega(\omega^2+4)\left[(4-\omega^2)^2+9\omega^2\right]}$$

$$= \frac{(1-\omega^2)(-3\omega-j(\omega^2+10))(4-\omega^2+j3\omega)}{\omega(\omega^2+4)\left[(4-\omega^2)^2+9\omega^2\right]}$$

$$= \frac{-(1-\omega^2)\left(3\omega(4-\omega^2)-3\omega(\omega^2+10)+j(\omega^2+10)(4-\omega^2)+j9\omega^2\right)}{\omega(\omega^2+4)\left[(4-\omega^2)^2+9\omega^2\right]}$$

$$= \frac{+(1-\omega^2)\left(6\omega(3+\omega^2)+j\left(\omega^4-3\omega^2-40\right)\right)}{\omega(\omega^2+4)\left[(4-\omega^2)^2+9\omega^2\right]}$$

$$\mathrm{Re}\,G(j\omega) = \frac{(1-\omega^2)\,6(3+\omega^2)}{(\omega^2+4)\left[(4-\omega^2)^2+9\omega^2\right]}$$

$$\mathrm{Im}\,G(j\omega) = \frac{(1-\omega^2)(\omega^4-3\omega^2-40)}{\omega(\omega^2+4)\left[(4-\omega^2)^2+9\omega^2\right]}$$

$$\mathrm{Re}\,G(j0) = \frac{1\cdot 6\cdot 3}{4\cdot 4^2} = \frac{9}{32}$$

$$G(j\infty) = -j0$$

$\mathrm{Im}\,G(j\omega)=0$ for $\omega = 1$ and for ω which solves

$$\omega^4-3\omega^2-40=0$$

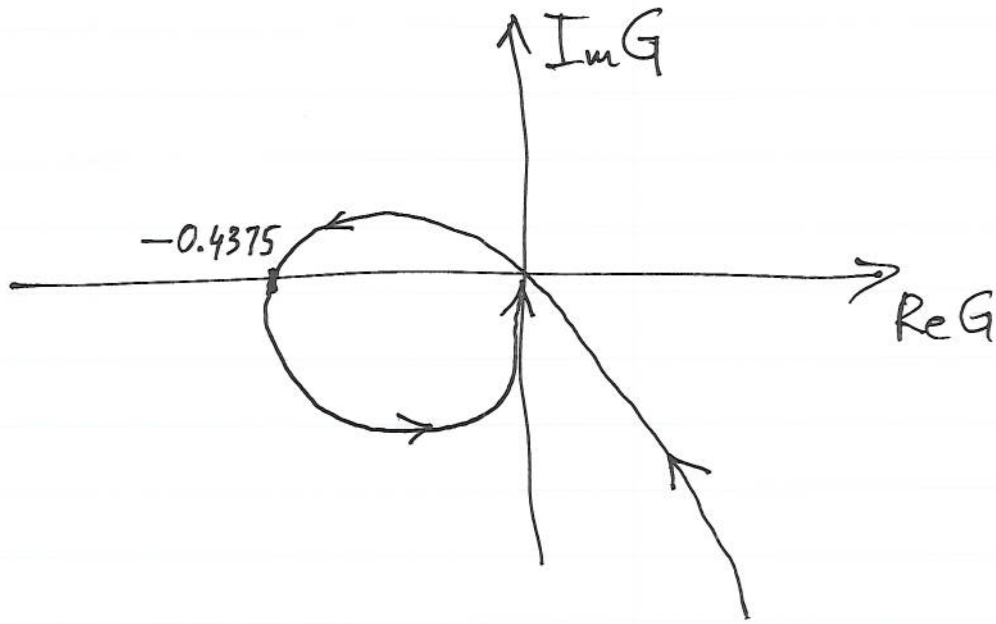
$$(\omega^2)_{1/2} = \frac{3\pm\sqrt{9+160}}{2}$$

$$\underline{(\omega^*)^2 = 8} \longrightarrow \omega^* = \sqrt{8} = 2.83$$

$$\mathrm{Re}\, G(j\omega^*) = \frac{(1-8)\,6\,(3+8)}{(8+4)\left((4-8)^2+9\cdot 8\right)}$$

$$= -\frac{7\cdot 6\cdot 11}{12\,(\underbrace{16+72}_{88})} = -\frac{7\cdot 6\cdot 11}{12\cdot 8\cdot 11}$$

$$= -\frac{7}{16} = -0.4375$$



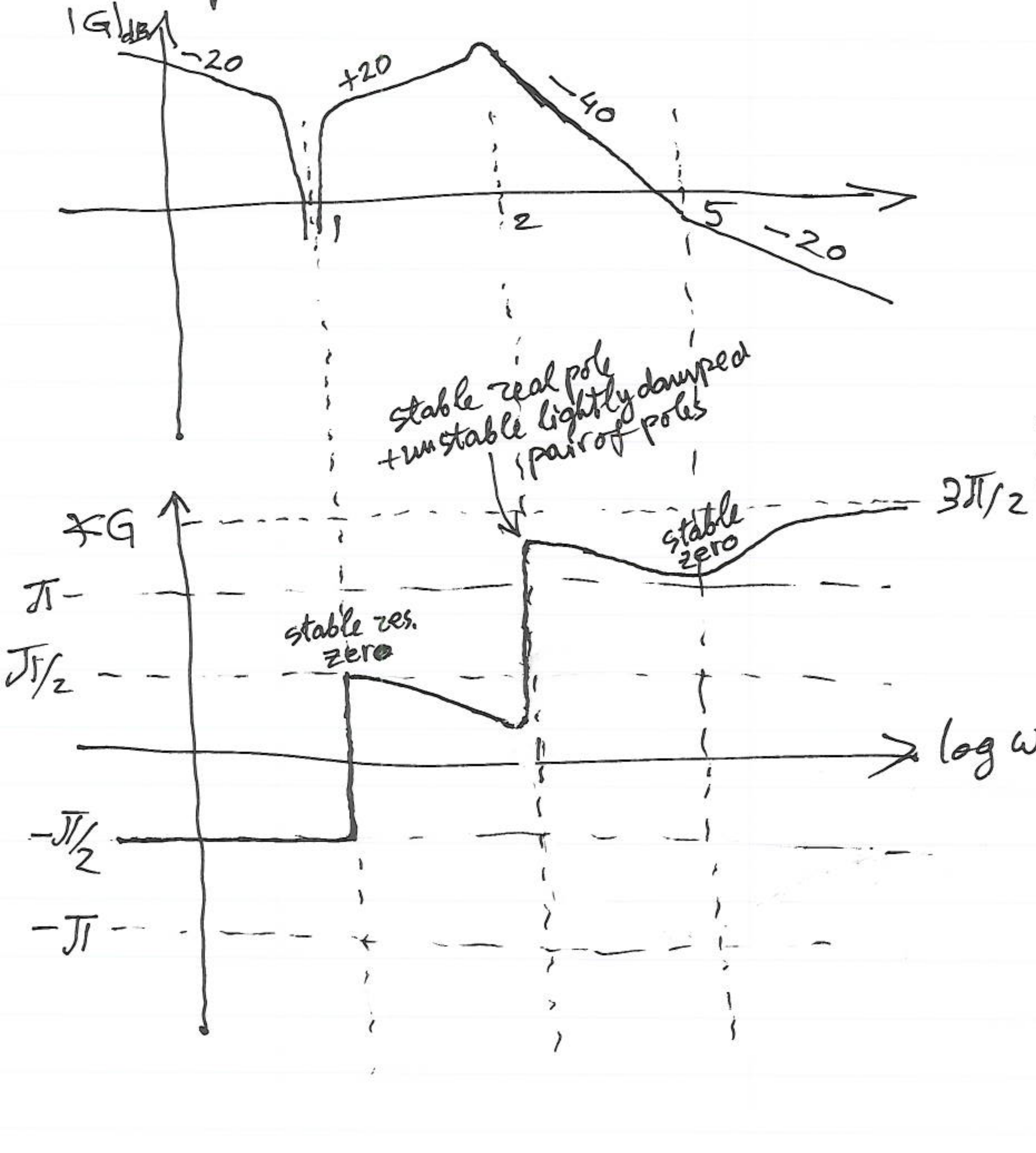
The open-loop system has <u>two unstable poles</u> (the roots of s^2-3s+4).

If $K > \frac{1}{0.4375} = \frac{16}{7} = 2.3$, then the critical point becomes encircled <u>once</u>, in the c.c.w. direction.

By Nyquist's criterion, the closed-loop system will be stable for

$$\underline{K > 2.3}$$

Bode plots



(Actual phase plot in Matlab quite different.)