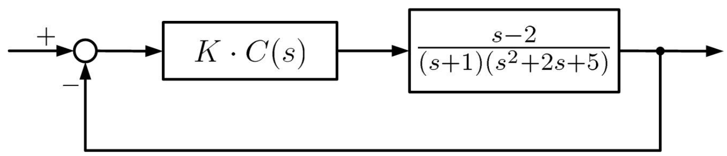
MAE 143B
LINEAR CONTROL Prof. M. Krstic

FINAL EXAM

June 13, 2013

- One sheet of hand-written notes (two pages). Write answers only in the blue book.
- Present your reasoning and calculations clearly. Inconsistent etchings will not be graded.
- Total points: 60. Time: (3 hours).

Problem 1: (10 points) Consider the following system:



(a) (3 points) Suppose $C(s) = 1$. Sketch the root locus of the closed-loop system as K varies from 0 to ∞.

(b) (3 points) Suppose $C(s) = \frac{s+4}{s+20}$, which is a phase-lead compensator. Sketch the root locus of the closed-loop system.

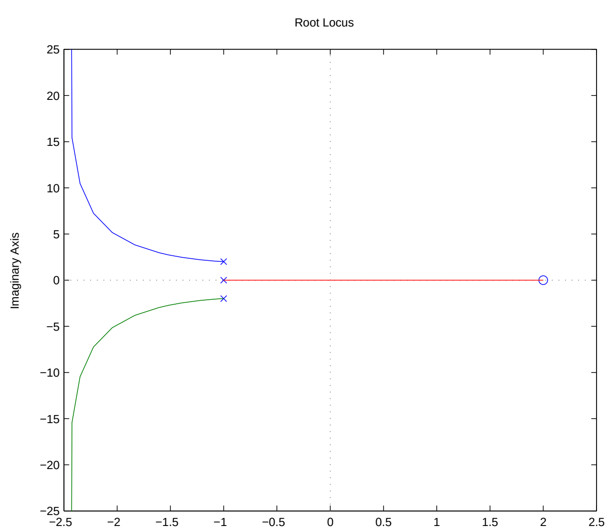
(c) (4 points) For **(b)**, find the range of the gain K $(K > 0)$ for which the feedback system is asymptotically stable.

Solution:

(a) Consider

$$G(s) = \frac{s-2}{(s+1)(s^2+2s+5)}.$$

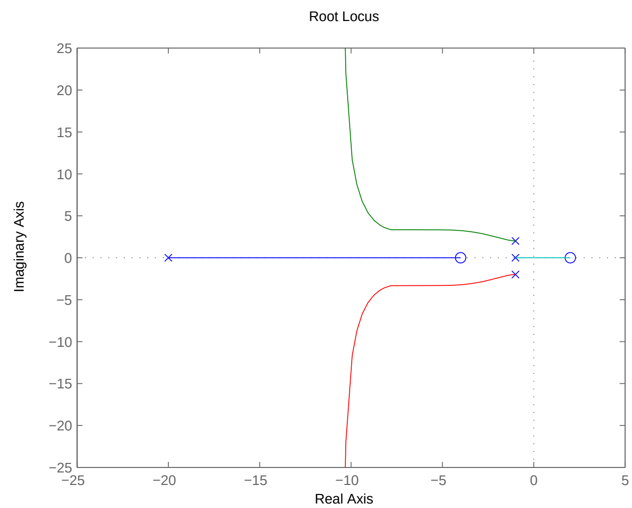
Its poles are $-2, -1+2j, -1-2j$, and its only zero is 2. The relative degree is 2. Then there are 2 asymptotes with angles $\pm\frac{\pi}{2}$ and centered at -2.5. The root locus is then shown as follows:



(b) Consider

$$C(s)G(s) = \frac{(s+4)(s-2)}{(s+20)(s+1)(s^2+2s+5)}.$$

In this case, one more pole -20 and one more zero is added. The relative degree is still 2. We have 2 asymptotes with angles $\pm\frac{\pi}{2}$ and centered at -10.5. The root locus is shown in the following figure.



(c) The characteristic polynomial of the closed-loop system is

$$1 + KC(s)G(s) = 0,$$

which is equivalent to

$$s^4 + 23s^3 + (67+K)s^2 + (145+2K)s + 100 - K = 0.$$

The Routh array is as follows:

s^4	1	$67+K$	$100-8K$
s^3	23	$145+2K$	
s^2	$\frac{1396+21K}{23}$	$100-8K$	
s^1	$\frac{200210+6021K+42K^2}{1396+21K}$		
s^0	$100-8K$		

According to the Routh's Criterion, to guarantee the asymptotical stability, we need $0 < K < 12.5$.

Problem 2: (15 points) Sketch the Bode plots for the following open-loop transfer functions:

(a) (5 points)

$$G(s) = \frac{45(s+1)(s+2)}{(s^2+4s+9)(s-5)}$$

(b) (5 points)

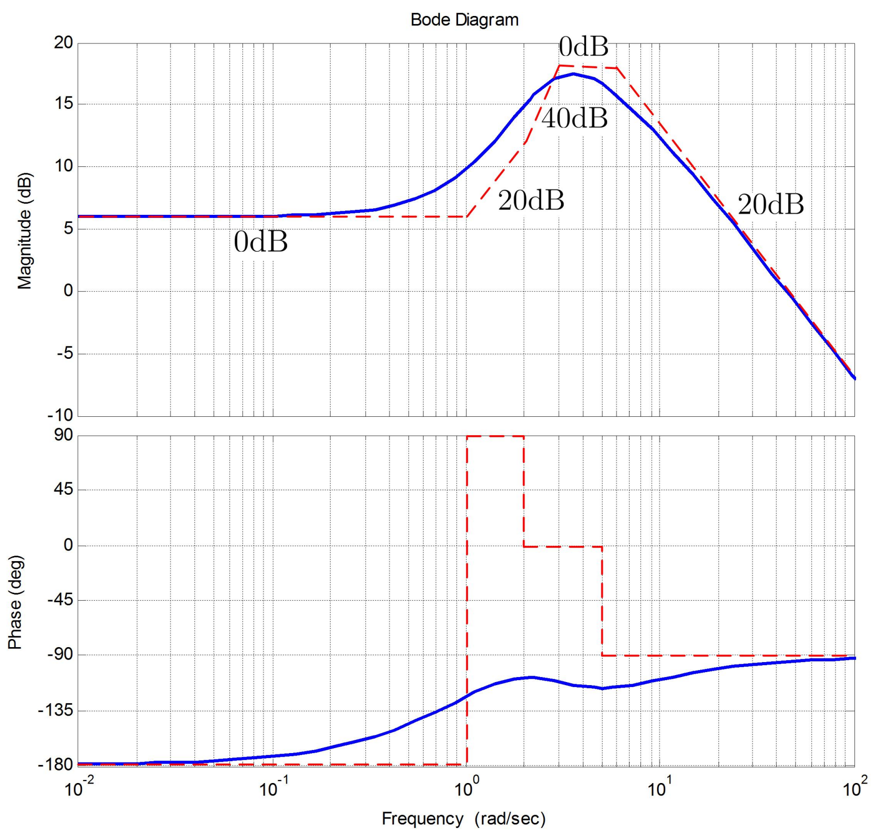
$$G(s) = \frac{s-10}{s^2(s^2+100)}$$

(c) (5 points)

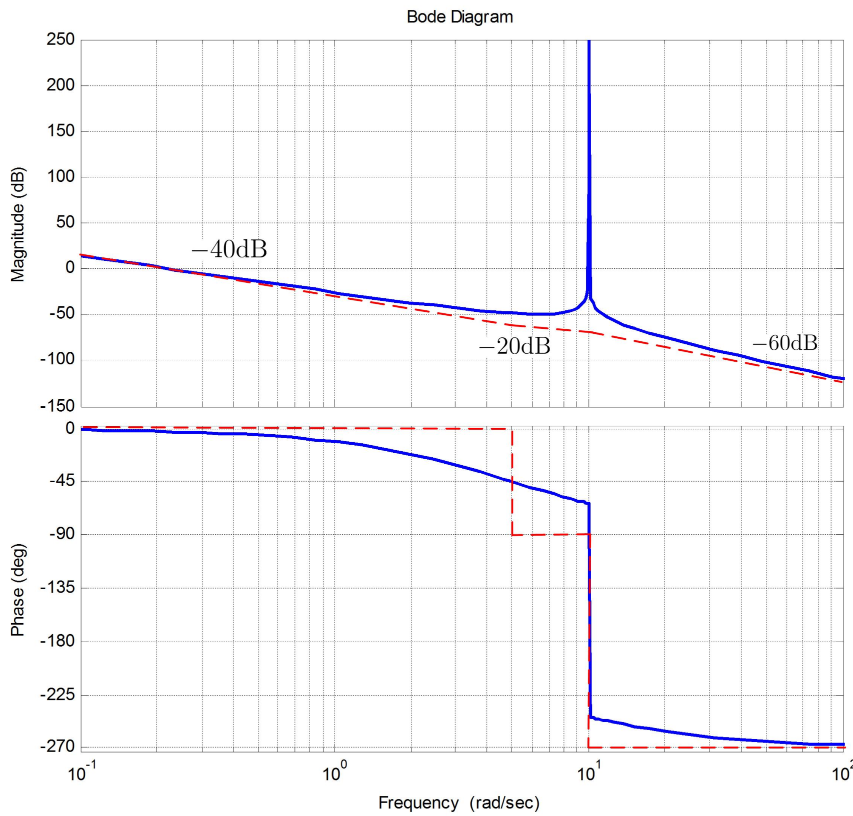
$$G(s) = \frac{(s-1)^3}{s(s^2+20s+100)}$$

Solution:

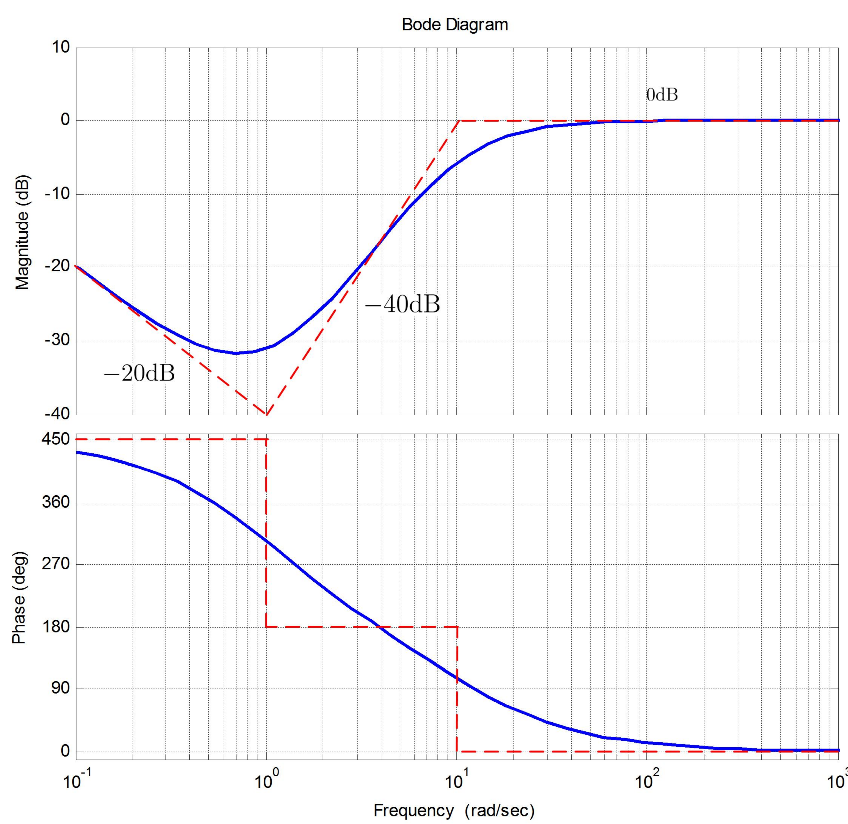
(a) See the following:



(b) See the following:



(c) See the following:



Problem 3: (10 points)Consider the open-loop transfer function:

$$G(s) = \frac{s^2 + 20s + 100}{(s-3)(s^2 + 2s + 17)}$$

(a) (5 points) Plot its Nyquist diagram of the system.

(b) (5 points) Use the Nyquist stability criterion, determine exactly the interval (or intervals) for the gain K ($K > 0$) such that the negative feedback loop of $G(s)$ and K is asymptotically stable.

Solution:

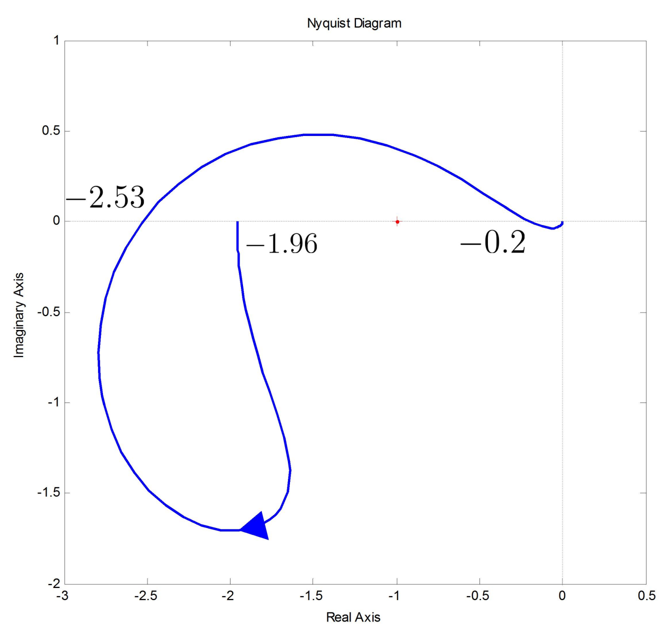
(a) Let $s = j\omega$. We have

$$\begin{aligned} G(j\omega) &= \frac{100 - \omega^2 + j20\omega}{(j\omega - 3)(17 - \omega^2 + j2\omega)} \\ &= \frac{(100 - \omega^2 + j20\omega)(-j\omega - 3)(17 - \omega^2 - j2\omega)}{(\omega^2 + 9)\left[(\omega^2 - 17)^2 + 4\omega^2\right]} \\ &= \frac{(-21\omega^4 + 371\omega^2 - 5100) + j\omega(-\omega^4 + 131\omega^2 - 2120)}{(\omega^2 + 9)\left[(\omega^2 - 17)^2 + 4\omega^2\right]}. \end{aligned}$$

It is seen that

- $G(j0+) = -1.96$ when $\omega \to 0+$,
- $G(j\infty) = -j0$ when $\omega \to \infty$,
- Im $G(j\omega) = 0$ and Re $G(j\omega) = -2.53$ when $\omega = 4.35$,
- Im $G(j\omega) = 0$ and Re $G(j\omega) = -0.20$ when $\omega = 10.59$.

Then the Nyquist plot is as follows.



(b) According to the Nyquist stability criterion, the closed-loop system will be asymptotically stable when $K > 1/0.2$, i.e., $K > 5$.

Problem 4: (10 points) Consider a negative unity feedback system with the following open-loop transfer function:

$$G(s) = \frac{K}{(0.01s+1)^3}.$$

(a) (2 points) Plot the Nyquist diagram of $G(s)$.

(b) (6 points) Find K ($K > 0$) such that the phase margin (PM) is 45 degrees, and determine the gain margin (GM) in this case.

(c) (2 points) Determine the range of K ($K > 0$) such that the negative feedback loop of $G(s)$ and K is asymptotically stable.

Solution:

(a) It is seen that

$$G(s) = \frac{10^6 K}{(s+10^2)^3}.$$

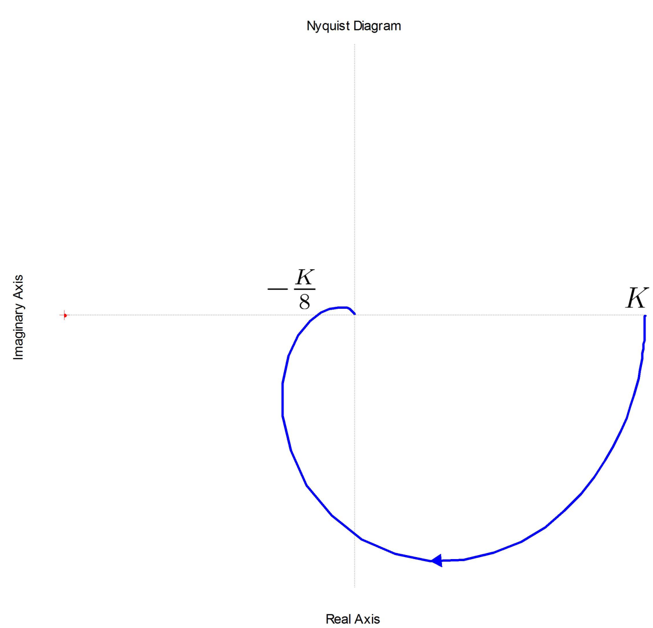
Let $s = j\omega$

$$G(s) = \frac{10^6 K\left[(10^6 - 3\times 10^2\omega^2) - j\omega(3\times 10^4 - \omega^2)\right]}{(\omega^2+10^4)^3}.$$

We see that

- $G(j0+) = K$ when $\omega \to 0+$,
- $G(j\infty) = j0$ when $\omega \to \infty$,
- $\text{Im } G(j\omega) = 0$ and $\text{Re } G(j\omega) = -K/8$ when $\omega = \sqrt{3}\times 10^2$, i.e., $\omega^2 = 3\times 10^4$,
- $\text{Re } G(j\omega) = 0$ and $\text{Im } G(j\omega) = -j\frac{3\sqrt{3}}{8}K$ for any $\omega = \frac{\sqrt{3}}{3}\times 10^2$.

The Nyquist plot is illustrated in the following figure.



(a) The PM of 45 degrees happens when $\omega = 100$, as is seen from

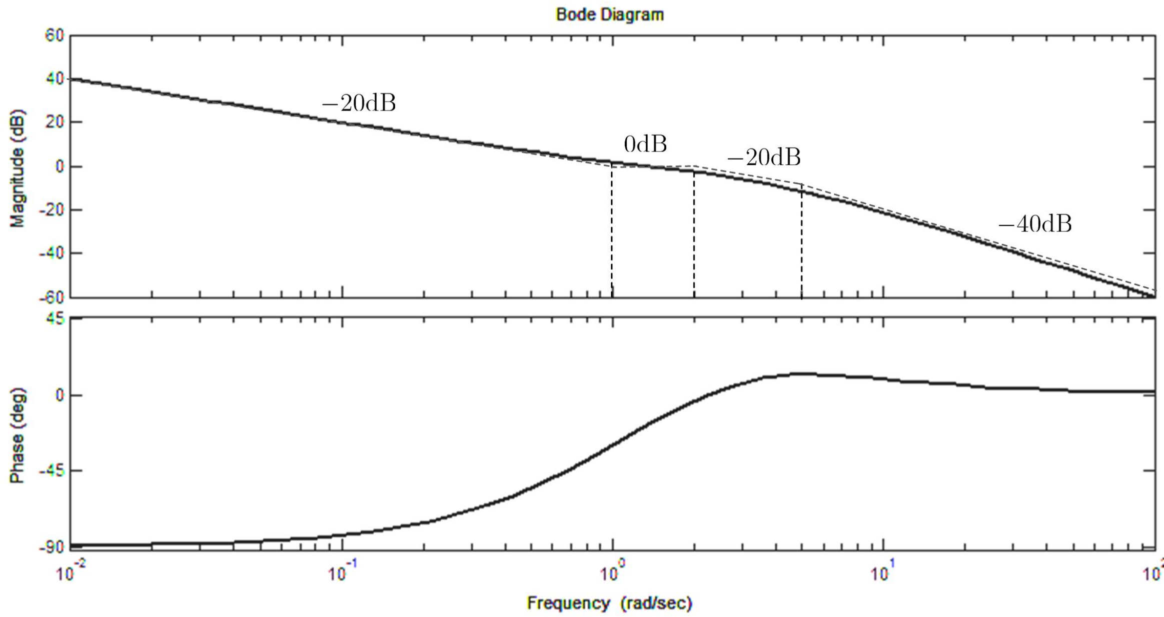
$$G(j\omega) = \frac{K}{\left(\frac{j\omega}{100} + 1\right)^3}.$$

Because $|G(j100)| = 1$ is needed, we can determine that $K = 2\sqrt{2}$.

The GM will be equal to $8/K$. That is, $\text{GM} = 2\sqrt{2}$.

(b) According to the Nyquist stability criterion, $0 < K < 8$ is needed to guarantee the asymptotic stability.

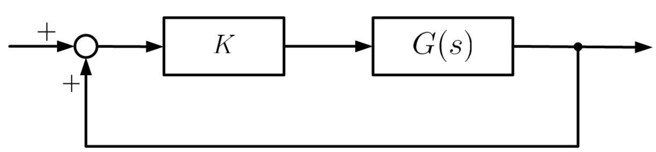
Problem 5: (15 points) Suppose we perform an experiment on a dynamic system and obtain the following Bode plot:



(a) (5 points) Determine the transfer function $G(s)$ of the system.

(b) (5 points) Sketch the Nyquist plot based on $G(s)$ obtained in **(a)**.

(c) (5 points) Sketch the root locus of the following **positive** feedback system with K varying from 0 to ∞.



Solution:

(a) From the given Bode plot, the transfer function $G(s)$ is given by

$$G(s) = \frac{-10(s+1)}{s(s-2)(s+5)}.$$

(b) Let s be replaced by $j\omega$. We have

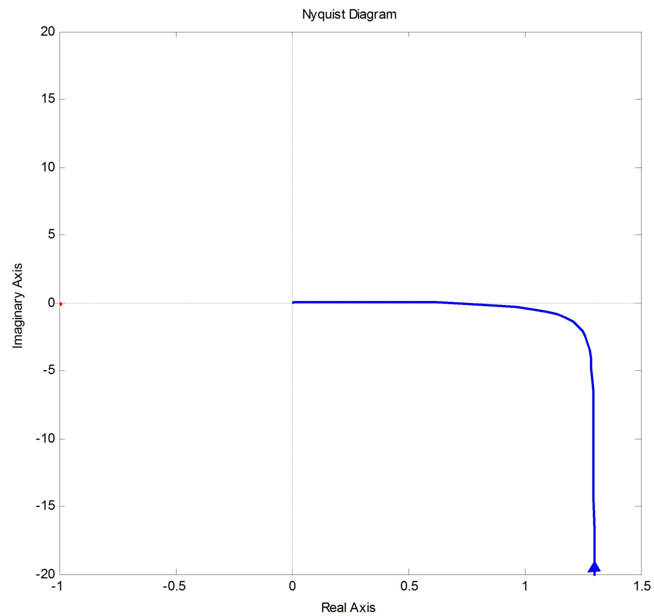
$$\begin{aligned} G(j\omega) &= \frac{-10(j\omega+1)}{j\omega(j\omega-2)(j\omega+5)} \\ &= \frac{-10\left[-\omega^2(\omega^2+13)+j2\omega(-\omega^2+5)\right]}{\omega^2(\omega^2+4)(\omega^2+25)}. \end{aligned}$$

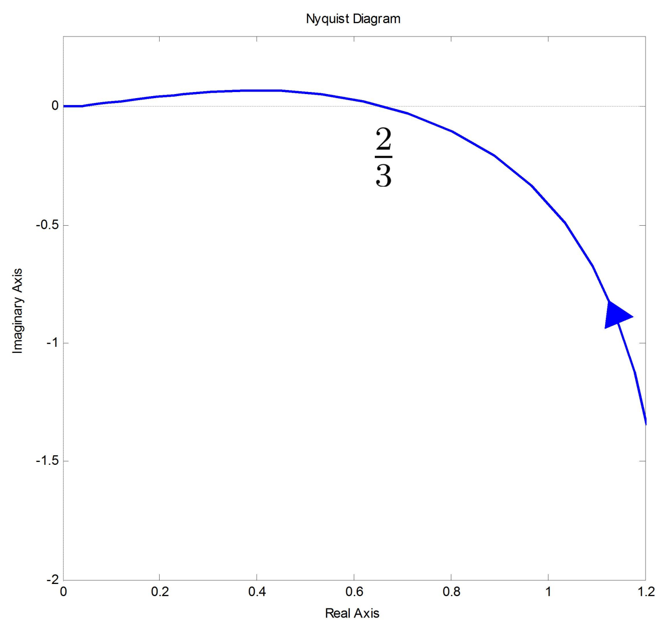
We note that

- $G(j0+) = 1?3 - j\infty$ when $\omega \to 0+$,

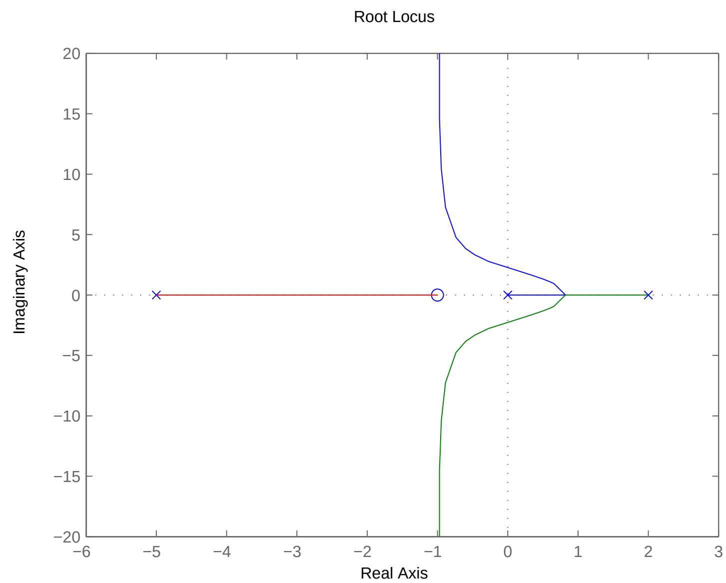
- $G(j\infty) = j0$ when $\omega \to \infty$,
- $\text{Im}\, G(j\sqrt{5}) = 0$ and $\text{Re}\, G(j\sqrt{5}) = 2/3$ when $\omega = \sqrt{5}$, i.e., $\omega^2 = 5$,
- $\text{Re}\, G(j\omega) \neq 0$ for any $0 < \omega < \infty$.

The following is the Nyquist plot and its enlarged view.





(c) This is a positive feedback system. To plot the root locus with respect to K varying from 0 to ∞, we only need to consider $-G(s)$. The relative degree of $-G(s)$ is 2. The 2 asymptotes of the root locus are with angles $\pm\frac{\pi}{2}$ and centered at -1.


Root Locus
Imaginary Axis
Real Axis
20
15
10
5
0
−5
−10
−15
−20
−6
−5
−4
−3
−2
−1
0
1
2
3