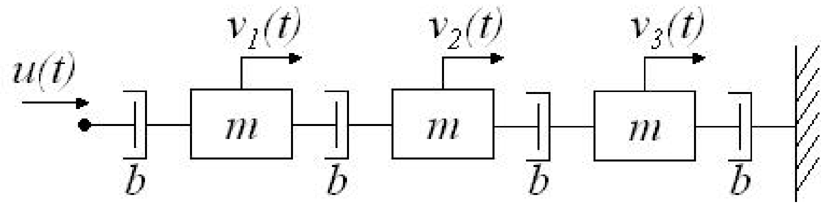
Problem 1. (7 points)

Consider the following mass-damper system:



where $u(t)$ is a forcing velocity, m is the mass of each of the three mass elements, b is the resistance of each of the dampers, and v_1, v_2, v_3 are respectively, the velocities of, the left, middle, and right mass, in the rightward reference direction. Derive the state space model with $u(t)$ as the input, $v = [v_1, v_2, v_3]^T$ as the state, and the force acting on the middle mass (in the rightward direction) as the output y.

Solution:

Applying Newton's law to the three masses, we can write three equations in terms of v_1, v_2, v_3, and u

$$\begin{aligned} m\dot{v}_1 &= b(u - v_1) + b(v_2 - v_1) \\ m\dot{v}_2 &= b(v_1 - v_2) + b(v_3 - v_2) \\ m\dot{v}_3 &= b(v_2 - v_3) - bv_3. \end{aligned}$$

We are concerned with the force applied to the middle mass therefore $y = b(v_1 - v_2) + b(v_3 - v_2)$. Rearranging the four equations we get

$$\begin{aligned} \dot{v}_1 &= -\frac{2b}{m}v_1 + \frac{b}{m}v_2 + \frac{b}{m}u \\ \dot{v}_2 &= \frac{b}{m}v_1 - \frac{2b}{m}v_2 + \frac{b}{m}v_3 \\ \dot{v}_3 &= \frac{b}{m}v_2 - \frac{2b}{m}v_3 \\ y &= bv_1 - 2bv_2 + bv_3. \end{aligned}$$

Since the system is linear it can be written as

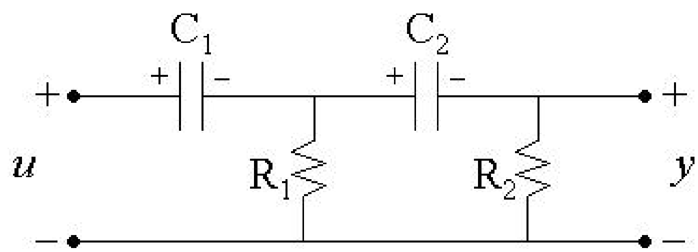
$$\begin{aligned} \dot{v} &= Fv + Gu \\ y &= Hv + Ju \end{aligned}$$

where

$$F = \frac{b}{m}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, G = \frac{b}{m}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, H = b\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}, J = [0].$$

Problem 2. (8 points)

Develop a state space model of the system below, i.e., for voltages across the capacitors as the state variables, find F, G, H, J (note that in this example J may by nonzero).



Solution:

Call v_1 and v_2 the voltages across the capacitors C_1 and C_2, respectively, according to the polarity in the figure. Applying KCL in the two nodes of the circuit, we get the following equations:

$$\begin{aligned} C_1\dot{v}_1 &= \frac{u - v_1}{R_1} + \frac{u - v_1 - v_2}{R_2} \\ C_2\dot{v}_2 &= \frac{u - v_1 - v_2}{R_2}. \end{aligned}$$

From KVL we get $y = u - v_1 - v_2$. Manipulating the above equations we get a state space representation of the system as

$$\begin{aligned} \dot{v}_1 &= -\frac{R_1 + R_2}{C_1R_1R_2}v_1 - \frac{1}{C_1R_2}v_2 + \frac{R_1 + R_2}{C_1R_1R_2}u \\ \dot{v}_2 &= -\frac{1}{C_2R_2}v_1 - \frac{1}{C_2R_2}v_2 + \frac{1}{C_2R_2}u \\ y &= -v_1 - v_2 + u. \end{aligned}$$

Since the system is linear it can be written as

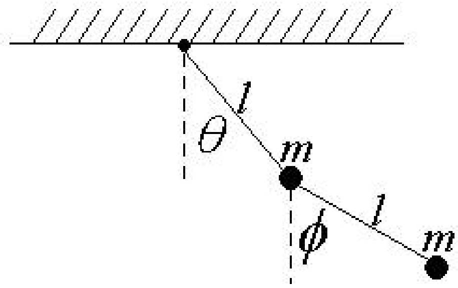
$$\begin{aligned} \dot{v} &= Fv + Gu \\ y &= Hv + Ju \end{aligned}$$

where

$$F = \begin{bmatrix} -\frac{R_1+R_2}{C_1R_1R_2} & -\frac{1}{C_1R_2} \\ -\frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix}, G = \begin{bmatrix} \frac{R_1+R_2}{C_1R_1R_2} \\ \frac{1}{C_2R_2} \end{bmatrix}, H = \begin{bmatrix} -1 & -1 \end{bmatrix}, J = [1].$$

Problem 3. (10 points)

Consider the "double pendulum" problem with two links of equal length l, two equal masses m, as shown in the figure,



and with $l = g$ (acceleration of gravity). The equation of motion can be derived as:

$$\ddot{\theta} = \frac{1}{1+\sin^2(\theta-\phi)}\left[-\dot{\theta}^2\sin(\theta-\phi)\cos(\theta-\phi) - \dot{\phi}^2\sin(\theta-\phi) + \cos\phi\sin(\phi-\theta) - \sin\theta\right]$$
$$\ddot{\phi} = \frac{1}{1+\sin^2(\theta-\phi)}\left[2\dot{\theta}^2\sin(\theta-\phi) + \dot{\phi}^2\sin(\theta-\phi)\cos(\theta-\phi) + 2\cos\theta\sin(\theta-\phi)\right]$$

Let the state vector be chosen as

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix}.$$

Verify that the following four vectors are equilibria of the system:

$$x_{dd} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_{ud} = \begin{bmatrix} \pi \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_{du} = \begin{bmatrix} 0 \\ 0 \\ \pi \\ 0 \end{bmatrix}, \quad x_{uu} = \begin{bmatrix} \pi \\ 0 \\ \pi \\ 0 \end{bmatrix}.$$

It can be shown that around each of the four equibria the system behaves approximately as

$$\ddot{\theta} \approx \cos\phi\sin(\phi-\theta) - \sin\theta$$
$$\ddot{\phi} \approx 2\cos\theta\sin(\theta-\phi).$$

Write the state space model of this nonlinear system as $\dot{x} = f(x)$. Then, find the linearization of the system around each of the equilibria, namely, find the system matrices F_{dd}, F_{ud}, F_{du}, F_{uu}.

Solution:

To verify that x_{dd}, x_{ud}, x_{du}, and x_{uu} are equilibria simply plug them in to the equations of motion and show that $\ddot{\theta} = \ddot{\phi} = 0$.

Using the approximate motion equations we can write the nonlinear state space system in terms of x

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \cos x_3 \sin(x_3 - x_1) - \sin x_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= 2\cos x_1 \sin(x_1 - x_3).
\end{aligned}$$

To linearize about the equilibria first we need to compute the Jacobian:

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\cos x_3 \cos(x_3 - x_1) - \cos(x_1) & 0 & -\sin x_3 \sin(x_3 - x_1) + \cos x_3 \cos(x_3 - x_1) & 0 \\ 0 & 0 & 0 & 1 \\ -2\sin x_1 \sin(x_1 - x_3) + 2\cos x_1 \cos(x_1 - x_3) & 0 & -2\cos x_1 \cos(x_1 - x_3) & 0 \end{bmatrix}$$

For small displacements δx near the equilibria, the system can be approximated as the linear system $\delta\dot{x} = F\delta x$, where

$$F = \tfrac{\partial f(x)}{\partial x}|_{x = x_{equ}} .$$

From this we get

$$F_{dd} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}, F_{ud} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix},$$

$$F_{du} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 2 & 0 \end{bmatrix}, F_{uu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 2 & 0 \end{bmatrix}.$$