MAE 143A
SIGNALS AND SYSTEMS — Prof. M. Krstic

Solutions for Homework 4

Problem 1. Find the $\mathcal{Z}$ transform of the functions

(a) $y_k = \left(\frac{2}{3}\right)^{k-2} 1_k$

(b) $y_k = \left(\frac{-1}{2}\right)^{k+2} 1_{k-1}$

(c) $y_k = k\left(\frac{1}{2}\right)^{2k} 1_{k-1}$

Solution:

(a)

$$y_k = \left(\frac{2}{3}\right)^{k-2} 1_k = \frac{9}{4}\left(\frac{2}{3}\right)^{k} 1_k$$

$$Y(z) = \frac{9}{4}\frac{z}{z-2/3} = \frac{27z}{12z-8}$$

(b)

$$y_k = \left(\frac{-1}{2}\right)^{k+2} 1_{k-1} = -\frac{1}{8}\left(\frac{-1}{2}\right)^{k-1} 1_{k-1}$$

$$Y(z) = -\frac{1}{8}z^{-1}\frac{z}{z+1/2} = -\frac{1}{8z+4}$$

(c)

$$y_k = k\left(\frac{1}{2}\right)^{2k} 1_{k-1} = k\left(\frac{1}{4}\right)^{k} 1_{k-1} = k\left(\frac{1}{4}\right)^{k} 1_k$$

$$Y(z) = \frac{z/4}{(z-1/4)^2} = \frac{4z}{(4z-1)^2}$$

Problem 2. Compute the *pulse* response of the following systems:

(a) $H(z) = \dfrac{z^{-3}}{z^2-1}$

(b) $H(z) = \dfrac{1}{z^2-4z+8}$

Solution:

(a)

$$H(z) = \frac{z^{-3}}{(z-1)(z+1)} = z^{-3}\left(C_1\frac{1}{z-1} + C_2\frac{1}{z+1}\right)$$

$$C_1 = \left.\frac{1}{z+1}\right|_{z=1} = \frac{1}{2}, \qquad C_2 = \left.\frac{1}{z-1}\right|_{z=-1} = -\frac{1}{2}$$

$$H(z) = \frac{1}{2}z^{-4}\left(\frac{z}{z-1} - \frac{z}{z+1}\right)$$

$$h_k = \frac{1}{2}(1-(-1)^k)1_{k-4}$$

(b) Recall that

$$\mathcal{Z}\left\{a^k \sin(\omega k)\right\} = \frac{az\sin(\omega)}{z^2 - 2az\cos(\omega) + a^2}$$

We have $a = \sqrt{8}$, and $2a\cos(\omega) = 4$, so that $\cos(\omega) = 1/\sqrt{2}$ and $\omega = \pi/4$.

$$H(z) = \frac{1}{z^2-4z+8} = \frac{z^{-1}}{a\sin(\omega)}\frac{az\sin(\omega)}{z^2-4z+8} = \frac{z^{-1}}{2}\mathcal{Z}\left\{(\sqrt{8})^k \sin\left(\frac{\pi}{4}k\right)\right\}$$

$$h_k = \frac{1}{2}(\sqrt{8})^{k-1}\sin\left(\frac{\pi}{4}(k-1)\right)1_{k-1}$$

Problem 3. Solve the difference equation

$$8y_{k+2} + 2y_{k+1} - y_k = 0, \quad y_0 = 3,\ y_1 = 2\,.$$

Solution:

$$8(z^2Y(z) - zy_1 - z^2y_0) + 2(zY(z) - zy_0) - Y(z) = 0$$

$$(8z^2+2z-1)Y(z) = 24z^2 + 22z$$

$$Y(z) = \frac{24z^2+22z}{8z^2+2z-1} = \frac{24z^2+22z}{8(z+\frac{1}{2})(z-\frac{1}{4})} = \frac{3z^2+\frac{11}{4}z}{(z+\frac{1}{2})(z-\frac{1}{4})} = z\left(\frac{C_1}{z+\frac{1}{2}} + \frac{C_2}{z-\frac{1}{4}}\right)$$

$$\begin{aligned} C_1 &= \left(z+\frac{1}{2}\right)\frac{1}{z}Y(z)|_{z=-\frac{1}{2}} = \frac{-\frac{3}{2}+\frac{11}{4}}{-\frac{1}{2}-\frac{1}{4}} = -\frac{5}{3} \\ C_2 &= \left(z-\frac{1}{4}\right)\frac{1}{z}Y(z)|_{z=\frac{1}{4}} = \frac{\frac{3}{4}+\frac{11}{4}}{\frac{1}{4}+\frac{1}{2}} = \frac{14}{3} \end{aligned}$$

$$Y(z) = \frac{-\frac{5}{3}z}{z+\frac{1}{2}} + \frac{\frac{14}{3}z}{z-\frac{1}{4}}$$

$$y_k = -\frac{5}{3}\left(-\frac{1}{2}\right)^k 1_k + \frac{14}{3}\left(\frac{1}{4}\right)^k 1_k$$

Problem 4. Find the solution of the discrete system

$$y_{k+2} + \frac{1}{4}y_k = u_k\,.$$

The input $u_k = 3(\frac{1}{2})^k - 2\delta_k$ for $k = 0, 1, 2, \ldots$ and initial conditions are $y_0 = 2, y_1 = 1$.

Solution:

$$U(z) = \frac{3z}{z-1/2} - 2$$

$$z^2Y(z) - z^2y_0 - zy_1 + \frac{1}{4}Y(z) = U(z)$$

$$\begin{aligned} Y(z) &= \frac{2z^2+z}{z^2+1/4} + \frac{3z}{(z-1/2)(z^2+1/4)} - \frac{2}{z^2+1/4} \\ &= \frac{2z^2+z-2}{z^2+1/4} + z\cdot\frac{3}{(z-1/2)(z^2+1/4)} \end{aligned}$$

$$\frac{3}{(z-1/2)(z^2+1/4)} = \frac{A}{z-1/2} + \frac{Bz+C}{z^2+1/4}$$

$$A = \left.\frac{3}{z^2+1/4}\right|_{z=1/2} = 6$$

$$6(z^2+1/4) + (Bz+C)(z-1/2) = 3, \quad \Rightarrow \quad B = -6\,, \;\; C = -3$$

$$\begin{aligned} Y(z) &= \frac{2z^2+z-2}{z^2+1/4} + \frac{6z}{z-1/2} - \frac{6z^2}{z^2+1/4} - \frac{3z}{z^2+1/4} \\ &= \frac{6z}{z-1/2} + \frac{-4z^2-2z-2}{z^2+1/4} = \frac{6z}{z-1/2} + \frac{4z^2-2z}{z^2+1/4} - 8 \end{aligned}$$

$$\begin{aligned} y_k &= 6\left(\frac{1}{2}\right)^k + 4\left(\frac{1}{2}\right)^k \cos\left(\frac{\pi}{2}k\right) - 4\left(\frac{1}{2}\right)^k \sin\left(\frac{\pi}{2}k\right) - 8\delta_k \\ &= -8\delta_k + \left(\frac{1}{2}\right)^{k-2}\left[\frac{3}{2} + \cos\left(\frac{\pi}{2}k\right) - \sin\left(\frac{\pi}{2}k\right)\right] \end{aligned}$$