MAE 143A

Signals and Systems — Prof. M. Krstic

MIDTERM

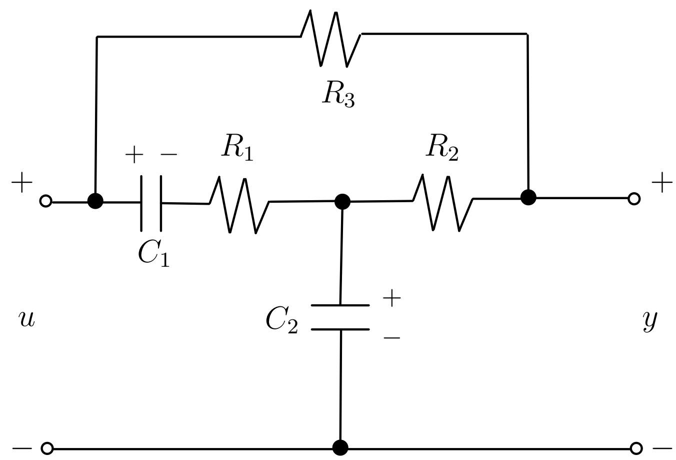
July 15, 2008

NAME: SOLUTIONS

- Closed book. One sheet (both sides) of handwritten notes allowed.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 45.
- Time: 2 hours.

Problem 1. (10 points)

Find the state space representation for the following circuit:



Solution: Denote the voltages on the capacitors by v_1 and v_2. We have

$$C_1\dot{v}_1 - \frac{u - v_1 - v_2}{R_1} = 0$$
$$C_2\dot{v}_2 - \frac{u - v_1 - v_2}{R_1} - \frac{u - v_2}{R_2 + R_3} = 0$$
$$\frac{v_2 - y}{R_2} + \frac{u - y}{R_3} = 0$$

Solving for y, we obtain

$$C_1\dot{v}_1 = -\frac{1}{R_1}v_1 - \frac{1}{R_1}v_2 + \frac{1}{R_1}u$$
$$C_2\dot{v}_2 = -\frac{1}{R_1}v_1 - \left(\frac{1}{R_1} + \frac{1}{R_2 + R_3}\right)(v_2 - u)$$

$$y = \frac{R_3}{R_2 + R_3}v_2 + \frac{R_2}{R_2 + R_3}u$$

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1R_1} & -\frac{1}{C_1R_1} \\ -\frac{1}{C_2R_1} & -\left(\frac{1}{C_2R_1} + \frac{1}{C_2(R_2+R_3)}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1R_1} \\ \left(\frac{1}{C_2R_1} + \frac{1}{C_2(R_2+R_3)}\right) \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \frac{R_3}{R_2+R_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \frac{R_2}{R_2 + R_3}u$$

Problem 2. (10 points)

One of the state-of-the-art HIV models is (Wodarz'99 with specific values of the parameters)

$$\begin{aligned} \dot{x} &= 2 - x - 3xy(1-u) \\ \dot{y} &= -y - zy + 3xy(1-u) \\ \dot{w} &= 2xyw - yw - \frac{1}{3}w \\ \dot{z} &= yw - z\,, \end{aligned}$$

where u is the input (drug concentration) and x, y, w, and z are concentrations of healthy cells, infected cells, memory cells, and killer cells, respectively.

For $u = 0$ (no treatment), there exist 3 equilibria in this system. One corresponds to a healthy person, another one corresponds to a person with AIDS, and the third one corresponds to a long-term non-progressor, i.e. a person with HIV who never develops AIDS.

Find the three equilibria.

Solution: Equilibria of the system for $u = 0$ are determined from the equations

$$\begin{aligned} 2 - x - 3xy &= 0 && (1) \\ -y - zy + 3xy &= 0 && (2) \\ 2xyw - yw - \frac{1}{3}w &= 0 && (3) \\ yw - z &= 0 && (4) \end{aligned}$$

From (3) we have either

$$w = 0 \qquad (5)$$

or

$$(2x-1)y = \frac{1}{3} \qquad (6)$$

Let us start with (5). From (4) we get $z = 0$. Substituting $z = 0$ into (2) gives $(3x-1)y = 0$, i.e. either $y = 0$ or $x = 1/3$. If $y = 0$, then from (1) $x = 2$. If $x = 1/3$, then from (1) $y = 5/3$. Therefore, we obtained two equilibria:

$$\begin{aligned} X_1 &= [2 \quad 0 \quad 0 \quad 0]^T && (\text{“healthy”})\,, && (7) \\ X_2 &= [1/3 \quad 5/3 \quad 0 \quad 0]^T && (\text{“AIDS”})\,. && (8) \end{aligned}$$

To find the third equilibrium, we go back to the split point (5), (6) and now assume that $w \neq 0$, i.e. (6) holds. Substituting $y = 1/(3(2x-1))$ into (1), we get $(2-x)(2x-1) = x$, or $x^2 - 2x + 1 = 0$, which gives $x = 1$ and $y = 1/3$. From (2) we get $z = 3x - 1 = 2$ and from (4) $w = z/y = 6$. Therefore, the third equilibrium is

$$X_3 = [1 \quad 1/3 \quad 6 \quad 2]^T\,, \qquad (9)$$

which corresponds to a long-term non-progressor (even though infected cells are present, memory cells and killer cells still exist and fight other infections despite the presence of HIV).

Problem 3. (8 points)

For each equilibrium of the system from Problem 1, find the linearization around that equilibrium (i.e. find F and G).

Solution: $X = [x, y, w, z]^T$, for $u = 0$ we have

$$\dot{X} = f(X,u), \quad \frac{\partial f}{\partial X} = \begin{bmatrix} -1-3y & -3x & 0 & 0 \\ 3y & 3x-1-z & 0 & -y \\ 2yw & 2xw - w & 2xy - y - 1/3 & 0 \\ 0 & w & y & -1 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 3yx \\ -3yx \\ 0 \\ 0 \end{bmatrix}$$

Now we can write for equilibrium X_i:

$$\delta \dot{X}_i = \left.\frac{\partial f}{\partial X}\right|_{X=X_i} \delta X_i + \left.\frac{\partial f}{\partial u}\right|_{X=X_i} \delta u = F_i \, \delta X_i + G_i \, \delta u, \quad i = 1, 2, 3\,,$$

where

$$F_1 = \begin{bmatrix} -1 & -6 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -6 & -1 & 0 & 0 \\ 5 & 0 & 0 & -5/3 \\ 0 & 0 & -8/9 & 0 \\ 0 & 0 & 5/3 & -1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 5/3 \\ -5/3 \\ 0 \\ 0 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} -2 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1/3 \\ 4 & 6 & 0 & 0 \\ 0 & 6 & 1/3 & -1 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Problem 4. (8 points)

Find Laplace transforms of the following functions:

(a) (4 points) $f(t) = t\mathrm{e}^{-2t}\sin(5t-15)1(t-3)$.

(b) (4 points) $f(t) = \sin(t)\cosh(3t)$.

$$\left(\text{Reminder:} \qquad \cosh(x) = \frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}; \qquad \sin(x) = \frac{\mathrm{e}^{jx}-\mathrm{e}^{-jx}}{2j} \right)$$

Solution:

(a)

$$\begin{aligned} F(s) &= -\frac{d}{ds}\left(\frac{5}{(s+2)^2+25}\mathrm{e}^{-3(s+2)}\right) \\ &= \frac{10(s+2)}{((s+2)^2+25)^2}\mathrm{e}^{-3(s+2)} + \frac{15}{(s+2)^2+25}\mathrm{e}^{-3(s+2)} \end{aligned}$$

(b)

$$\begin{aligned} f(t) &= \frac{1}{4j}(\mathrm{e}^{jt}-\mathrm{e}^{-jt})(\mathrm{e}^{3t}+\mathrm{e}^{-3t}) \\ &= \frac{1}{4j}(\mathrm{e}^{(3+j)t}+\mathrm{e}^{(j-3)t}-\mathrm{e}^{(3-j)t}-\mathrm{e}^{-(3+j)t}), \end{aligned}$$

$$\begin{aligned} F(s) &= \frac{1}{4j}\left[\frac{1}{s-3-j}+\frac{1}{s+3-j}-\frac{1}{s-3+j}-\frac{1}{s+3+j}\right] \\ &= \frac{1}{4j}\left[\frac{2j}{(s-3-j)(s-3+j)}+\frac{2j}{(s+3-j)(s+3+j)}\right] \\ &= \frac{1}{2}\left[\frac{1}{(s-3)^2+1}+\frac{1}{(s+3)^2+1}\right]. \end{aligned}$$

Problem 5. (9 points)

Let the Laplace transform of $f(t)$ be $F(s)$.

(a) (4 points) Find the Laplace transform of the following function:

$$g(t) = \int_0^t \int_0^\xi \tau \mathrm{e}^{-a\tau} f(2\tau)\, d\tau\, d\xi.$$

(b) (5 points) Using the result obtained in (a) and Final Value Theorem, find the integral

$$I = \int_0^\infty \int_0^\xi \tau f(2\tau)\, d\tau\, d\xi$$

if the signal $f(t)$ is such that

$$F(s) = \frac{s^3 + 2s^2}{s^4 + 2s^3 + 3s^2 + s + 1}.$$

(all poles of this $F(s)$ are in the left half plane).

Solution:

(a) Denote

$$h(t) = \int_0^t \tau \mathrm{e}^{-a\tau} f(2\tau)\, d\tau.$$

Using the properties of Laplace transform, we get

$$H(s) = -\frac{1}{2s}\frac{d}{ds}F((s+a)/2).$$

Since

$$g(t) = \int_0^t h(\xi)\, d\xi,$$

we obtain

$$G(s) = \frac{1}{s}H(s) = -\frac{1}{2s^2}\frac{d}{ds}F((s+a)/2)$$

(b)

$$\begin{aligned} I &= \lim_{s\to 0} s\left[-\frac{1}{2s^2}\frac{d}{ds}F(s/2)\right] \\ &= -\lim_{s\to 0}\frac{(\frac{3}{8}s^2+s)((\frac{s}{2})^4+2(\frac{s}{2})^3+3(\frac{s}{2})^2+\frac{s}{2}+1)-(\frac{s^3}{8}+\frac{s^2}{2})(\frac{1}{4}s^3+\frac{3}{4}s^2+\frac{3}{2}s+\frac{1}{2})}{2s((\frac{s}{2})^4+2(\frac{s}{2})^3+3(\frac{s}{2})^2+\frac{s}{2}+1)^2} \\ &= -\frac{1}{2} \end{aligned}$$