MAE 143A

SIGNALS AND SYSTEMS — Prof. M. Krstic

Homework 2

Problem 1. Find the Laplace transform of the following functions:

(a) $f(t) = 5e^{-2t} - 3(t+1)^7,\ \ t \geq 0.$

(b) $f(t) = t\sin(t+2), \quad t \geq 0.$

(c) $f(t) = \sinh(t)\cos(t), \quad t \geq 0.$

Solution:

(a) $f(t) = 5e^{-2t} - 3(t^7 + 7t^6 + 21t^5 + 35t^4 + 35t^3 + 21t^2 + 7t + 1),$

$$F(s) = \frac{5}{s+2} - 3\left(\frac{7!}{s^8} + 7\frac{6!}{s^7} + 21\frac{5!}{s^6} + 35\frac{4!}{s^5} + 35\frac{3!}{s^4} + 21\frac{2!}{s^3} + \frac{7}{s^2} + \frac{1}{s}\right)$$

(b) $f(t) = t(\sin(t)\cos(2) + \cos(t)\sin(2)),$

$$F(s) = -\frac{d}{ds}\left(\frac{\cos(2)}{s^2+1} + \frac{\sin(2)s}{s^2+1}\right) = \frac{2s\cos(2)}{(s^2+1)^2} + \frac{\sin(2)(s^2-1)}{(s^2+1)^2}$$

(c)

$$f(t) = \frac{1}{4}\left(e^t - e^{-t}\right)\left(e^{jt} + e^{-jt}\right) = \frac{1}{4}\left(e^{t(1+j)} - e^{-t(1-j)} - e^{-t(j+1)} + e^{t(1-j)}\right)$$

$$\begin{aligned} F(s) &= \frac{1}{4}\left(\frac{1}{s-1-j} - \frac{1}{s+1-j} - \frac{1}{s+1+j} + \frac{1}{s+j-1}\right) \\ &= \frac{1}{4}\left(\frac{2s-2}{(s-1)^2+1} - \frac{2s+2}{(s+1)^2+1}\right) = \frac{s^2-2}{(s^2+2)^2-4s^2} \end{aligned}$$

Problem 2. Using the theorem about the Laplace transform of a derivative of a time function, find the Laplace transforms of the functions

$$\begin{aligned} f(t) &= te^{-at}\sin(bt), \quad t \geq 0 \\ f(t) &= te^{-at}\cos(bt), \quad t \geq 0, \end{aligned}$$

where $a > 0$ and $b > 0$ are constants.

Solution:

(a)

$$F(s) = -\frac{d}{ds}\left(\mathcal{L}\left\{e^{-at}\sin(bt)\right\}\right) = -\frac{d}{ds}\left(\frac{b}{(s+a)^2+b^2}\right) = \frac{2b(s+a)}{((s+a)^2+b^2)^2}$$

(b)

$$F(s) = -\frac{d}{ds}\left(\mathcal{L}\left\{e^{-at}\cos(bt)\right\}\right) = -\frac{d}{ds}\left(\frac{s+a}{(s+a)^2+b^2}\right) = \frac{(s+a)^2-b^2}{((s+a)^2+b^2)^2}$$

Problem 3. Find the Laplace transform of the following function:

$$f(t) = \frac{e^{-t}}{t}(\sin \omega t)^2, \quad t \geq 0\,.$$

Hints: (a) Use the identity $tg(t) \mapsto -\frac{d}{ds}G(s)$.

(b) Use the Initial Value Theorem to determine a constant of integration.

Solution:

$$\begin{aligned}
tf(t) &= e^{-t}\sin^2(\omega t)\\
-\frac{d}{ds}F(s) &= \mathcal{L}\left\{e^{-t}\left(\frac{1}{2}-\frac{1}{2}\cos(2\omega t)\right)\right\}\\
F(s) &= -\frac{1}{2}\int \frac{1}{s+1}\,ds + \frac{1}{2}\int \frac{s+1}{(s+1)^2+4\omega^2}\,ds + C\\
&= -\frac{1}{2}\log(s+1) + \frac{1}{4}\log\left((s+1)^2+4\omega^2\right) + C\\
&= \frac{1}{4}\log\left(1+\frac{4\omega^2}{(s+1)^2}\right) + C\,.
\end{aligned}$$

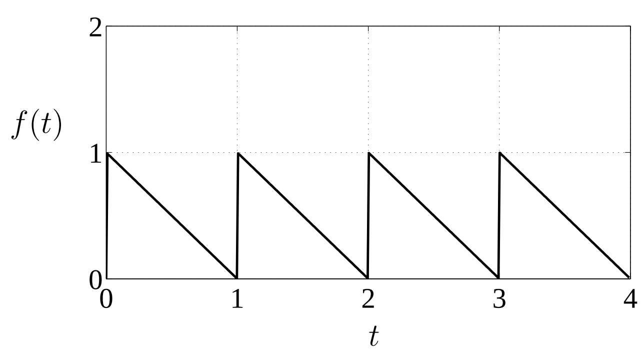
Since $f(0)=0$, by Initial Value Theorem we have

$$\begin{aligned}
&\lim_{s\to\infty} sF(s) = 0\\
&\lim_{s\to\infty} s\left(\frac{1}{4}\log\left(1+\frac{4\omega^2}{(s+1)^2}\right)+C\right) = \lim_{s\to\infty} sC = 0,
\end{aligned}$$

from which it follows that $C=0$. Finally,

$$F(s) = \frac{1}{4}\log\left(1+\frac{4\omega^2}{(s+1)^2}\right).$$

Problem 4. Find the Laplace transform of the following function of time:



This periodic signal is turned on at $t=0$ ($f(t)\equiv 0$ for $t<0$) and stays on for all $t>0$ (even though it is only shown up to $t=4$ in this figure).

This problem is given to show you that finding Laplace transforms symbolically in MATLAB is not always an option (after solving this problem you can be proud that you did something MATLAB can't do).

Hints: (a) Represent the function as a sum of delayed step functions.

(b) Use the formula $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for any $-1 < q < 1$.

Solution:

$$f(t) = (1-t)1(t) + 1(t-1) + 1(t-2) + \ldots$$

$$\begin{aligned} F(s) &= \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \ldots \\ &= -\frac{1}{s^2} + \frac{1}{s}\left(1 + e^{-s} + e^{-2s} + \ldots\right) = -\frac{1}{s^2} + \frac{1}{s(1-e^{-s})} \end{aligned}$$