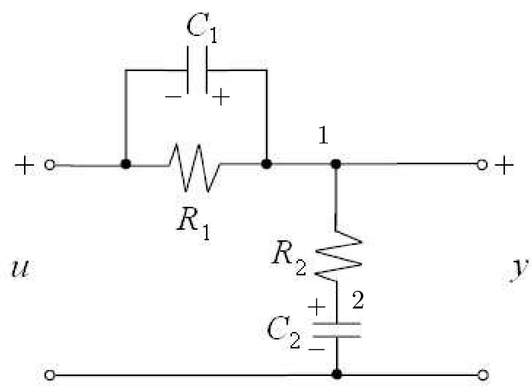
MAE 143A

SIGNALS AND SYSTEMS — Prof. M. Krstic

Solutions for Homework 1

Problem 1. Derive the state space model for each of the following circuits, using voltages on the capacitors as the state variables:

(a)



Solution:

KVL at node 1: $y = v_1 + u$

KCL at node 1: $C_1 \frac{dv_1}{dt} + \frac{v_1}{R_1} + \frac{y - v_2}{R_2} = 0$

KCL at node 2: $C_2 \frac{dv_2}{dt} - \frac{y - v_2}{R_2} = 0$

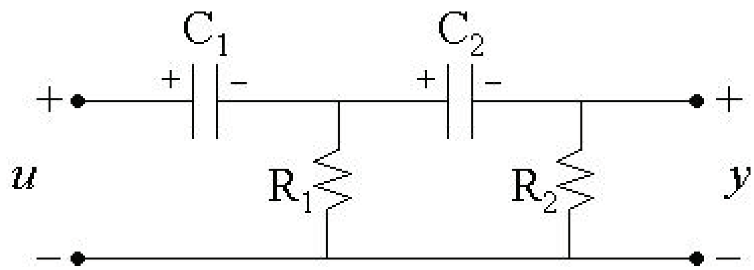
We get

$$C_1 \dot{v}_1 + \frac{1}{R_1} v_1 + \frac{1}{R_2}(v_1 + u - v_2) = 0$$
$$C_2 \dot{v}_2 + \frac{1}{R_2}(v_2 - v_1 - u) = 0$$
$$y = v_1 + u$$

$$\begin{bmatrix} \dot{v}_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + u$$

(b)



Solution:

Applying KCL in the two nodes of the circuit, we get the following equations:

$$\begin{aligned} C_1\dot{v}_1 &= \frac{u-v_1}{R_1} + \frac{u-v_1-v_2}{R_2} \\ C_2\dot{v}_2 &= \frac{u-v_1-v_2}{R_2}\,. \end{aligned}$$

From KVL we get $y = u - v_1 - v_2$. Manipulating the above equations we get a state space representation of the system:

$$\begin{aligned} \dot{v}_1 &= -\frac{R_1+R_2}{C_1R_1R_2}v_1 - \frac{1}{C_1R_2}v_2 + \frac{R_1+R_2}{C_1R_1R_2}u \\ \dot{v}_2 &= -\frac{1}{C_2R_2}v_1 - \frac{1}{C_2R_2}v_2 + \frac{1}{C_2R_2}u \\ y &= -v_1 - v_2 + u. \end{aligned}$$

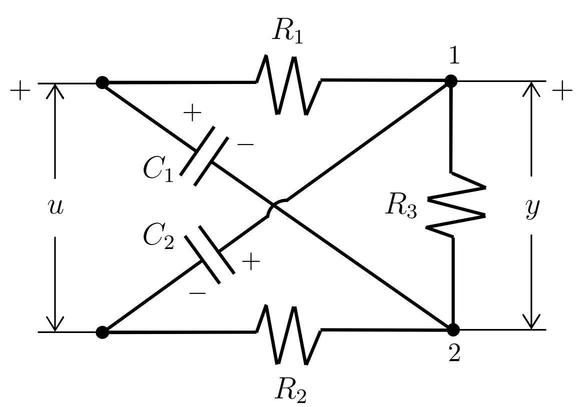
Since the system is linear it can be written as

$$\begin{aligned} \dot{v} &= Fv + Gu \\ y &= Hv + Ju \end{aligned}$$

where

$$F = \begin{bmatrix} -\frac{R_1+R_2}{C_1R_1R_2} & -\frac{1}{C_1R_2} \\ -\frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix},\ G = \begin{bmatrix} \frac{R_1+R_2}{C_1R_1R_2} \\ \frac{1}{C_2R_2} \end{bmatrix},\ H = [\text{-1}\ \ \text{-1}],\ J = 1\,.$$

(c)



Solution: KVL: $u = v_1 - y + v_2$, therefore $y = v_1 + v_2 - u$, $H = [1 \;\; 1]$, $J = -1$.

Currents through node 1:

$$\frac{u - v_2}{R_1} = C_2 \dot{v}_2 + \frac{y}{R_3}$$

Currents through node 2:

$$\frac{u - v_1}{R_2} = C_1 \dot{v}_1 + \frac{y}{R_3}$$

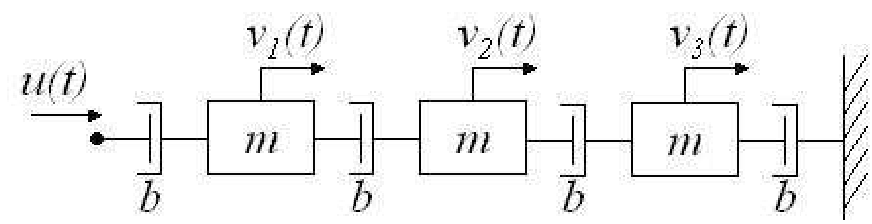
Using the expression for y, from the last two equations we get

$$\dot{v}_1 = \frac{1}{C_1}\left[-\left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_1 - \frac{1}{R_3} v_2 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right) u\right]$$
$$\dot{v}_2 = \frac{1}{C_2}\left[-\frac{1}{R_3} v_1 - \left(\frac{1}{R_1} + \frac{1}{R_3}\right) v_2 + \left(\frac{1}{R_1} + \frac{1}{R_3}\right) u\right]$$

Therefore,

$$F = \begin{bmatrix} -\dfrac{R_2 + R_3}{C_1 R_2 R_3} & -\dfrac{1}{C_1 R_3} \\ \\ -\dfrac{1}{C_2 R_3} & -\dfrac{R_1 + R_3}{C_2 R_1 R_3} \end{bmatrix}, \quad G = \begin{bmatrix} \dfrac{R_2 + R_3}{C_1 R_2 R_3} \\ \\ \dfrac{R_1 + R_3}{C_2 R_1 R_3} \end{bmatrix}.$$

Problem 2. Consider the following mass-damper system



where $u(t)$ is a forcing velocity, m is the mass of each of the three mass elements, b is the resistance of each of the dampers, and v_1, v_2, v_3 are, respectively, the velocities of the left, middle, and right mass, in the rightward reference direction. Derive the state space model with $u(t)$ as the input, $v = [v_1, v_2, v_3]^T$ as the state, and the force acting on the middle mass (in the rightward direction) as the output y.

Solution: Applying Newton's law to the three masses, we can write three equations in terms of v_1, v_2, v_3, and u

$$m\dot{v}_1 = b(u - v_1) + b(v_2 - v_1)$$
$$m\dot{v}_2 = b(v_1 - v_2) + b(v_3 - v_2)$$
$$m\dot{v}_3 = b(v_2 - v_3) - bv_3\,.$$

We are concerned with the force applied to the middle mass, therefore $y = b(v_1 - v_2) + b(v_3 - v_2)$. Rearranging the four equations we get

$$\begin{aligned}
\dot{v}_1 &= -\frac{2b}{m}v_1 + \frac{b}{m}v_2 + \frac{b}{m}u \\
\dot{v}_2 &= \frac{b}{m}v_1 - \frac{2b}{m}v_2 + \frac{b}{m}v_3 \\
\dot{v}_3 &= \frac{b}{m}v_2 - \frac{2b}{m}v_3 \\
y &= bv_1 - 2bv_2 + bv_3 \,.
\end{aligned}$$

Since the system is linear, it can be written as

$$\begin{aligned}
\dot{v} &= Fv + Gu \\
y &= Hv + Ju
\end{aligned}$$

where

$$F = \frac{b}{m}\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \; G = \frac{b}{m}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; H = b[1 \;\; \text{-2} \;\; 1], \; J = 0\,.$$

Problem 3. Consider the system

$$\begin{aligned}
\ddot{\theta} + \theta - \theta^2 &= \sin u \\
\dot{\zeta} + \zeta &= \dot{\theta} + (\zeta + \theta)u
\end{aligned}$$

(This system does not come from any physical application but its structure and its nonlinear terms mimic phenomena that arise in mechanical and bio-chemaical systems.)

(a) Treating θ as the output and u as the input, derive a state space representation of the system

(b) For $u = 0$, find all the equilibria of the system.

(c) For each equilibrium, find the linearization around that equilibrium.

Solution: (a) Denote $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = \zeta$, then we have

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + x_1^2 + \sin u \\
\dot{x}_3 &= x_2 - x_3 + (x_1 + x_3)u
\end{aligned}$$

(b)

$$\begin{aligned}
&\dot{x}_1 = x_2 = 0 \quad \Rightarrow \quad x_2 = 0 \\
&\dot{x}_2 = -x_1 + x_1^2 = 0 \quad \Rightarrow \quad x_1 = 0 \text{ and } x_1 = 1 \\
&\dot{x}_3 = x_2 - x_3 = 0 \quad \Rightarrow \quad x_3 = 0
\end{aligned}$$

Equilibria:

$$E_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(c)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ 2x_1 - 1 & 0 & 0 \\ u & 1 & u-1 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ \cos u \\ x_1 + x_3 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$H = [1 \ \ 0 \ \ 0], \quad J = 0$$

Problem 4. One of the state-of-the-art HIV models is (Wodarz'99 with specific values of the parameters)

$$\begin{aligned} \dot{x} &= 2 - x - 3xy(1-u) \\ \dot{y} &= -y - zy + 3xy(1-u) \\ \dot{w} &= 2xyw - yw - \frac{1}{3}w \\ \dot{z} &= yw - z\,, \end{aligned}$$

where u is the input (drug concentration) and x, y, w, and z are concentrations of healthy cells, infected cells, memory cells, and killer cells, respectively.

For $u = 0$ (no treatment), there exist 3 equilibria in this system. One corresponds to a healthy person, another one corresponds to a person with AIDS, and the third one corresponds to a long-term non-progressor, i.e. a person with HIV who never develops AIDS.

(a) Find the three equilibria.

(b) For each equilibrium, find the linearization around that equilibrium.

Solution:

(a) Equilibria of the system for $u = 0$ are determined from the equations

$$2 - x - 3xy = 0 \tag{1}$$

$$-y - zy + 3xy = 0 \tag{2}$$

$$2xyw - yw - \frac{1}{3}w = 0 \tag{3}$$

$$yw - z = 0 \tag{4}$$

From (3) we have either

$$w = 0 \tag{5}$$

or

$$(2x-1)y = \frac{1}{3} \tag{6}$$

Let us start with (5). From (4) we get $z = 0$. Substituting $z = 0$ into (2) gives $(3x-1)y = 0$, i.e. either $y = 0$ or $x = 1/3$. If $y = 0$, then from (1) $x = 2$. If $x = 1/3$, then from (1) $y = 5/3$. Therefore, we obtained two equilibria:

$$X_1 = [2 \quad 0 \quad 0 \quad 0]^T \qquad (\text{"healthy"}), \tag{7}$$

$$X_2 = [1/3 \quad 5/3 \quad 0 \quad 0]^T \qquad (\text{"AIDS"}). \tag{8}$$

To find the third equilibrium, we go back to the split point (5), (6) and now assume that $w \neq 0$, i.e. (6) holds. Substituting $y = 1/(3(2x-1))$ into (1), we get $(2-x)(2x-1) = x$, or $x^2 - 2x + 1 = 0$, which gives $x = 1$ and $y = 1/3$. From (2) we get $z = 3x - 1 = 2$ and from (4) $w = z/y = 6$. Therefore, the third equilibrium is

$$X_3 = [1 \quad 1/3 \quad 6 \quad 2]^T, \tag{9}$$

which corresponds to a long-term non-progressor (even though infected cells are present, memory cells and killer cells still exist and fight other infections despite the presence of HIV).

(b) $X = [x, y, w, z]^T$, for $u = 0$ we have

$$\dot{X} = f(X, u), \quad \frac{\partial f}{\partial X} = \begin{bmatrix} -1-3y & -3x & 0 & 0 \\ 3y & 3x-1-z & 0 & -y \\ 2yw & 2xw - w & 2xy - y - 1/3 & 0 \\ 0 & w & y & -1 \end{bmatrix}, \quad \frac{\partial f}{\partial u} = \begin{bmatrix} 3yx \\ -3yx \\ 0 \\ 0 \end{bmatrix}$$

Now we can write for equilibrium X_i:

$$\delta \dot{X}_i = \left.\frac{\partial f}{\partial X}\right|_{X=X_i} \delta X_i + \left.\frac{\partial f}{\partial u}\right|_{X=X_i} \delta u = F_i\, \delta X_i + G_i\, \delta u, \quad i = 1, 2, 3,$$

where

$$F_1 = \begin{bmatrix} -1 & -6 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -6 & -1 & 0 & 0 \\ 5 & 0 & 0 & -5/3 \\ 0 & 0 & -8/9 & 0 \\ 0 & 0 & 5/3 & -1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 5/3 \\ -5/3 \\ 0 \\ 0 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} -2 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1/3 \\ 4 & 6 & 0 & 0 \\ 0 & 6 & 1/3 & -1 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$