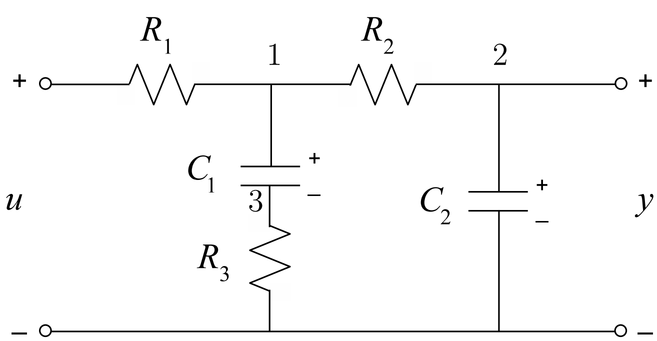
MAE 143A
Signals and Systems — Prof. M. Krstic

MIDTERM SOLUTIONS

July 14, 2010

- One sheet (both sides) of *handwritten* notes allowed.
- No graphing calculators.
- Present your reasoning and calculations clearly. Inconsistent etchings will not be graded.
- Write answers only in the blue book.
- Total points: 25. Time: 1 hour 15 minutes.

Problem 1. (9 points) Find the state space representation for the following circuit:



Solution: Let's denote voltages on the capacitors by v_1 and v_2, respectively, and voltage between node 1 and "–" by v_0. We have $y = v_2$ and

KCL at node 1:

$$\frac{v_0 - u}{R_1} + C_1\dot{v}_1 + \frac{v_0 - v_2}{R_2} = 0 \tag{1}$$

KCL at node 2:

$$C_2\dot{v}_2 + \frac{v_2 - v_0}{R_2} = 0 \tag{2}$$

KCL at node 3:

$$C_1\dot{v}_1 + \frac{v_1 - v_0}{R_3} = 0 \tag{3}$$

Subtracting (3) from (1) we get

$$\frac{v_0 - u}{R_1} + \frac{v_0 - v_2}{R_2} - \frac{v_1 - v_0}{R_3} = 0 \quad \Rightarrow \quad v_0 = \frac{R_1R_2R_3}{R_1R_2 + R_2R_3 + R_1R_3}\left(\frac{v_1}{R_3} + \frac{v_2}{R_2} + \frac{u}{R_1}\right) \tag{4}$$

State equations are:

$$C_1\dot{v}_1 = \frac{v_0 - v_1}{R_3} = \frac{-(R_1+R_2)v_1 + R_1v_2 + R_2u}{R_1R_2 + R_2R_3 + R_1R_3}$$
$$C_2\dot{v}_2 = \frac{v_0 - v_2}{R_2} = \frac{R_1v_1 - (R_1+R_3)v_2 + R_3u}{R_1R_2 + R_2R_3 + R_1R_3}$$

Therefore,

$$\dot{v} = Fv + Gu, \quad y = Hv + Ju, \quad v = [v_1 \;\; v_2]^T$$

$$F = \frac{1}{R}\begin{bmatrix} -\frac{R_1+R_2}{C_1} & \frac{R_1}{C_1} \\ \frac{R_1}{C_2} & -\frac{R_1+R_3}{C_2} \end{bmatrix}, \quad G = \frac{1}{R}\begin{bmatrix} \frac{R_2}{C_1} \\ \frac{R_3}{C_2} \end{bmatrix}, \quad H = [0 \;\; 1], \quad J = 0\,,$$

where $R = R_1R_2 + R_2R_3 + R_1R_3$.

Problem 2. (8 points) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1(2x_1 + x_2) \\ \dot{x}_2 &= -x_3 + 5 + x_2 - x_2^2 - 2x_1x_2 \\ \dot{x}_3 &= x_2 - 1 - \left(\frac{1}{3} + u\right)x_3 \end{aligned}$$

Denote the state vector as $x = [x_1, x_2, x_3]^T$. For $u = 0$, find all the equilibria of the system.

(This model is inspired by a model of the rotating stall instability in axial flow compressors that are used in gas turbine/jet engines. The variables, (very) roughly, are: x_3-pressure rise, x_2-mean flow through compressor, x_1-the amplitude of the rotating stall instability, and u-control through downstream valve/throttle.) For $u = 0$, find all the equilibria of the system.

Solution: Equilibria of the system:

$$\begin{aligned} &x_1(2x_1 + x_2) = 0 \\ &-x_3 + 5 + x_2 - x_2^2 - 2x_1x_2 = 0 \\ &x_2 - 1 - \frac{1}{3}x_3 = 0 \end{aligned}$$

From the top equation we have:

1) $x_1 = 0 \quad \Rightarrow \quad x_2^2 + 2x_2 - 8 = 0 \quad \Rightarrow \quad x_2 = 2$ and $x_2 = -4 \quad \Rightarrow \quad x_3 = 3$ and $x_3 = -15$

2) $2x_1 + x_2 = 0 \quad \Rightarrow -x_3 + 5 + x_2 = 0$ and $x_3 = 3x_2 - 3 \quad \Rightarrow \quad x_2 = 4 \quad \Rightarrow \quad x_3 = 9, \;\; x_1 = -2$

So, we have 3 equilibria: $E_1 = [0 \quad 2 \quad 3]^T$, $E_2 = [0 \;\; -4 \;\; -15]^T$, $E_3 = [-2 \quad 4 \quad 9]^T$.

Problem 3. (8 points) For each equilibrium of the system from Problem 2, find the linearization around that equilibrium (i.e. find F and G).

Solution:

$$\dot{x} = f(x,u), \qquad \frac{\partial f}{\partial x} = \begin{bmatrix} 4x_1 + x_2 & x_1 & 0 \\ -2x_2 & 1 - 2x_1 - 2x_2 & -1 \\ 0 & 1 & -1/3 - u \end{bmatrix}, \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ -x_3 \end{bmatrix}$$

Now we can write for equilibrium E_i: $\delta \dot{x}_i = \frac{\partial f}{\partial x}\Big|_{x=E_i} \delta x_i + \frac{\partial f}{\partial u}\Big|_{x=x_i} \delta u = F_i\, \delta x_i + G_i\, \delta u, \quad i = 1, 2, 3.$

$$F_1 = \begin{bmatrix} 2 & 0 & 0 \\ -4 & -3 & -1 \\ 0 & 1 & -1/3 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -4 & 0 & 0 \\ 8 & 9 & -1 \\ 0 & 1 & -1/3 \end{bmatrix}, \quad F_3 = \begin{bmatrix} -4 & -2 & 0 \\ -8 & -3 & -1 \\ 0 & 1 & -1/3 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 \\ 0 \\ -9 \end{bmatrix}$$