Problem 1. (15 points)

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_1(x_1+x_2)\\ \dot{x}_2 &= -x_3+4+x_2-x_2^2-x_1x_2\\ \dot{x}_3 &= x_2+1-\left(\frac{1}{2}+u\right)x_3\end{aligned}$$

Denote the state vector as $X=[x_1,x_2,x_3]^T$. For $u=0$, find all the equilibria of the system. Then find the linearization at all those equilibria.

(This model is inspired by a model of the rotating stall instability in axial flow compressors that are used in gas turbine/jet engines. The variables, (very) roughly, are: x_3-pressure rise, x_2-mean flow through compressor, x_1-the amplitude of the rotating stall instability, and u-control through downstream valve/throttle.)

Solution: Equilibria of the system:

$$\begin{aligned}&x_1(x_1+x_2)=0\\ &-x_3+4+x_2-x_2^2-x_1x_2=0\\ &x_2+1-\frac{x_3}{2}=0\end{aligned}$$

From the top equation we have:

1) $x_1=0 \quad\Rightarrow\quad x_2^2+x_2-2=0 \quad\Rightarrow\quad x_2=1$ and $x_2=-2 \quad\Rightarrow\quad x_3=4$ and $x_3=-2$

2) $x_1+x_2=0 \quad\Rightarrow\quad x_2-2=0 \quad\Rightarrow\quad x_2=2 \quad\Rightarrow\quad x_3=6,\;\; x_1=-2$

So, we have 3 equilibria: $X_1=[0\quad 1\quad 4]^T$, $X_2=[0\ -2\ -2]^T$, $X_3=[-2\quad 2\quad 6]^T$.

$$\dot{X}=f(X,u),\qquad \frac{\partial f}{\partial X}=\begin{bmatrix}2x_1+x_2 & x_1 & 0\\ -x_2 & 1-x_1-2x_2 & -1\\ 0 & 1 & -1/2-u\end{bmatrix},\qquad \frac{\partial f}{\partial u}=\begin{bmatrix}0\\0\\-x_3\end{bmatrix}$$

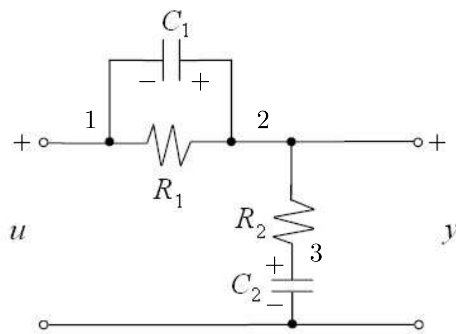
Now we can write for equilibrium X_i: $\delta\dot{X}_i=\frac{\partial f}{\partial X}\big|_{X=X_i}\delta X_i+\frac{\partial f}{\partial u}\big|_{X=X_i}\delta u=F_i\,\delta X_i+G_i\,\delta u$, $i=1,2,3$.

$$F_1=\begin{bmatrix}1&0&0\\-1&-1&-1\\0&1&-1/2\end{bmatrix},\quad F_2=\begin{bmatrix}-2&0&0\\2&5&-1\\0&1&-1/2\end{bmatrix},\quad F_3=\begin{bmatrix}-2&-2&0\\-2&-1&-1\\0&1&-1/2\end{bmatrix}$$

$$G_1=\begin{bmatrix}0\\0\\-4\end{bmatrix},\quad G_2=\begin{bmatrix}0\\0\\2\end{bmatrix},\quad G_3=\begin{bmatrix}0\\0\\-6\end{bmatrix}$$

Problem 2. (10 points)

Consider the circuit



Find the state representation for this system.

Solution:

$$\begin{aligned} &\text{KVL at node 2:} && y = v_{C_1} + u \\ &\text{KCL at node 2:} && C_1 \frac{dv_{C_1}}{dt} + \frac{v_{C_1}}{R_1} + \frac{y - v_{C_2}}{R_2} = 0 \\ &\text{KCL at node 3:} && C_2 \frac{dv_{C_2}}{dt} - \frac{y - v_{C_2}}{R_2} = 0 \end{aligned}$$

We get

$$C_1 \dot{v}_{C_1} + \frac{1}{R_1} v_{C_1} + \frac{1}{R_2}(v_{C_1} + u - v_{C_2}) = 0$$

$$C_2 \dot{v}_{C_2} + \frac{1}{R_2}(v_{C_2} - v_{C_1} - u) = 0$$

$$y = v_{C_1} + u$$

$$\dot{\begin{bmatrix} v_{C_1} \\ v_{C_2} \end{bmatrix}} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_{C_1} \\ v_{C_2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_{C_1} \\ v_{C_2} \end{bmatrix} + u$$