Problem 1. (8 points)

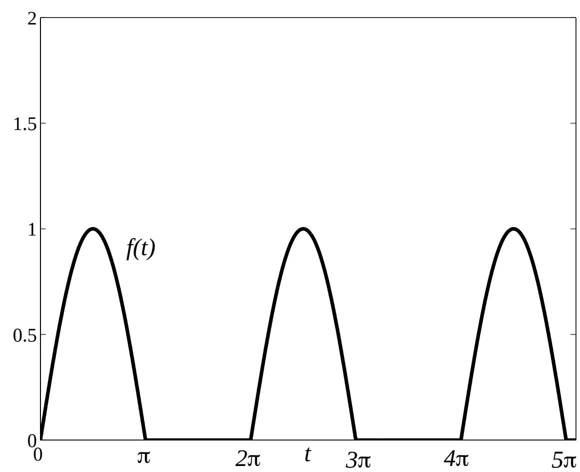
The time-delay rule allows computation of Laplace transforms of many signals of engineering interesting. As an illustration, you are asked to compute the Laplace Transform of the following functions, which appear as the output of basic rectifier circuits.

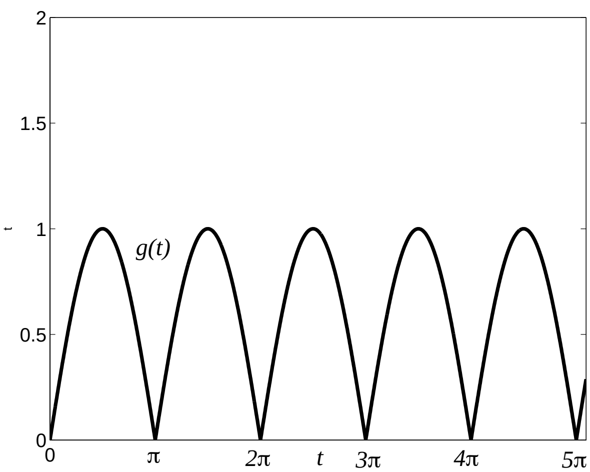
(a) (4 points) The function $f(t) = \max\{\sin(t), 0\}1(t)$, i.e., the function that is equal to $\sin(t)$ whenever it is positive and zero otherwise, as shown in the left figure.

Hint: Notice that you can write the equation $f(t) - f(t-\pi) = \sin(t)1(t)$. Apply to both sides of this equation the Laplace Transform and solve for $F(s)$.

(b) (4 points) The function $g(t) = \text{abs}(\sin(t)1(t)$, i.e., the absolute value of the function $\sin(t)1(t)$ as shown in the right figure.

Hint: As in (a), try to write $g(t)$ as the sum of $f(t)$ and a properly modified version of $f(t)$, then take Laplace Transform and use the solution of (a) to find the solution for $G(s)$.





Solution:

(a) We start from the equation

$$f(t) - f(t-\pi) = \sin(t)1(t).$$

Applying the Laplace Transform to both sides, we get

$$F(s) - \mathrm{e}^{-\pi s}F(s) = \frac{1}{s^2+1}.$$

Solving for $F(s)$:

$$F(s) = \frac{1}{1-\mathrm{e}^{-\pi s}}\frac{1}{s^2+1}.$$

(b) It is clear that $g(t)$ can be written as

$$g(t) = f(t) + f(t-\pi).$$

Applying the Laplace Transform to both sides, we get

$$G(s) = F(s) + \mathrm{e}^{-\pi s} F(s).$$

Solving for $G(s)$:

$$G(s) = \left(1 + \mathrm{e}^{-\pi s}\right) F(s).$$

Plugging $F(s)$ from (a) into the above equation,

$$G(s) = \left(\frac{1+\mathrm{e}^{-\pi s}}{1-\mathrm{e}^{-\pi s}}\right)\frac{1}{s^2+1}.$$

Note (not required in the exam): using the fact that $\tanh\left(\frac{\pi}{2}s\right) = \left(\frac{1-\mathrm{e}^{-\pi s}}{1+\mathrm{e}^{-\pi s}}\right)$, we can write

$$G(s) = \frac{1}{\tanh\left(\frac{\pi}{2}s\right)}\frac{1}{s^2+1}.$$

That's the way it's shown in tables.

Some people tried to solve the problem using infinite series (the hard way, in fact nobody succeeded). For this, note that

$$\begin{aligned} f(t) &= \sin(t)1(t) + \sin(t-\pi)1(t-\pi) + \sin(t-2\pi)1(t-2\pi) + ... \\ &= \sum_{i=0}^{\infty} \sin(t-i\pi)1(t-i\pi). \end{aligned}$$

Then,

$$\begin{aligned} F(s) &= \sum_{i=0}^{\infty} \frac{\mathrm{e}^{-i\pi s}}{s^2+1} \\ &= \frac{1}{s^2+1}\sum_{i=0}^{\infty}\left(\mathrm{e}^{-\pi s}\right)^i \\ &= \frac{1}{s^2+1}\frac{1}{1-\mathrm{e}^{-\pi s}}, \end{aligned}$$

using the sum of the geometric series. Similarly,

$$\begin{aligned} f(t) &= \sin(t)1(t) + 2\sin(t-\pi)1(t-\pi) + 2\sin(t-2\pi)1(t-2\pi) + ... \\ &= \sin(t)1(t) + 2\sum_{i=1}^{\infty} \sin(t-i\pi)1(t-i\pi). \end{aligned}$$

Then,

$$
\begin{aligned}
F(s) &= \frac{1}{s^2+1} + 2\sum_{i=1}^{\infty} \frac{\mathrm{e}^{-i\pi s}}{s^2+1} \\
&= \frac{1}{s^2+1} + \frac{2}{s^2+1} \sum_{i=1}^{\infty} \left(\mathrm{e}^{-\pi s}\right)^i \\
&= \frac{1}{s^2+1} + \frac{2\mathrm{e}^{-\pi s}}{s^2+1} \sum_{i=0}^{\infty} \left(\mathrm{e}^{-\pi s}\right)^i \\
&= \frac{1}{s^2+1} \frac{1+\mathrm{e}^{-\pi s}}{1-\mathrm{e}^{-\pi s}}.
\end{aligned}
$$

Problem 2. (8 points)

The Laplace Transform is not only useful for solving ODE's and in control design, but can also be used to easily compute difficult improper integrals that sometimes appear in applications. This problem illustrates how to do it.

(a) (4 points) Compute the Inverse Laplace Transform of the function $F(s) = \arctan\left(\frac{a}{s}\right)$.
Hint: carefully compute $F'(s)$ using the chain rule and remembering $\arctan(x)' = \frac{1}{1+x^2}$. Then remember that $\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$.

(b) (4 points) Use the solution of (a) to find the value of the following integral, which appears in sampling theory.

$$\int_0^\infty \frac{\sin(\tau)}{\tau} d\tau.$$

Hint: write $g(t) = \int_0^t \frac{\sin(\tau)}{\tau} d\tau$. Try to find $G(s)$ from the solution of (a) and remembering that $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$; then apply the Final Value Theorem, remembering that $\tan(\pi/2) = \infty$. **NOTE**: When applying the Final Value Theorem in this problem, you don't need to verify the hypothesis (stability of $sG(s)$).

Solution:

(a) Computing the derivative of $F(s)$, we get

$$F'(s) = \frac{1}{1 + \left(\frac{a}{s}\right)^2} \frac{-a}{s^2} = -\frac{a}{s^2 + a^2}.$$

Taking Inverse Laplace Transform and using the fact that $\mathcal{L}^{-1}\{F'(s)\} = -tf(t)$, we get

$$tf(t) = \sin(at)1(t),$$

hence

$$f(t) = \frac{\sin(at)}{t} 1(t).$$

(b) Writing $g(t) = \int_0^t \frac{\sin(\tau)}{\tau} d\tau$, then, using that $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$ and the solution of (a) we get that

$$G(s) = \frac{\arctan\left(\frac{1}{s}\right)}{s}.$$

Applying the Final Value Theorem,

$$\int_0^\infty \frac{\sin(\tau)}{\tau} d\tau = \lim_{t\to\infty} g(t) = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \arctan\left(\frac{1}{s}\right),$$

and since $\tan(\pi/2) = \infty$, then $\arctan(\infty) = \pi/2$, so that

$$\int_0^\infty \frac{\sin(\tau)}{\tau} d\tau = \lim_{s\to 0} \arctan\left(\frac{1}{s}\right) = \arctan(\infty) = \pi/2.$$

Problem 3. (7 points)

Compute the *impulse* response of the following system.

$$H(s) = \mathrm{e}^{-4s} \frac{3}{s^2 + 6s + 13}$$

Solution: To compute the impulse response, we just need to compute $\mathcal{L}^{-1}\{H(s)\}$. Now,

$$\frac{3}{s^2 + 6s + 13} = 3\frac{1}{(s+3)^2 + 4} = \frac{3}{2}\frac{2}{(s+3)^2 + 2^2},$$

hence

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 6s + 13}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2 + 2^2}\right\} = \frac{3}{2}\mathrm{e}^{-3t}\sin(2t)1(t),$$

and then, using the time delay property, the impulse response of the system is

$$\mathcal{L}^{-1}\{H(s)\} = \frac{3}{2}\mathrm{e}^{-3(t-4)}\sin(2(t-4))1(t-4),$$

which can be written as (not required in the exam):

$$\frac{3\mathrm{e}^{12}}{2}\mathrm{e}^{-3t}\sin(2t-8)1(t-4),$$

Problem 4. (8 points)

Compute the *step* response of the following system.

$$H(s) = \frac{2}{(s+1)(s^2+4)}$$

Solution: To compute the step response, we just need to compute $\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$. Now,

$$\frac{H(s)}{s} = \frac{2}{s(s+1)(s^2+4)},$$

and doing partial fraction decomposition,

$$\frac{2}{s(s+1)(s^2+4)} = \frac{2}{s(s+1)(s+j2)(s-j2)} = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+j2} + \frac{C_4}{s-j2}.$$

Now,

$$\begin{aligned}
C_1 &= \left(s\frac{H(s)}{s}\right)_{s=0} = \frac{2}{4} = \frac{1}{2}, \\
C_2 &= \left((s+1)\frac{H(s)}{s}\right)_{s=-1} = \frac{2}{(-1)5} = \frac{-2}{5}, \\
C_3 &= \left((s+j2)\frac{H(s)}{s}\right)_{s=-j2} = \frac{2}{-j2(-j2+1)(-j4)} = \frac{-1}{4(1-j2)}, \\
C_4 &= \left((s-j2)\frac{H(s)}{s}\right)_{s=j2} = \frac{2}{j2(j2+1)(j4)} = \frac{-1}{4(1+j2)} = \bar{C}_3.
\end{aligned}$$

Then,

$$\frac{2}{s(s+1)(s^2+4)} = \frac{1}{2}\frac{1}{s} - \frac{2}{5}\frac{1}{s+1} - \frac{1}{4(1+j2)}\frac{1}{s+j2} - \frac{1}{4(1+j2)}\frac{1}{s-j2}.$$

Working on the last two fractions, we have that

$$\begin{aligned}
-\frac{1}{4(1-j2)}\frac{1}{s+j2} - \frac{1}{4(1+j2)}\frac{1}{s-j2} &= -\frac{(1+j2)(s-j2)+(1-j2)(s+j2)}{4(1+j2)(1-j2)(s+j2)(s-j2)} = -\frac{s+4}{10(s^2+4)} \\
&= -\frac{1}{10}\left(\frac{s}{s^2+4} + 2\frac{2}{s^2+4}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s} - \frac{2}{5}\frac{1}{s+1} - \frac{1}{10}\left(\frac{s}{s^2+4} + 2\frac{2}{s^2+4}\right)\right\} \\
&= \left(\frac{1}{2} - \frac{2}{5}\mathrm{e}^{-t} - \frac{1}{10}(\cos(4t) + 2\sin(4t))\right)1(t) \\
&= \left(\frac{1}{2} - \frac{2}{5}\mathrm{e}^{-t} - \frac{\sqrt{5}}{10}\sin(4t + \arctan(\frac{1}{2}))\right)1(t)
\end{aligned}$$

The last step (expressing the sine+cosine as a single sine) is not required in the exam, but is useful to get a compact and simple expression.

A maybe easier alternative, using matching coefficients in the partial fraction decomposition,

$$\frac{2}{s(s+1)(s^2+4)} = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{As+B}{s^2+4}.$$

As before, $C_1 = \frac{1}{2}$ and $C_2 = \frac{-2}{5}$. Then, expanding the fraction:

$$\begin{aligned}
\frac{2}{s(s+1)(s^2+4)} &= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{As+B}{s^2+4} \\
&= \frac{C_1(s^3+s^2+4s+4) + C_2(s^3+4s) + (As+B)(s^2+s)}{s(s+1)(s^2+4)} \\
&= \frac{s^3(C_1+C_2+A) + s^2(C_1+A+B) + s(4C_1+4C_2+B) + 4C_1}{s(s+1)(s^2+4)},
\end{aligned}$$

hence,

$$\begin{aligned}
C_1 + C_2 + A &= 0, \\
C_1 + A + B &= 0, \\
4C_1 + 4C_2 + B &= 0, \\
4C_1 &= 2,
\end{aligned}$$

from where we get $C_1 = 1/2$ as we knew. Then, $A = -C_1 - C_2 = -1/2 + 2/5 = -1/10$, $B = -A - C_1 = 1/10 - 1/2 = -4/10$, and the third equation is verified thus ensuring that our solution is correct. We get

$$\frac{2}{s(s+1)(s^2+4)} = \frac{1}{2}\frac{1}{s} - \frac{2}{5}\frac{1}{s+1} - \frac{1}{10}\left(\frac{s}{s^2+4} + \frac{4}{s^2+4}\right)$$

as before.

Problem 5. (8 points)

(a) (4 points) Let the persistent forcing signal $u(t) = \sin(2t)1(t)$ drive the system

$$Y(s) = \frac{2s^2 + 8}{s(s^2 + 2s + 15)} U(s).$$

Does this system, despite persistent forcing, reach a steady state? If so, what is $\lim_{t\to\infty} y(t)$?

(b) (4 points) Same questions as in (a) for the system

$$Y(s) = \frac{2s^2 + 8}{s(s^2 + 2s - 15)} U(s).$$

Solution: Since $U(s) = \frac{2}{s^2+4}$, then

(a)

$$Y(s) = \frac{2s^2 + 8}{s(s^2 + 2s + 15)} U(s) = \frac{4}{s(s^2 + 2s + 15)} = \frac{4}{s((s+1)^2 + 14)},$$

and then $sY(s) = \frac{4}{(s+1)^2+14}$ has all poles in the LHP, so the FVT can be applied and

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{4}{1^2 + 14} = \frac{4}{15}.$$

(b)

$$Y(s) = \frac{2s^2 + 8}{s(s^2 + 2s - 15)} U(s) = \frac{4}{s(s^2 + 2s - 15)} = \frac{4}{s(s+5)(s-3)},$$

and then $sY(s) = \frac{4}{(s+5)(s-3)}$ has a pole in the RHP, so the FVT cannot be applied: the system does not have a steady state (actually, it diverges to infinity).

Problem 6. (10 points)

Solve the differential equation

$$\ddot{y} + 5\dot{y} + 4y = \mathrm{e}^{-t}$$

for initial conditions $y(0) = 1$, $\dot{y}(0) = -5$.

Solution: Transforming the differential equation to the Laplace domain, we get

$$s^2 Y(s) - \dot{y}(0) - sy(0) + 5sY(s) - 5y(0) + 4Y(s) = \frac{1}{s+1},$$

and substituting $\dot{y}(0)$ and $y(0)$, and solving for $Y(s)$:

$$Y(s) = \frac{1 + s(s+1)}{(s+1)(s^2+5s+4)} = \frac{s^2+s+1}{(s+1)^2(s+4)} = \frac{C_1}{(s+1)^2} + \frac{C_2}{s+1} + \frac{C_3}{s+4},$$

where

$$\begin{aligned}
C_1 &= \left((s+1)^2 Y(s)\right)_{s=-1} = \frac{1}{3}, \\
C_2 &= \left(\frac{d}{ds}\left((s+1)^2 Y(s)\right)\right)_{s=-1} = \left(\frac{d}{ds}\left(\frac{s^2+s+1}{s+4}\right)\right)_{s=-1} \\
&= \left(\frac{(s+4)(2s+1) - (s^2+s+1)}{(s+4)^2}\right)_{s=-1} = \left(\frac{s^2+8s+3}{(s+4)^2}\right)_{s=-1} \\
&= \frac{1-8+3}{3^2} = -\frac{4}{9}, \\
C_3 &= \left((s+4)Y(s)\right)_{s=-4} = \frac{13}{(-3)^2} = \frac{13}{9},
\end{aligned}$$

hence

$$Y(s) = \frac{1}{3}\frac{1}{(s+1)^2} - \frac{4}{9}\frac{1}{s+1} + \frac{13}{9}\frac{1}{s+4},$$

and

$$y(t) = \left(\frac{1}{3}\mathrm{e}^{-t}t - \frac{4}{9}\mathrm{e}^{-t} + \frac{13}{9}\mathrm{e}^{-4t}\right) 1(t).$$

Note that this solution verifies the initial conditions and differential equation, hence we know our solution is right.

Problem 7. (9 points)

Find the $\mathcal{Z}$ transform of the functions

(a) (3 points) $y_k = \left(\frac{1}{2}\right)^{k+2} 1_k$.

(b) (3 points) $y_k = \left(\frac{-1}{2}\right)^{k-1} 1_{k-1}$.

(c) (3 points) $y_k = (-1)^k k 5^{k+1} 1_{k-1}$.

Solution:

(a)

$$y_k = \left(\frac{1}{2}\right)^{k+2} 1_k = \left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{k} 1_k = \frac{1}{4}\left(\frac{1}{2}\right)^{k} 1_k$$

hence

$$Y(z) = \frac{1}{4}\frac{z}{z-\frac{1}{2}} = \frac{z}{4z-2}$$

(b)

$$y_k = \left(\frac{-1}{2}\right)^{k-1} 1_{k-1}$$

is a version of

$$y_k = \left(\frac{-1}{2}\right)^{k} 1_k$$

delayed one time unit, hence

$$Y(z) = \frac{1}{z}\frac{z}{z+\frac{1}{2}} = \frac{2}{2z+1}$$

(c)

$$\begin{aligned} y_k &= (-1)^k k 5^{k+1} 1_{k-1} = (-1)(5^2)(-1)^{k-1}5^{k-1}(k-1+1)1_{k-1} \\ &= -25\left((-5)^{k-1}(k-1) + (-5)^{k-1}\right)1_{k-1}, \end{aligned}$$

which is a version of

$$y_k = -25\left((-5)^k k + (-5)^k\right)1_k$$

delayed one time unit, hence

$$Y(z) = -25\frac{1}{z}\left(-\frac{5z}{(z+5)^2} + \frac{z}{z+5}\right) = -25\left(\frac{-5}{(z+5)^2} + \frac{1}{z+5}\right) = -\frac{25z}{(z+5)^2}.$$

Another way to reach the solution: if you realize that $k1_{k-1} = k1_k$ (because at $k=0$ the value is zero in both cases and for $k \geq 1$ the value is the same), then

$$y_k = (-1)^k k 5^{k+1} 1_{k-1} = 5(-5)^k k 1_k,$$

whose transform is

$$Y(z) = 5\frac{-5z}{(z+5)^2} = -\frac{25z}{(z+5)^2}.$$

Problem 8. (9 points)

For the discrete time system with a transfer function

$$H(z) = \frac{2z-1}{z^2+1}$$

(a) (4 points) Find the difference equation governing the relationship between the input $u(k)$ and the output $y(k)$, assuming zero initial conditions.

(b) (5 points) For the input $u(k) = \left(\frac{1}{2}\right)^{k-1} 1_{k-1}$, and zero initial conditions, i.e., $y_0 = 0$, $y_1 = 0$, find the output $y(k)$.

Solution:

(a) Since

$$Y(z) = \frac{2z-1}{z^2+1} U(z),$$

then

$$(z^2+1)Y(z) = (2z-1)U(z),$$

hence,

$$y_{k+2} + y_k = 2u_{k+1} - u_k.$$

(b) Since $u(k) = \left(\frac{1}{2}\right)^{k-1} 1_{k-1}$, then

$$U(z) = \frac{1}{z}\frac{z}{z-\frac{1}{2}} = \frac{2}{2z-1}.$$

Plugging $U(z)$ into the relationship

$$Y(z) = \frac{2z-1}{z^2+1} U(z),$$

we get

$$Y(z) = \frac{2z-1}{z^2+1}\frac{2}{2z-1} = \frac{2}{z^2+1} = 2\frac{1}{z}\frac{z}{z^2+1}$$

hence

$$y_k = 2\sin\left(\frac{\pi}{2}(k-1)\right) 1_{k-1}.$$

Alternatively we can write

$$Y(z) = \frac{2}{z^2+1} = 2\frac{1+z^2-z^2}{z^2+1} = 2 - \frac{z^2}{z^2+1},$$

so

$$y_k = 2\left(\delta_k - \cos\left(\frac{\pi}{2}k\right) 1_k\right),$$

an expression equivalent to the previous one.

Problem 9. (8 points)

Consider the following discrete autonomous system,

$$y_{k+2} = \frac{y_{k+1} + y_k}{2},$$

with initial conditions $y_0 = 1$, $y_1 = 0$. Using the $\mathcal{Z}$ transform, write the solution for y_k and compute y_{100}.

Solution: Applying the $\mathcal{Z}$ transform, we get

$$z^2 Y(z) - z y_1 - z^2 y_0 = \frac{zY(z) - z y_0 + Y(z)}{2},$$

so

$$\left(z^2 - \frac{z+1}{2}\right) Y(z) = z^2 + \frac{-z}{2},$$

and solving for $Y(z)$,

$$Y(z) = \frac{z^2 - \frac{z}{2}}{z^2 - \frac{z}{2} - \frac{1}{2}} = z \frac{z - \frac{1}{2}}{(z-1)\left(z + \frac{1}{2}\right)}.$$

Using partial fraction decomposition,

$$Y(z) = z\left(\frac{C_1}{z-1} + \frac{C_2}{z + \frac{1}{2}}\right),$$

where

$$C_1 = \left((z-1)\frac{z - \frac{1}{2}}{(z-1)\left(z + \frac{1}{2}\right)}\right)_{z=1} = \left(\frac{z - \frac{1}{2}}{z + \frac{1}{2}}\right)_{z=1} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$$

and

$$C_1 = \left((z + \frac{1}{2})\frac{z - \frac{1}{2}}{(z-1)\left(z + \frac{1}{2}\right)}\right)_{z=-1/2} = \left(\frac{z - \frac{1}{2}}{z - 1}\right)_{z=-1/2} = \frac{-1}{-\frac{3}{2}} = \frac{2}{3}.$$

Hence,

$$Y(z) = z\left(\frac{1/3}{z-1} + \frac{2/3}{z + \frac{1}{2}}\right) = \frac{1}{3}\left(\frac{z}{z-1} + 2\frac{z}{z + \frac{1}{2}}\right),$$

and

$$y_k = \frac{1}{3}\left(1 + 2\left(-\frac{1}{2}\right)^k\right) 1_k = \frac{1}{3}\left(1 + \frac{(-1)^k}{2^{k-1}}\right) 1_k.$$

Plugging $k = 100$ into the expression for y_k,

$$y_{100} = \frac{1}{3}\left(1 + \frac{(-1)^{100}}{2^{99}}\right) 1_{100} \approx \frac{1}{3}.$$