MAE 143A
SIGNALS AND SYSTEMS

Prof. M. Krstic

FINAL EXAM

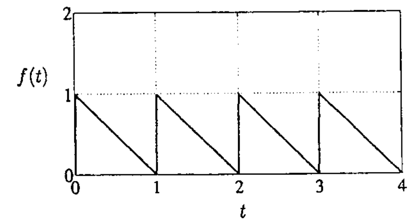
July 27, 2005

NAME: Solutions

- Open books and notes.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 60.
- Time: 5:00–7:50pm

Problem 1. (9 points)

Find the Laplace transform of the following function of time:



This periodic signal is turned on at $t = 0$ ($f(t) \equiv 0$ for $t < 0$) and stays on for all $t > 0$ (even though it is only shown up to $t = 4$ in this figure).

This problem is given to show you that finding Laplace transforms symbolically in MATLAB is not always an option (after solving this problem you can be proud that you did something MATLAB can't do).

Hints: (a) Represent $f(t)$ as a sum of a negative ramp function and delayed step functions.

(b) Use the formula $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for any $-1 < q < 1$.

$$f(t) = (1-t)\,1(t) + 1(t-1) + 1(t-2) + \dots$$

$$F(s) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} + \dots$$

$$= -\frac{1}{s^2} + \frac{1}{s}\left(1 + e^{-s} + e^{-2s} + \dots\right)$$

$$= -\frac{1}{s^2} + \frac{1}{s}\cdot\frac{1}{1-e^{-s}}$$

Problem 2 (10 points)

Find the Laplace transform of the following function:

$$f(t) = \frac{e^{-t}}{t}(\sin \omega t)^2$$

Hints: (a) Use the identity $tg(t) \mapsto -\frac{d}{ds}G(s)$.

(b) $\cos 2\theta = 1 - 2(\sin\theta)^2$

(c) Use the Initial Value Theorem to determine a constant of integration.

$$t f(t) = e^{-t}\sin^2 \omega t$$

$$-\frac{d}{ds}F(s) = \mathcal{L}\left\{e^{-t}\left(\frac{1}{2} - \frac{1}{2}\cos 2\omega t\right)\right\}$$

$$F(s) = -\int \frac{1}{2(s+1)}\,ds + \int \frac{s+1}{2((s+1)^2+4\omega^2)}\,ds + C$$

$$= -\frac{1}{2}\log(s+1) + \frac{1}{4}\log((s+1)^2+4\omega^2) + C$$

$$= \frac{1}{4}\log\left(1 + \frac{4\omega^2}{(s+1)^2}\right) + C$$

Since $f(0) = 0$, by IVT

we have $\lim_{s\to\infty} sF(s) = 0$

so that $F(\infty) = 0$.

$$F(\infty) = \frac{1}{4}\log 1 + C = C = 0$$

Finally $F(s) = \frac{1}{4}\log\left(1 + \frac{4\omega^2}{(s+1)^2}\right)$

Problem 3. (5 points)

Find the *impulse* response of the system

$$H(s) = e^{-5s}\frac{1}{(s+2)^2+3}$$

$$H(s) = \frac{e^{-5s}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{(s+2)^2+(\sqrt{3})^2}$$

$$h(t) = \frac{1}{\sqrt{3}} e^{-2(t-5)} \sin(\sqrt{3}(t-5))1(t-5)$$

Problem 4 (11 points)

Find the *step* response of the system

$$H(s) = \frac{1}{(s+1)^3}$$

$$F(s) = \frac{1}{s} H(s) = \frac{1}{s(s+1)^3}$$

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} + \frac{D}{(s+1)^3}$$

$$A = \left.\frac{1}{(s+1)^3}\right|_{s=0} = 1 \qquad D = \left.\frac{1}{s}\right|_{s=-1} = -1$$

$$C = \left.\frac{d}{ds}\left(\frac{1}{s}\right)\right|_{s=-1} = \left.-\frac{1}{s^2}\right|_{s=-1} = -1$$

$$B = \frac{1}{2}\left.\frac{d^2}{ds^2}\left(\frac{1}{s}\right)\right|_{s=-1} = \left.\frac{1}{2}\left(\frac{2}{s^3}\right)\right|_{s=-1} = -1$$

$$F(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}$$

$$f(t) = \left(1 - e^{-t} - te^{-t} - \frac{t^2}{2}e^{-t}\right)1(t)$$

Problem 5. (5 points)

Let the persistent forcing signal $u(t) = (1 + \cos(3t))1(t)$ drive the system

$$Y(s) = \frac{s^2+9}{s^2+4s+3}U(s).$$

Does this system, despite persistent forcing, reach a steady state? If so, what is $\lim_{t\to\infty} y(t)$?

$$U(s) = \frac{1}{s} + \frac{s}{s^2+9} = \frac{2s^2+9}{s(s^2+9)}$$

$$Y(s) = \frac{s^2+9}{s^2+4s+3} \cdot \frac{2s^2+9}{s(s^2+9)} = \frac{2s^2+9}{s(s^2+4s+3)}$$

By FVT, $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{2s^2+9}{s^2+4s+3} = 3$

Problem 6. (4 points)

For the discrete-time system with a transfer function

$$H(z) = \frac{z+2}{z(z^2+3z+1)},$$

find the difference equation governing the relationship between the input $u(k)$ and the output $y(k)$.

$$Y(z) = H(z)U(z)$$

$$(z^3 + 3z^2 + z)Y(z) = (z+2)U(z)$$

$$y_{k+3} + 3y_{k+2} + y_{k+1} = u_{k+1} + 2u_k$$

Problem 7. (7 points)

Solve the differential equation

$$\ddot{y} + 4\dot{y} + 3y = 0$$

for initial conditions $y(0) = -1, \dot{y}(0) = 6$.

$$s^2 Y(s) - s y(0) - \dot{y}(0) + 4(sY(s) - y(0)) + 3Y(s) = 0$$

$$(s^2 + 4s + 3) Y(s) = -s + 2$$

$$Y(s) = -\frac{s+2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = -\frac{s-2}{s+3}\Big|_{s=-1} = \frac{3}{2} \qquad B = -\frac{s-2}{s+1}\Big|_{s=-3} = -\frac{5}{2}$$

$$Y(s) = \frac{3/2}{s+1} - \frac{5/2}{s+3}$$

$$y(t) = \frac{3}{2}e^{-t} - \frac{5}{2}e^{-3t}$$

Problem 8. (9 points)

Find the solution of the discrete system

$$y_{k+2} + \frac{1}{4} y_k = u_k .$$

The input $u_k = 3\left(\frac{1}{2}\right)^k - 2\delta_k$ for $k = 0, 1, 2, \ldots$ and initial conditions $y_0 = 2$, $y_1 = 1$.

Hint: $$\mathcal{Z}\left\{a^k \cos\left(\frac{\pi}{2}k\right)\right\} = \frac{z^2}{z^2 + a^2}, \qquad \mathcal{Z}\left\{a^k \sin\left(\frac{\pi}{2}k\right)\right\} = \frac{az}{z^2 + a^2}.$$

$$U(z) = \frac{3z}{z - \frac{1}{2}} - 2$$

$$z^2 Y(z) - z^2 y_0 - z y_1 + \frac{1}{4} Y(z) = U(z)$$

$$Y(z) = \frac{2z^2 + z}{z^2 + \frac{1}{4}} + \frac{3z}{(z - \frac{1}{2})(z^2 + \frac{1}{4})} - \frac{2}{z^2 + \frac{1}{4}}$$

$$= \frac{10z^2 + z}{z^2 + \frac{1}{4}} - 8 + z \cdot \frac{3}{(z - \frac{1}{2})(z^2 + \frac{1}{4})}$$

$$\frac{3}{(z - \frac{1}{2})(z^2 + \frac{1}{4})} = \frac{A}{z - \frac{1}{2}} + \frac{Bz + C}{z^2 + \frac{1}{4}}$$

$$A = \frac{3}{\frac{1}{4} + \frac{1}{4}} = 6 \ ; \qquad 6\left(z^2 + \frac{1}{4}\right) + (Bz + C)\left(z - \frac{1}{2}\right) = 3$$

$$B = -6, \quad C = -3$$

$$Y(z) = \frac{10z^2 + z}{z^2 + \frac{1}{4}} - 8 + \frac{6z}{z - \frac{1}{2}} - \frac{6z^2}{z^2 + \frac{1}{4}} - \frac{3z}{z^2 + \frac{1}{4}}$$

$$= \frac{4z^2}{z^2 + \frac{1}{4}} - \frac{2z}{z^2 + \frac{1}{4}} + \frac{6z}{z - \frac{1}{2}} - 8$$

$$y_k = 4\left(\frac{1}{2}\right)^k \cos\left(\frac{\pi}{2}k\right) - 4\left(\frac{1}{2}\right)^k \sin\left(\frac{\pi}{2}k\right) + 6\left(\frac{1}{2}\right)^k - 8\delta_k, \quad k = 0, 1, \ldots$$