Problem 1. (7 points)

Consider system

$$\begin{aligned} \ddot{\theta} + \theta - \theta^2 &= \sin u \\ \dot{\zeta} + \zeta &= \dot{\theta} + (\zeta + \theta)u\,. \end{aligned}$$

(This system does not come from any physical application but its structure and its nonlinear terms mimic phenomena that arise in mechanical and bio-chemical systems.) Treating θ as the output and u as the input, derive a state space representation of the system.

$x_1 = \theta, \quad x_2 = \dot{\theta}, \quad x_3 = \zeta$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + x_1^2 + \sin u$$

$$\dot{x}_3 = x_2 - x_3 + (x_1 + x_3)u$$

Problem 2. (5 points)

For the system in Problem 1, let $u = 0$ and find all the equilibrium points of the state space system.

$$\dot{x}_1 = x_2 = 0 \longrightarrow x_2 = 0$$

$$\dot{x}_2 = -x_1 + x_1^2 = 0 \longrightarrow x_1 = 0 \text{ and } x_1 = 1$$

$$\dot{x}_3 = x_2 - x_3 = 0 \longrightarrow x_3 = 0$$

Equilibria:

$$E_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Problem 3. (8 points)

At the equilibria found in Problem 2, calculate the linearization (in the state space form, i.e., find F, G, H, J) of the system from Problem 1.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ 2x_1 - 1 & 0 & 0 \\ u & 1 & u-1 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \cos u \\ x_1 + x_3 \end{bmatrix}$$

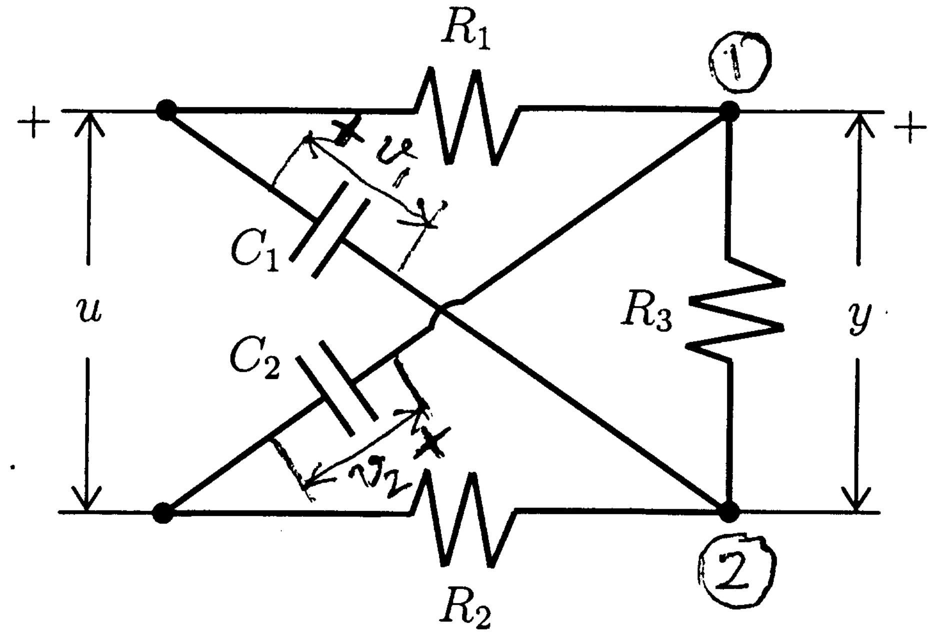
$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 \\ \phi \\ 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad J = 0$$

Problem 4. (10 points)

Derive the state space model for the following circuit



(The wiggle in the branch C_2 where it crosses the branch C_1 means the branches have no contact—one branch passes "over" or "under" the other.)

KVL: $u = v_1 - y + v_2 \longrightarrow y = v_1 + v_2 - u$

$\longrightarrow H = [1 \; 1], \; J = -1$

KCL:

Currents through node 1:

$$\frac{u - v_2}{R_1} = C_2 \dot{v}_2 + \frac{y}{R_3} \qquad (A)$$

Currents through node 2:

$$\frac{u - v_1}{R_2} = C_1 \dot{v}_1 + \frac{y}{R_3} \qquad (B)$$

Substituting $y = v_1 + v_2 - u$ into equations (A) and (B) and solving for $\dot{v}_1, \dot{v}_2$, we get

$$\dot{v}_1 = \frac{1}{C_1}\left[-\left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_1 - \frac{1}{R_3} v_2 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right) u \right)$$

$$\dot{v}_2 = \frac{1}{C_2}\left(-\frac{1}{R_3} v_1 - \left(\frac{1}{R_1} + \frac{1}{R_3}\right) v_2 + \left(\frac{1}{R_1} + \frac{1}{R_3}\right) u \right)$$

so

$$F = \begin{bmatrix} -\dfrac{R_2 + R_3}{C_1 R_2 R_3} & -\dfrac{1}{C_1 R_3} \\[2ex] -\dfrac{1}{C_2 R_3} & -\dfrac{R_1 + R_3}{C_2 R_1 R_3} \end{bmatrix}$$

$$G = \begin{bmatrix} \dfrac{R_2 + R_3}{C_1 R_2 R_3} \\[2ex] \dfrac{R_1 + R_3}{C_2 R_1 R_3} \end{bmatrix}$$