MAE 143A

Signals and Systems — Prof. M. Krstic

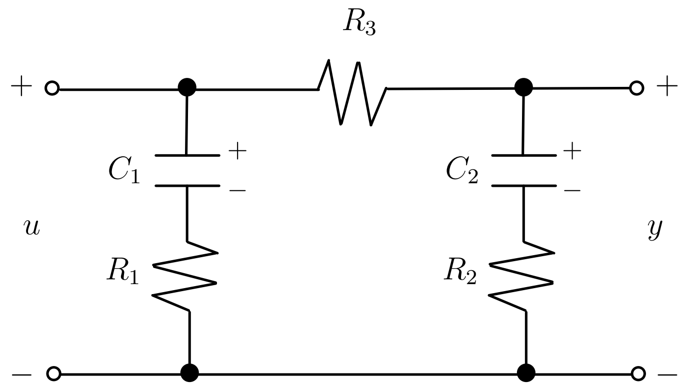
MIDTERM

July 20, 2009

NAME: SOLUTIONS

- Closed book. One sheet (both sides) of handwritten notes allowed.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write answers only in the blue book.
- The problems are *not* ordered by difficulty.
- Total points: 50.
- Time: 2.5 hours.

Problem 1. (8 points) Find the state space representation for the following circuit:



Solution: Denote the voltages on the capacitors by v_1 and v_2. We have

$$C_1\dot{v}_1 - \frac{u - v_1}{R_1} = 0 \tag{1}$$

$$C_2\dot{v}_2 - \frac{y - v_2}{R_2} = 0 \tag{2}$$

$$C_2\dot{v}_2 - \frac{u - y}{R_3} = 0 \tag{3}$$

From (2) and (3) we get

$$\begin{aligned} \frac{y - v_2}{R_2} &= \frac{u - y}{R_3} \\ y &= \frac{R_3}{R_2 + R_3} v_2 + \frac{R_2}{R_2 + R_3} u \end{aligned}$$

Using this expression for y, we get

$$\begin{aligned} C_1\dot{v}_1 &= -\frac{1}{R_1} v_1 + \frac{1}{R_1} u \\ C_2\dot{v}_2 &= -\frac{1}{R_2 + R_3} v_2 + \frac{1}{R_2 + R_3} u \end{aligned}$$

State space representation is

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1 R_1} & 0 \\ 0 & -\frac{1}{C_2(R_2+R_3)} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1 R_1} \\ \frac{1}{C_2(R_2+R_3)} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \frac{R_3}{R_2+R_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \frac{R_2}{R_2 + R_3} u$$

Problem 2. (8 points) Consider the following system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2^2(1+u)\\ \dot{x}_2 &= -x_2 + \frac{2x_1(9-2x_3)}{x_2^2+9} + x_1 u\\ \dot{x}_3 &= -4x_3 + 3 + x_2\end{aligned}$$

(This system does not come from any physical application but its nonlinearities are typical for chemical and biological systems.)

For $u = 0$, find all the equilibria of the system.

Solution: Equilibria of the system for $u = 0$ are determined from the equations

$$-2x_1 + x_2^2 = 0 \tag{4}$$

$$-x_2 + \frac{2x_1(9-2x_3)}{x_2^2+9} = 0 \tag{5}$$

$$-4x_3 + 3 + x_2 = 0\,. \tag{6}$$

From (4) and (6) we have

$$x_1 = \frac{1}{2}x_2^2, \qquad x_3 = \frac{3}{4} + \frac{1}{4}x_2\,. \tag{7}$$

Substituting these expressions into (5), we get

$$x_2(x_2^2+9) = x_2^2\left(\frac{15}{2} - \frac{1}{2}x_2\right). \tag{8}$$

One solution of (8) is $x_2 = 0$, which gives $x_1 = 0$ and $x_3 = 3/4$. If $x_2 \neq 0$, dividing (8) by x_2 we get

$$\frac{3}{2}x_2^2 - \frac{15}{2}x_2 + 9 = 0\,, \tag{9}$$

or

$$x_2^2 - 5x_2 + 6 = 0\,. \tag{10}$$

This equation has solutions $x_2 = 2$ (which gives $x_1 = 2$, $x_3 = 5/4$) and $x_2 = 3$ (which gives $x_1 = 9/2$, $x_3 = 3/2$).

In summary, 3 equilibria are:

$$E_1 = [0 \quad 0 \quad 3/4]^T\,, \tag{11}$$

$$E_2 = [2 \quad 2 \quad 5/4]^T\,, \tag{12}$$

$$E_3 = [9/2 \quad 3 \quad 3/2]^T \tag{13}$$

Problem 3. <u>(8 points)</u> For each equilibrium of the system from Problem 2, find the linearization around that equilibrium (i.e. find F and G).

Solution: $X = [x_1, x_2, x_3]^T$, we have

$$\dot{X} = f(X,u), \quad \frac{\partial f}{\partial X} = \begin{bmatrix} -2 & 2x_2 & 0 \\ \dfrac{2(9-2x_3)}{x_2^2+9} & -1 - \dfrac{4x_1x_2(9-2x_3)}{(x_2^2+9)^2} & -\dfrac{4x_1}{x_2^2+9} \\ 0 & 1 & -4 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} x_2^2 \\ x_1 \\ 0 \end{bmatrix}$$

Now we can write for equilibrium E_i:

$$\delta \dot{X}_i = \left.\frac{\partial f}{\partial X}\right|_{X=E_i} \delta X_i + \left.\frac{\partial f}{\partial u}\right|_{X=E_i} \delta u = F_i\,\delta X_i + G_i\,\delta u, \quad i = 1,2,3\,,$$

where

$$F_1 = \begin{bmatrix} -2 & 0 & 0 \\ 5/3 & -1 & 0 \\ 0 & 1 & -4 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -2 & 4 & 0 \\ 1 & -21/13 & -8/13 \\ 0 & 1 & -4 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} -2 & 6 & 0 \\ 2/3 & -2 & -1 \\ 0 & 1 & -4 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 9 \\ 9/2 \\ 0 \end{bmatrix}$$

Problem 4. (10 points) Find Laplace transforms of the following functions:

(a) (4 points) $f(t) = te^{-3t}\sin(2t-4)1(t-2)$

(b) (4 points) $f(t) = t^2\cos(2t)1(t)$

(c) (2 points) For both (a) and (b) compute $f(\infty)$ using the Final Value Theorem and compare the result with the direct computation from $f(t)$. Explain a possible discrepancy between the direct and FVT computations.

Solution:

(a)

$$\begin{aligned} F(s) &= -\frac{d}{ds}\left(\frac{2}{(s+3)^2+4}\mathrm{e}^{-2(s+3)}\right) \\ &= \frac{4(s+3)}{((s+3)^2+4)^2}\mathrm{e}^{-2(s+3)} + \frac{4}{(s+3)^2+4}\mathrm{e}^{-2(s+3)} \\ &= \frac{4(s^2+7s+16)}{((s+3)^2+4)^2}\mathrm{e}^{-2(s+3)} \end{aligned}$$

(b)

$$\begin{aligned} F(s) &= -\frac{d}{ds}\left(-\frac{d}{ds}\left(\frac{s}{s^2+4}\right)\right) \\ &= \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right) \\ &= \frac{2s^3-24s}{(s^2+4)^3} \end{aligned}$$

(c.a) Using FVT:

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{4(s^2+7s+16)}{((s+3)^2+4)^2}\mathrm{e}^{-2(s+3)} = 0$$

Directly:

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{t\to\infty} te^{-3t}\sin(2t-4)1(t-2) = 0$$

Both FVT and direct computation give the same result.

(c.b) Using FVT:

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{2s^3-24s}{(s^2+4)^3} = 0$$

Directly: no limit exists since $|f(t)|$ grows to infinity with time while the sign of $f(t)$ is switched infinitely many times. FVT gives the wrong answer because there are poles on the imaginary axis (triple poles $\pm 2j$).

Problem 5. <u>(8 points)</u> Compute the *impulse* response of the following systems:

(a) (3 points) $H(s) = \dfrac{e^{-2s}}{(3s+5)^4}$

(b) (5 points) $H(s) = \dfrac{s^2+5}{(s+2)^2(s+3)^2}$

Solution:

(a)

$$H(s) = \frac{e^{-2s}}{81(s+5/3)^4}$$

$$h(t) = \frac{1}{81}e^{-5/3(t-2)}\frac{(t-2)^3}{6}1(t-2) = \frac{(t-2)^3}{486}\mathrm{e}^{-5/3(t-2)}1(t-2)$$

(b)

$$H(s) = \frac{s^2+5}{(s+2)^2(s+3)^2} = \frac{C_1}{s+2} + \frac{C_2}{(s+2)^2} + \frac{C_3}{s+3} + \frac{C_4}{(s+3)^2}$$

$$\begin{aligned}
C_1 &= \frac{d}{ds}\left(\frac{s^2+5}{(s+3)^2}\right)\bigg|_{s=-2} = \frac{6s-10}{(s+3)^3}\bigg|_{s=-2} = -22 \\
C_2 &= \frac{s^2+5}{(s+3)^2}\bigg|_{s=-2} = 9 \\
C_3 &= \frac{d}{ds}\left(\frac{s^2+5}{(s+2)^2}\right)\bigg|_{s=-3} = \frac{4s-10}{(s+2)^3}\bigg|_{s=-3} = 22 \\
C_4 &= \frac{s^2+5}{(s+2)^2}\bigg|_{s=-3} = 14
\end{aligned}$$

$$h(t) = (-22e^{-2t} + 9te^{-2t} + 22e^{-3t} + 14te^{-3t})1(t)$$

Problem 6. (8 points) The step response of some linear time invariant system is given by

$$f(t) = 1 - 2e^{-t} + e^{-2t}.$$

Under some unknown input the system's output is

$$y(t) = 2(1 - \cos(t))e^{-t}.$$

Find this unknown input.

Solution:

$$F(s) = H(s)\frac{1}{s},$$

where $H(s)$ is the transfer function of the system. Therefore,

$$H(s) = sF(s) = s\left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right) = \frac{2}{(s+1)(s+2)}$$

The Laplace transform of the output $y(t)$ is

$$Y(s) = \frac{2}{s+1} - \frac{2(s+1)}{(s+1)^2+1}$$

Since $Y(s) = H(s)U(s)$, we have

$$\begin{aligned} U(s) = \frac{Y(s)}{H(s)} &= \left(\frac{2}{s+1} - \frac{2(s+1)}{(s+1)^2+1}\right)\frac{(s+1)(s+2)}{2} \\ &= s+2 - \frac{(s+1)^2(s+2)}{(s+1)^2+1} \\ &= \frac{s+2}{(s+1)^2+1} \\ &= \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \end{aligned}$$

Finally,

$$u(t) = (\cos(t) + \sin(t))e^{-t}1(t)$$