MAE 143A

Signals and Systems

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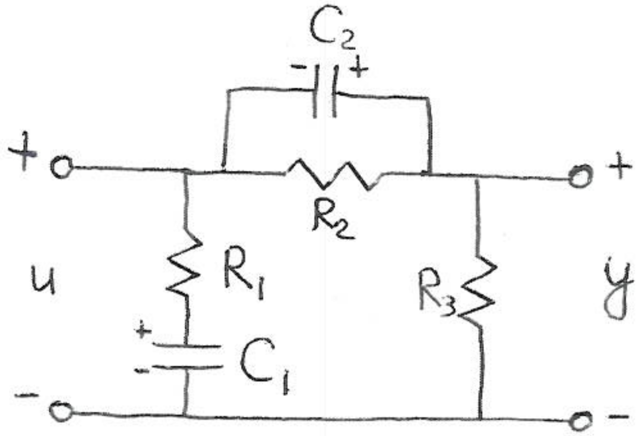
MIDTERM

October 27, 2009

NAME: SOLUTIONS

- Closed book. One sheet (both sides) of handwritten notes allowed.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write answers only in the blue book.
- The problems are *not* ordered by difficulty.
- Total points: 35.
- Time: 1 hour 10 minutes.

Problem 1. (11 points) Find the state space representation for the following circuit:



Solution: Denote the voltages on the capacitors by v_1 and v_2.
We have:

$$C_1 \dot{v}_1 = \frac{u - v_1}{R_1}$$

$$C_2 \dot{v}_2 = -\frac{v_2}{R_2} - \frac{y}{R_3}$$

$$y = u + v_2$$

State-space representation is:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1 C_1} & 0 \\ 0 & -\frac{1}{C_2}\left(\frac{1}{R_2} + \frac{1}{R_3}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1 R_1} \\ -\frac{1}{C_2 R_3} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + 1 \cdot u$$

Problem 2. (12 points) Consider the system

$$\dot{x}_1 = (1-x_1)x_1 - \frac{2x_1x_2}{1+x_1^2} + x_2x_1u\,,$$
$$\dot{x}_2 = \left(2 - \frac{x_2}{1+x_1^2}\right)x_2 + x_1^2u\,.$$

(This system does not come from any physical application but its nonlinearities are typical for chemical and biological systems.)

For $u=0$, find all the equilibria of the system.

Solution:

$$(1-x_1)x_1 - \frac{2x_1x_2}{1+x_1^2} = 0 \qquad (1)$$

$$\left(2 - \frac{x_2}{1+x_1^2}\right)x_2 = 0 \qquad (2)$$

1. $x_1 = 0 \overset{(2)}{\Rightarrow} (2-x_2)x_2 = 0 \Rightarrow x_2 = 0$ or $x_2 = 2$, so two eq. points: $(0,0)$ and $(0,2)$.

2. $x_1 \neq 0 \overset{(1)}{\Rightarrow} 1 - x_1 = \frac{2x_2}{1+x_1^2} \overset{(2)}{\Rightarrow} \left(2 + \frac{x_1-1}{2}\right)x_2 = 0$

$\Rightarrow \quad x_1 = -3 \Rightarrow x_2 = \frac{4(1+3^2)}{2} = 20$

or

$x_2 = 0 \Rightarrow x_1 = 1$

3. From (2) we have either $x_2 = 0$ or $x_2 = 2(1+x_1^2)$.
It is easy to see that these equations do not produce new equilibria.

So, there are 4 equilibria:

$$E_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} -3 \\ 20 \end{bmatrix}$$

Problem 3. (12 points) For each equilibrium of the system from Problem 2, find the linearization around that equilibrium (i.e. find F and G).

Solution: Denote $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\dot{X} = f(X,u), \quad \frac{\partial f}{\partial X} = \begin{bmatrix} 1 - 2x_1 - \frac{2x_2}{1+x_1^2} + \frac{4x_1^2 x_2}{(1+x_1^2)^2} & \frac{-2x_1}{1+x_1^2} \\ \frac{2x_1 x_2^2}{(1+x_1^2)^2} & 2 - \frac{2x_2}{1+x_1^2} \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} x_2 x_1 \\ x_1^2 \end{bmatrix}; \quad \delta \dot{X}_i = \left.\frac{\partial f}{\partial X}\right|_{X=E_i} \delta X_i + \left.\frac{\partial f}{\partial u}\right|_{X=E_i} \delta u$$

$$= F_i \delta X_i + G_i \delta u, \quad i = 1,2,3,4.$$

$$F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}, \quad F_3 = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 51/5 & 3/5 \\ -24 & -2 \end{bmatrix};$$

$$G_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G_4 = \begin{bmatrix} -60 \\ 9 \end{bmatrix}$$