MAE 143A
SIGNALS AND SYSTEMS

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FINAL EXAM

September 1, 2004

NAME: Solutions

- Open books and notes.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 50.
- Time: 5–7pm (2.4 minutes/point)

Problem 1. (4 points)

For the time function

$$x(t) = 5e^{-2t} - 3t, \qquad t \geq 0$$

(a) (2 points) Find the Laplace transform $X(s) = \mathcal{L}\{x(t)\}$.
(b) (2 points) Based on point (a), and the initial value theorem, find $x(0)$.

(a) $X(s) = \frac{5}{s+2} - \frac{3}{s^2}$

(b) $x(0) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \left(\frac{5s}{s+2} - \frac{3}{s}\right) = 5$

Problem 2. (3 points)

Find the *impulse* response of the system

$$H(s) = e^{-2s}\frac{s+1}{(s+1)^2+9}$$

$$h(t) = \mathcal{L}^{-1}\left\{e^{-2s}\frac{s+1}{(s+1)^2+9}\right\}$$

$$= e^{-(t-2)}\cos(3(t-2))\,1(t-2)$$

Problem 3. (6 points)

Find the *step* response of the system

$$H(s) = 2\frac{s^2+s+1}{s^2+3s+2}$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}H(s)\right\} = \mathcal{L}^{-1}\left\{\frac{2(s^2+s+1)}{s(s+1)(s+2)}\right\} =$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s+1} + \frac{3}{s+2}\right\}$$

$$= \left(1 - 2e^{-t} + 3e^{-2t}\right)1(t)$$

Problem 4. (6 points)

Using the theorem about the Laplace transform of a derivative of a time function, find the Laplace transforms of the functions

$$x(t) = te^{-at}\sin(bt)\,, \qquad t \geq 0$$

$$x(t) = te^{-at}\cos(bt)\,, \qquad t \geq 0$$

where $a \geq 0$ and $b > 0$ are constants.

(1) $$X(s) = -\frac{d}{ds}\left(\mathcal{L}\{e^{-at}\sin bt\}\right) =$$

$$= -\frac{d}{ds}\left(\frac{b}{(s+a)^2+b^2}\right) = \frac{2b(s+a)}{((s+a)^2+b^2)^2}$$

(2) $$X(s) = -\frac{d}{ds}\left(\mathcal{L}\{e^{-at}\cos bt\}\right) =$$

$$= -\frac{d}{ds}\left(\frac{s+a}{(s+a)^2+b^2}\right)$$

$$= \frac{(s+a)^2-b^2}{((s+a)^2+b^2)^2}$$

Problem 5. (7 points)

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+4)^2}.$$

My objective with this problem is to show you why the method of *matching coefficients* cannot be used in general (in the form in which it is typically taught to you). This problem will teach you the proper way of using this method for repeated complex poles. Postulate the partial fraction expansion of $F(s)$ in the form

$$F(s) = A\frac{1}{s} + B\frac{2}{s^2+4} + C\frac{s}{s^2+4} + D\frac{4s}{(s^2+4)^2} + E\frac{s^2-4}{(s^2+4)^2}$$

and find the coefficients A, B, C, D, E. Then use the result of Problem 4. You can also apply the method for partial fraction expansion that I taught in class to solve this problem, however I want you to solve it here using the method matching coefficients.

The message I want you to retain is that the method of matching coefficients requires the knowledge of the solution to Problem 4 (and extensions of this result to higher powers of t), in addition to the special form of the fraction terms in $F(s)$ above. In contrast, the method I taught you in class employs just two formulae and the fact that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+p)^n}\right\} = \frac{1}{(n-1)!}t^{n-1}e^{-pt}\,,$$

whether p is real or complex, and applies to complex poles of any multiplicity.

$$F(s) = \frac{A(s^4+8s^2+16)+2B(s^3+4s)+C(s^4+4s^2)+4Ds^2+E(s^3-4s)}{s(s^2+4)^2}$$

s^4: $A+C=0$, s^2: $8A+4C+4D=0$, s^0: $16A=1$

s^3: $2B+E=0$, s^1: $8B-4E=0$

$A=\frac{1}{16}$, $B=0$, $C=-\frac{1}{16}$, $D=-\frac{1}{16}$, $E=0$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{16}\cdot\frac{1}{s} - \frac{1}{16}\frac{s}{s^2+4} - \frac{1}{16}\cdot\frac{4s}{(s^2+4)^2}\right\} =$$

$$= \frac{1}{16}\left(1-\cos 2t - t\sin 2t\right)1(t)$$

Problem 6. (3 points)

Let the persistent forcing signal $u(t) = (3 + \sin(2t))1(t)$ drive the system

$$Y(s) = \frac{s^2+4}{s^2+5s+6}U(s).$$

Does this system, despite persistent forcing, reach a steady state? If so, what is $\lim_{t\to\infty} y(t)$?

$$U(s) = \frac{3}{s} + \frac{2}{s^2+4} = \frac{3s^2+2s+12}{s(s^2+4)}$$

$$Y(s) = \frac{3s^2+2s+12}{s(s^2+5s+6)}.$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{3s^2+2s+12}{s^2+5s+6} = 2$$

Problem 7. (2 points)

For the system with a transfer function

$$H(s) = \frac{s+1}{s\left(s-\frac{1}{2}\right)},$$

find the differential equation governing the relationship between the input $u(t)$ and the output $y(t)$.

$$Y(s) = \frac{s+1}{s(s-\frac{1}{2})}U(s)$$

$$\left(s^2 - \frac{s}{2}\right)Y(s) = (s+1)U(s)$$

$$\ddot{y} - \frac{1}{2}\dot{y} = \dot{u} + u$$

Problem 8. (9 points)

Solve the differential equation

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = 0$$

for initial conditions $y(0) = 1, \dot{y}(0) = 2, \ddot{y}(0) = 3$.

$$s^3 Y(s) - s^2 y(0) - s\dot{y}(0) - \ddot{y}(0)$$
$$+ 3(s^2 Y(s) - s y(0) - \dot{y}(0))$$
$$+ 2(s Y(s) - y(0)) = 0$$

$$Y(s) = \frac{s^2 + 5s + 11}{s(s+1)(s+2)} = \frac{11/2}{s} - \frac{7}{s+1} + \frac{5/2}{s+2}$$

$$y(t) = \frac{11}{2} - 7e^{-t} + \frac{5}{2}e^{-2t}$$

Problem 9. (10 points)

Find the solution of the discrete system

$$y_{k+2} - \frac{1}{2}y_{k+1} = u_{k+1} + u_k .$$

the input $u_k = \sin\left(\frac{\pi}{2}k\right) 1_k$ and initial conditions $y_0 = y_1 \cancel{= u_0} = 1$. $u_0 = 0$

$$z^2 Y(z) - z^2 - z - \frac{1}{2}(z Y(z) - z) = z U(z) + U(z)$$

$$\left(z^2 - \frac{z}{2}\right) Y(z) = z^2 + \frac{z}{2} + (z+1)\frac{z}{z^2+1}$$

$$Y(z) = \frac{z + 1/2}{z - 1/2} + \frac{z+1}{(z - 1/2)(z^2+1)}$$

$$= 1 + \frac{1}{z - 1/2} + \frac{A}{z - 1/2} + \frac{Bz + C}{z^2+1}$$

$$A(z^2+1) + (Bz + C)(z - \tfrac{1}{2}) = z + 1$$

$$A + B = 0, \quad -\frac{B}{2} + C = 1, \quad A - \frac{C}{2} = 1$$

$$A = 6/5, \quad B = -6/5, \quad C = \frac{2}{5}$$

$$Y(z) = 1 + \frac{11/5}{z - 1/2} - \frac{6/5\, z}{z^2+1} + \frac{2/5}{z^2+1}$$

$$y(k) = \mathcal{Z}^{-1}\{Y(z)\} =$$

$$= \begin{cases} 1, & k = 0 \\ \frac{11}{5}\left(\frac{1}{2}\right)^{k-1} - \frac{6}{5}\cos\left(\frac{\pi}{2}(k-1)\right) + \frac{2}{5}\sin\left(\frac{\pi}{2}(k-1)\right), & k = 1, 2, \ldots \end{cases}$$