MAE 143A

SIGNALS AND SYSTEMS — Prof. M. Krstic

Solutions for Homework 3

Problem 1. Compute the *impulse* response of the following systems:

(a) $H(s) = \mathrm{e}^{-5s}\dfrac{1}{(s+4)^2+2}$

(b) $H(s) = \dfrac{1}{(2s+3)^5}$

Solution:

(a) Impulse response is given by $\mathcal{L}^{-1}\{H(s)\}$. Since

$$H(s) = \frac{1}{\sqrt{2}}\mathrm{e}^{-5s}\frac{\sqrt{2}}{(s+4)^2+\sqrt{2}^2}$$

and

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+4)^2+\sqrt{2}^2}\right\} = \mathrm{e}^{-4t}\sin\left(\sqrt{2}t\right),$$

we get

$$\mathcal{L}^{-1}\{H(s)\} = \frac{1}{\sqrt{2}}\mathrm{e}^{-4(t-5)}\sin\left(\sqrt{2}(t-5)\right)1(t-5).$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{1}{(2s+3)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{32(s+3/2)^5}\right\} = \frac{1}{32}\cdot\frac{t^4}{4!}\mathrm{e}^{-\frac{3}{2}t}.$$

Problem 2. Compute the *step* response of the following systems:

(a) $H(s) = 2\dfrac{s^2+s+1}{s^2+3s+2}$

(b) $H(s) = \dfrac{1}{(s+1)^3}$

(c) $H(s) = \dfrac{2}{(s+1)(s^2+4)}$

Solution: (a)

$$F(s) = \frac{1}{s}H(s) = 2\frac{s^2+s+1}{s(s^2+3s+2)} = 2\frac{s^2+s+1}{s(s+1)(s+2)}$$

$$F(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+2}$$

$$C_1 = 2\frac{s^2+s+1}{(s+1)(s+2)}\bigg|_{s=0} = 1, \quad C_2 = 2\frac{s^2+s+1}{s(s+2)}\bigg|_{s=-1} = \frac{2}{-1} = -2$$

$$C_3 = 2\frac{s^2+s+1}{s(s+1)}\bigg|_{s=-2} = \frac{6}{(-2)(-1)} = 3$$

$$F(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{3}{s+2}$$

$$f(t) = \left(1 - 2e^{-t} + 3e^{-2t}\right)1(t)\,.$$

(b)

$$F(s) = \frac{1}{s}H(s) = \frac{1}{s(s+1)^3}$$

$$F(s) = \frac{A}{s} + \frac{B}{s} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$A = \frac{1}{(s+1)^3}\bigg|_{s=0} = 1, \quad D = \frac{1}{s}\bigg|_{s=-1} = -1$$

$$C = \frac{d}{ds}\left(\frac{1}{s}\right)\bigg|_{s=-1} = -\frac{1}{s^2}\bigg|_{s=-1} = -1$$

$$B = \frac{1}{2}\frac{d^2}{ds^2}\left(\frac{1}{s}\right)\bigg|_{s=-1} = \frac{1}{2}\left(\frac{2}{s^3}\right)\bigg|_{s=-1} = -1$$

$$F(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}$$

$$f(t) = \left(1 - e^{-t} - te^{-t} - \frac{t^2}{2}e^{-t}\right)1(t)$$

(c)

$$F(s) = \frac{1}{s}H(s) = \frac{2}{s(s+1)(s^2+4)},$$

and doing partial fraction decomposition,

$$\frac{2}{s(s+1)(s^2+4)} = \frac{2}{s(s+1)(s+j2)(s-j2)} = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+j2} + \frac{C_4}{s-j2}.$$

Now,

$$\begin{aligned}
C_1 &= s\frac{H(s)}{s}\bigg|_{s=0} = \frac{2}{4} = \frac{1}{2}, \\
C_2 &= (s+1)\frac{H(s)}{s}\bigg|_{s=-1} = \frac{2}{(-1)5} = \frac{-2}{5}, \\
C_3 &= (s+j2)\frac{H(s)}{s}\bigg|_{s=-j2} = \frac{2}{-j2(-j2+1)(-j4)} = \frac{-1}{4(1-j2)}, \\
C_4 &= (s-j2)\frac{H(s)}{s}\bigg|_{s=j2} = \frac{2}{j2(j2+1)(j4)} = \frac{-1}{4(1+j2)} = \bar{C}_3.
\end{aligned}$$

Then,

$$\frac{2}{s(s+1)(s^2+4)} = \frac{1}{2}\frac{1}{s} - \frac{2}{5}\frac{1}{s+1} - \frac{1}{4(1+j2)}\frac{1}{s+j2} - \frac{1}{4(1+j2)}\frac{1}{s-j2}.$$

Working on the last two fractions, we have that

$$\begin{aligned}
-\frac{1}{4(1-j2)}\frac{1}{s+j2} - \frac{1}{4(1+j2)}\frac{1}{s-j2} &= -\frac{(1+j2)(s-j2)+(1-j2)(s+j2)}{4(1+j2)(1-j2)(s+j2)(s-j2)} = -\frac{s+4}{10(s^2+4)} \\
&= -\frac{1}{10}\left(\frac{s}{s^2+4} + 2\frac{2}{s^2+4}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s} - \frac{2}{5}\frac{1}{s+1} - \frac{1}{10}\left(\frac{s}{s^2+4} + 2\frac{2}{s^2+4}\right)\right\} \\
&= \left(\frac{1}{2} - \frac{2}{5}\mathrm{e}^{-t} - \frac{1}{10}\left(\cos(2t) + 2\sin(2t)\right)\right)1(t) \\
&= \left(\frac{1}{2} - \frac{2}{5}\mathrm{e}^{-t} - \frac{\sqrt{5}}{10}\sin\left(2t + \arctan\left(\frac{1}{2}\right)\right)\right)1(t)
\end{aligned}$$

A maybe easier alternative, using matching coefficients in the partial fraction decomposition,

$$\frac{2}{s(s+1)(s^2+4)} = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{As+B}{s^2+4}.$$

As before, $C_1 = \frac{1}{2}$ and $C_2 = \frac{-2}{5}$. Then, expanding the fraction:

$$\begin{aligned}
\frac{2}{s(s+1)(s^2+4)} &= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{As+B}{s^2+4} \\
&= \frac{C_1(s^3+s^2+4s+4) + C_2(s^3+4s) + (As+B)(s^2+s)}{s(s+1)(s^2+4)} \\
&= \frac{s^3(C_1+C_2+A) + s^2(C_1+A+B) + s(4C_1+4C_2+B) + 4C_1}{s(s+1)(s^2+4)},
\end{aligned}$$

hence,

$$\begin{aligned} C_1 + C_2 + A &= 0, \\ C_1 + A + B &= 0, \\ 4C_1 + 4C_2 + B &= 0, \\ 4C_1 &= 2, \end{aligned}$$

from where we get $C_1 = 1/2$. Then, $A = -C_1 - C_2 = -1/2 + 2/5 = -1/10$, $B = -A - C_1 = 1/10 - 1/2 = -4/10$, and the third equation is verified thus ensuring that our solution is correct. We get

$$\frac{2}{s(s+1)(s^2+4)} = \frac{1}{2}\frac{1}{s} - \frac{2}{5}\frac{1}{s+1} - \frac{1}{10}\left(\frac{s}{s^2+4} + \frac{4}{s^2+4}\right)$$

as before.

Problem 3. For the system with a transfer function

$$H(s) = \frac{s+1}{3s^2+7} - \frac{1}{3s+2}$$

find the differential equation governing the relationship between the input $u(t)$ and the output $y(t)$, assuming zero initial conditions.

Solution:

$$Y(s) = H(s)U(s)$$

$$(3s^2+7)(3s+2)Y(s) = [(s+1)(3s+2) - (3s^2+7)]U(s)$$

$$(9s^3 + 6s^2 + 21s + 14)Y(s) = (5s - 5)U(s)$$

$$9\dddot{y} + 6\ddot{y} + 21\dot{y} + 14y = 5\dot{u} - 5u$$

Problem 4.

(a) Let the persistent forcing signal $u(t) = \sin(2t)1(t)$ drive the system

$$Y(s) = \frac{2s^2+8}{s(s^2+2s+15)}U(s)\,.$$

Does this system, despite persistent forcing, reach a steady state? If so, what is $\lim_{t\to\infty} y(t)$?

(b) Same questions as in (a) for the system

$$Y(s) = \frac{2s^2+8}{s(s^2+2s-15)}U(s)\,.$$

Solution: (a) Since $U(s) = \frac{2}{s^2+4}$,

$$Y(s) = \frac{2s^2+8}{s(s^2+2s+15)}U(s) = \frac{4}{s(s^2+2s+15)} = \frac{4}{s((s+1)^2+14)},$$

and then $sY(s) = \frac{4}{(s+1)^2+14}$ has all poles in the LHP, so the FVT can be applied and

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{4}{1^2+14} = \frac{4}{15}.$$

(b)

$$Y(s) = \frac{2s^2+8}{s(s^2+2s-15)}U(s) = \frac{4}{s(s^2+2s-15)} = \frac{4}{s(s+5)(s-3)},$$

and then $sY(s) = \frac{4}{(s+5)(s-3)}$ has a pole in the RHP, so the FVT cannot be applied: the system does not have a steady state (actually, it diverges to infinity).

Problem 5. Using the Laplace transform, solve the following differential equations:

(a) $\dddot{y} + 2\ddot{y} + 2\dot{y} = 0, \quad y(0) = 1,\ \dot{y}(0) = 2,\ \ddot{y}(0) = 3.$

(b) $\ddot{y} + 2\dot{y} + 4y = (1-t)\mathrm{e}^{-2t}, \quad y(0) = 1,\ \dot{y}(0) = 1.$

Solution:
(a) Applying the Laplace transform, we get

$$s^3Y(s) - s^2y(0) - s\dot{y}(0) - \ddot{y}(0) + 2(s^2Y(s) - sy(0) - \dot{y}(0)) + 2(sY(s) - y(0)) = 0$$

$$Y(s) = \frac{s^2+4s+9}{s(s^2+2s+2)} = \frac{s^2+4s+9}{s((s+1)^2+1)} = \frac{C_1}{s} + \frac{C_2s+C_3}{(s+1)^2+1}$$

$$C_1 = \left.\frac{s^2+4s+9}{(s+1)^2+1}\right|_{s=0} = \frac{9}{2}$$

$$\frac{9}{2}(s^2+2s+2) + C_2s^2 + C_3s = s^2+4s+9 \quad \Rightarrow \quad C_2 = -\frac{7}{2},\ C_3 = -5$$

$$Y(s) = \frac{9/2}{s} + \frac{(-7/2)s-5}{(s+1)^2+1} = \frac{9/2}{s} - \frac{7/2(s+1)}{(s+1)^2+1} - \frac{3/2}{(s+1)^2+1}$$

$$y(t) = \frac{9}{2} - \left(\frac{7}{2}\cos t + \frac{3}{2}\sin t\right)e^{-t}.$$

(b)

$$\mathcal{L}\left\{(1-t)e^{-2t}\right\} = \frac{1}{s+2} - \frac{1}{(s+2)^2} = \frac{s+1}{(s+2)^2}$$

Applying the Laplace transform to the ODE we get

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2(sY(s) - y(0)) + 4Y(s) = \frac{s+1}{(s+2)^2}$$

$$(s^2+2s+4)Y(s) = \frac{s+1}{(s+2)^2} + s + 3$$

$$Y(s) = \frac{s+1}{(s^2+2s+4)(s+2)^2} + \frac{s+3}{s^2+2s+4}$$

$$\frac{s+1}{(s^2+2s+4)(s+2)^2} = \frac{C_1}{s+2} + \frac{C_2}{(s+2)^2} + \frac{C_3 s + C_4}{s^2+2s+4}$$

$$\begin{aligned} C_1 &= \frac{d}{ds}\left(\frac{s+1}{s^2+2s+4}\right)\bigg|_{s=-2} = \frac{s^2+2s+4-(s+1)(2s+2)}{(s^2+2s+4)^2}\bigg|_{s=-2} = \frac{1}{8} \\ C_2 &= \frac{s+1}{s^2+2s+4}\bigg|_{s=-2} = -\frac{1}{4} \end{aligned}$$

Matching coefficients gives

$$\frac{1}{8}(s+2)(s^2+2s+4) - \frac{1}{4}(s^2+2s+4) + (C_3 s + C_4)(s+2)^2 = s+1$$

$$\frac{1}{8} + C_3 = 0, \quad 4C_4 = 1 \quad \Rightarrow \quad C_3 = -\frac{1}{8}, \quad C_4 = \frac{1}{4}$$

Going back to $Y(s)$,

$$\begin{aligned} Y(s) &= \frac{\frac{1}{8}}{s+2} + \frac{-\frac{1}{4}}{(s+2)^2} + \frac{-\frac{1}{8}s + \frac{1}{4}}{s^2+2s+4} + \frac{s+3}{s^2+2s+4} \\ &= \frac{\frac{1}{8}}{s+2} + \frac{-\frac{1}{4}}{(s+2)^2} + \frac{\frac{7}{8}s + \frac{13}{4}}{(s+1)^2+3} \\ &= \frac{\frac{1}{8}}{s+2} + \frac{-\frac{1}{4}}{(s+2)^2} + \frac{\frac{7}{8}(s+1) + \frac{19}{8}}{(s+1)^2+3} \end{aligned}$$

$$y(t) = \frac{1}{8}e^{-t}\left[7\cos\left(\sqrt{3}t\right) + \frac{19}{\sqrt{3}}\sin\left(\sqrt{3}t\right)\right] + \frac{1}{8}(1-2t)e^{-2t}$$