MAE 143A

Signals and Systems — Prof. M. Krstic

FINAL

August 1, 2008

NAME: SOLUTIONS

- Closed book. One sheet (both sides) of handwritten notes allowed.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 45.
- Time: 2 hours and 30 minutes.

Problem 1. (6 points)

Compute the *impulse* response of the following systems:

(a) (3 points) $H(s) = \mathrm{e}^{-5s}\dfrac{1}{(s+4)^2+2}$

(b) (3 points) $H(s) = \dfrac{1}{(2s+3)^5}$

Solution:

(a) Impulse response is given by $\mathcal{L}^{-1}\{H(s)\}$. Since

$$H(s) = \frac{1}{\sqrt{2}}\mathrm{e}^{-5s}\frac{\sqrt{2}}{(s+4)^2+\sqrt{2}^2}$$

and

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+4)^2+\sqrt{2}^2}\right\} = \mathrm{e}^{-4t}\sin\left(\sqrt{2}t\right)\,,$$

we get

$$\mathcal{L}^{-1}\{H(s)\} = \frac{1}{\sqrt{2}}\mathrm{e}^{-4(t-5)}\sin\left(\sqrt{2}(t-5)\right)1(t-5)\,.$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{1}{(2s+3)^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{32(s+3/2)^5}\right\} = \frac{1}{32}\cdot\frac{t^4}{4!}\,\mathrm{e}^{-\frac{3}{2}t}$$

Problem 2. (8 points)

Compute the *step* response of the system

$$H(s) = \frac{1}{(s+3)(s+2)^2}$$

Solution:

Step response is given by $\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\}$.

$$\frac{1}{s(s+3)(s+2)^2} = \frac{C_1}{s} + \frac{C_2}{s+3} + \frac{C_3}{s+2} + \frac{C_4}{(s+2)^2}$$

$$\begin{aligned}
C_1 &= \left.\frac{1}{(s+3)(s+2)^2}\right|_{s=0} = \frac{1}{12} \\
C_2 &= \left.\frac{1}{s(s+2)^2}\right|_{s=-3} = -\frac{1}{3} \\
C_3 &= \left.\frac{d}{ds}\left(\frac{1}{s(s+3)}\right)\right|_{s=-2} = \left.\frac{-2s-3}{s^2(s+3)^2}\right|_{s=-2} = \frac{1}{4} \\
C_4 &= \left.\frac{1}{s(s+3)}\right|_{s=-2} = -\frac{1}{2}
\end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \left[\frac{1}{12} - \frac{1}{3}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{-2t} - \frac{1}{2}t\mathrm{e}^{-2t}\right] 1(t)$$

Problem 3. (7 points)

For the system with a transfer function

$$H(s) = \frac{s+1}{3s^2+7} - \frac{1}{3s+2}$$

find the differential equation governing the relationship between the input $u(t)$ and the output $y(t)$, assuming zero initial conditions.

Solution:

$$Y(s) = H(s)U(s)$$

$$(3s^2+7)(3s+2)Y(s) = [(s+1)(3s+2) - (3s^2+7)]U(s)$$

$$(9s^3+6s^2+21s+14)Y(s) = (5s-5)U(s)$$

$$9\dddot{y} + 6\ddot{y} + 21\dot{y} + 14y = 5\dot{u} - 5u$$

Problem 4. (10 points)

Solve the differential equation

$$\ddot{y} + 2\dot{y} + 4 = (1-t)\mathrm{e}^{-2t}, \qquad y(0) = 1,\ \dot{y}(0) = 1$$

Solution:

$$\mathcal{L}\left\{(1-t)\mathrm{e}^{-2t}\right\} = \frac{1}{s+2} - \frac{1}{(s+2)^2} = \frac{s+1}{(s+2)^2}$$

Applying the Laplace transform to the ODE we get

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2(sY(s) - y(0)) + 4Y(s) = \frac{s+1}{(s+2)^2}$$

$$(s^2+2s+4)Y(s) = \frac{s+1}{(s+2)^2} + s + 3$$

$$Y(s) = \frac{s+1}{(s^2+2s+4)(s+2)^2} + \frac{s+3}{s^2+2s+4}$$

$$\frac{s+1}{(s^2+2s+4)(s+2)^2} = \frac{C_1}{s+2} + \frac{C_2}{(s+2)^2} + \frac{C_3 s + C_4}{s^2+2s+4}$$

$$\begin{aligned} C_1 &= \frac{d}{ds}\left(\frac{s+1}{s^2+2s+4}\right)\bigg|_{s=-2} = \frac{s^2+2s+4-(s+1)(2s+2)}{(s^2+2s+4)^2}\bigg|_{s=-2} = \frac{1}{8} \\ C_2 &= \frac{s+1}{s^2+2s+4}\bigg|_{s=-2} = -\frac{1}{4} \end{aligned}$$

Matching coefficients gives

$$\frac{1}{8}(s+2)(s^2+2s+4) - \frac{1}{4}(s^2+2s+4) + (C_3 s + C_4)(s+2)^2 = s+1$$

$$\frac{1}{8} + C_3 = 0, \quad 4C_4 = 1 \quad \Rightarrow \quad C_3 = -\frac{1}{8}, \quad C_4 = \frac{1}{4}$$

Going back to $Y(s)$,

$$\begin{aligned} Y(s) &= \frac{\frac{1}{8}}{s+2} + \frac{-\frac{1}{4}}{(s+2)^2} + \frac{-\frac{1}{8}s + \frac{1}{4}}{s^2+2s+4} + \frac{s+3}{s^2+2s+4} \\ &= \frac{\frac{1}{8}}{s+2} + \frac{-\frac{1}{4}}{(s+2)^2} + \frac{\frac{7}{8}s + \frac{13}{4}}{(s+1)^2+3} \\ &= \frac{\frac{1}{8}}{s+2} + \frac{-\frac{1}{4}}{(s+2)^2} + \frac{\frac{7}{8}(s+1) + \frac{19}{8}}{(s+1)^2+3} \end{aligned}$$

$$y(t) = \frac{1}{8}\mathrm{e}^{-t}\left[7\cos\left(\sqrt{3}t\right) + \frac{19}{\sqrt{3}}\sin\left(\sqrt{3}t\right)\right] + \frac{1}{8}(1-2t)\mathrm{e}^{-2t}$$

Problem 5. (6 points)

Find the $\mathcal{Z}$ transform of the functions

(a) (2 points) $\quad y_k = \left(\frac{2}{3}\right)^{k-2} 1_k$

(b) (2 points) $\quad y_k = \left(\frac{-1}{2}\right)^{k+2} 1_{k-1}$

(c) (2 points) $\quad y_k = k\left(\frac{1}{2}\right)^{2k} 1_{k-1}$

Solution:

(a)

$$y_k = \left(\frac{2}{3}\right)^{k-2} 1_k = \frac{9}{4}\left(\frac{2}{3}\right)^{k} 1_k$$

hence

$$Y(z) = \frac{9}{4}\frac{z}{z-2/3} = \frac{27z}{12z-8}$$

(b)

$$y_k = \left(\frac{-1}{2}\right)^{k+2} 1_{k-1} = -\frac{1}{8}\left(\frac{-1}{2}\right)^{k-1} 1_{k-1}$$

hence

$$Y(z) = -\frac{1}{8}z^{-1}\frac{z}{z+1/2} = -\frac{1}{8z+4}$$

(c)

$$y_k = k\left(\frac{1}{2}\right)^{2k} 1_{k-1} = k\left(\frac{1}{4}\right)^{k} 1_{k-1} = k\left(\frac{1}{4}\right)^{k} 1_k$$

hence

$$Y(z) = \frac{z/4}{(z-1/4)^2} = \frac{4z}{(4z-1)^2}$$

Problem 6. (8 points)

Solve the difference equation

$$8y_{k+2} + 2y_{k+1} - y_k = 0, \qquad y_0 = 3,\ y_1 = 2\,.$$

Solution:

$$8(z^2Y(z) - zy_1 - z^2y_0) + 2(zY(z) - zy_0) - Y(z) = 0$$

$$(8z^2 + 2z - 1)Y(z) = 24z^2 + 22z$$

$$Y(z) = \frac{24z^2 + 22z}{8z^2 + 2z - 1} = \frac{24z^2 + 22z}{8(z + \frac{1}{2})(z - \frac{1}{4})} = \frac{3z^2 + \frac{11}{4}z}{(z + \frac{1}{2})(z - \frac{1}{4})} = z\left(\frac{C_1}{z + \frac{1}{2}} + \frac{C_2}{z - \frac{1}{4}}\right)$$

$$\begin{aligned} C_1 &= \left(z + \frac{1}{2}\right)\frac{1}{z}Y(z)\Big|_{z=-\frac{1}{2}} = \frac{-\frac{3}{2} + \frac{11}{4}}{-\frac{1}{2} - \frac{1}{4}} = -\frac{5}{3} \\ C_2 &= \left(z - \frac{1}{4}\right)\frac{1}{z}Y(z)\Big|_{z=\frac{1}{4}} = \frac{\frac{3}{4} + \frac{11}{4}}{\frac{1}{4} + \frac{1}{2}} = \frac{14}{3} \end{aligned}$$

$$Y(z) = \frac{-\frac{5}{3}z}{z + \frac{1}{2}} + \frac{\frac{14}{3}z}{z - \frac{1}{4}}$$

$$y_k = -\frac{5}{3}\left(-\frac{1}{2}\right)^k 1_k + \frac{14}{3}\left(\frac{1}{4}\right)^k 1_k$$