MAE 143A

Signals and Systems — Prof. M. Krstic

FINAL

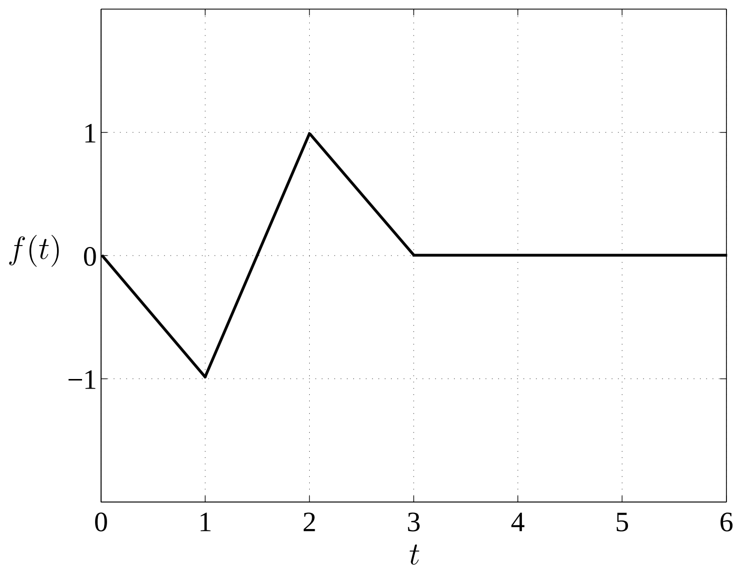
August 3, 2007

NAME: SOLUTIONS

- Open books and notes.
- Present your reasoning and calculations clearly. Random or inconsistent etchings will not be graded.
- Write only on the paper provided. If you run out of space for a given problem, continue on the pages at the end of the set and indicate "Continued on page X."
- The problems are *not* ordered by difficulty.
- Total points: 75.
- Time: 7:00–10:00.

Problem 1. (8 points)

Find the Laplace transform of the "heartbeat" function:



Solution:

$$f(t) = -t1(t) + 3(t-1)1(t-1) - 3(t-2)1(t-2) + (t-3)1(t-3)$$

$$F(s) = -\frac{1}{s^2} + \frac{3}{s^2}\mathrm{e}^{-s} - \frac{3}{s^2}\mathrm{e}^{-2s} + \frac{1}{s^2}\mathrm{e}^{-3s} = \frac{(\mathrm{e}^{-s}-1)^3}{s^2}$$

Problem 2. (8 points)

The Laplace transform is often used to compute the sums of infinite series. This problem will teach you how to do it. Start with the definition of Laplace transform:

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt\,.$$

(a) (4 points) Compute

$$\sum_{n=0}^{\infty} F(n)$$

and

$$\sum_{n=0}^{\infty} (-1)^n F(n)\,.$$

Hint: Use the formula $\displaystyle\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}\,, \quad -1 < q < 1.$

(b) (4 points) Use the result of (a) to find the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Solution:

$$\sum_{n=0}^{\infty} F(n) = \int_0^\infty f(t) \sum_{n=0}^{\infty} e^{-nt}\, dt = \int_0^\infty \frac{f(t)}{1-e^{-t}}\, dt$$

$$\sum_{n=0}^{\infty} (-1)^n F(n) = \int_0^\infty f(t) \sum_{n=0}^{\infty} (-e^{-t})^n\, dt = \int_0^\infty \frac{f(t)}{1+e^{-t}}\, dt$$

For the sum in (b),

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty} (-1)^n F(n) \quad \Rightarrow \quad F(s) = \frac{1}{s+1} \quad \Rightarrow \quad f(t) = e^{-t}$$

Therefore

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \int_0^\infty \frac{e^{-t}}{1+e^{-t}}\, dt = -\int_0^\infty \frac{d(1+e^{-t})}{1+e^{-t}} = -\ln(1+e^{-t})|_0^\infty = -\ln 1 + \ln 2 = \ln 2$$

Problem 3. (7 points)

Compute the *impulse* response of the following system

$$H(s) = \mathrm{e}^{-3s}\frac{s+3}{s^2+4s+9}$$

Solution:

$$\begin{aligned} H(s) &= \mathrm{e}^{-3s}\frac{s+3}{(s+2)^2+5} \\ &= \mathrm{e}^{-3s}\frac{s+2}{(s+2)^2+\sqrt{5}^2} + \mathrm{e}^{-3s}\frac{1}{(s+2)^2+\sqrt{5}^2} \\ &= \mathrm{e}^{-3s}\frac{s+2}{(s+2)^2+\sqrt{5}^2} + \frac{1}{\sqrt{5}}\mathrm{e}^{-3s}\frac{\sqrt{5}}{(s+2)^2+\sqrt{5}^2}. \end{aligned}$$

$$h(t) = \left[\cos(\sqrt{5}(t-3)) + \frac{1}{\sqrt{5}}\sin(\sqrt{5}(t-3))\right]\mathrm{e}^{-2(t-3)}1(t-3).$$

Problem 4. (10 points)

Compute the *ramp* response of the system

$$H(s) = \frac{1}{s(s+2)}$$

Solution:

$$Y(s) = H(s)U(s) = \frac{1}{s^3(s+2)} = \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s^3} + \frac{C_4}{s+2}$$

$$\begin{aligned}
C_3 &= \left(s^3 Y(s)\right)_{s=0} = \frac{1}{2} \\
C_2 &= \left(\frac{d}{ds}(s^3 Y(s))\right)_{s=0} = \left(-\frac{1}{(s+2)^2}\right)_{s=0} = -\frac{1}{4} \\
C_1 &= \frac{1}{2}\left(\frac{d^2}{ds^2}(s^3 Y(s))\right)_{s=0} = \frac{1}{2}\left(\frac{2}{(s+2)^3}\right)_{s=0} = \frac{1}{8} \\
C_4 &= ((s+2)Y(s))_{s=-2} = -\frac{1}{8}
\end{aligned}$$

$$y(t) = \left[\frac{1}{8} - \frac{1}{4}t + \frac{1}{4}t^2 - \frac{1}{8}\mathrm{e}^{-2t}\right] 1(t)$$

Problem 5. (6 points)

For the system with a transfer function

$$H(s) = \frac{s+1}{s^4 + 2s^2 + 1}$$

find the differential equation governing the relationship between the input $u(t)$ and the output $y(t)$, assuming zero initial conditions.

Solution:

$$Y(s) = H(s)U(s) = \frac{s+1}{s^4 + 2s^2 + 1}U(s)$$

$$(s^4 + 2s^2 + 1)Y(s) = (s+1)U(s)$$

$$y^{(4)} + 2\ddot{y} + y = \dot{u} + u$$

Problem 6. (10 points)

Solve the differential equation

$$\ddot{y} + 2\dot{y} + 5y = t\mathrm{e}^{-t}, \qquad y(0) = 3,\ \dot{y}(0) = -1\,.$$

Solution:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{1}{(s+1)^2}$$

$$(s^2 + 2s + 5)Y(s) - 3s + 1 - 2\cdot 3 = \frac{1}{(s+1)^2}$$

$$Y(s) = \frac{3s+5}{s^2+2s+5} + \frac{1}{(s^2+2s+5)(s+1)^2}$$

Partial fraction expansion:

$$\frac{1}{(s^2+2s+5)(s+1)^2} = \frac{C_1 s + C_2}{s^2+2s+5} + \frac{C_3}{s+1} + \frac{C_4}{(s+1)^2}$$

$$C_4 = \left(\frac{1}{s^2+2s+5}\right)_{s=-1} = \frac{1}{4}$$

$$C_3 = \left(\frac{d}{ds}\frac{1}{s^2+2s+5}\right)_{s=-1} = \left(-\frac{2s+2}{(s^2+2s+5)^2}\right)_{s=-1} = 0$$

$$(C_1 s + C_2)(s+1)^2 + \frac{1}{4}(s^2+2s+5) = 1 \quad \Rightarrow \quad C_1 = 0, \quad C_2 = -\frac{1}{4}$$

$$Y(s) = \frac{3s + \frac{19}{4}}{s^2+2s+5} + \frac{1}{4(s+1)^2} = \frac{3(s+1) + \frac{7}{4}}{(s+1)^2 + 4} + \frac{1}{4(s+1)^2}$$

$$y(t) = 3\mathrm{e}^{-t}\cos(2t) + \frac{7}{8}\mathrm{e}^{-t}\sin(2t) + \frac{1}{4}t\mathrm{e}^{-t}$$

Problem 7. (9 points)

Find the $\mathcal{Z}$ transform of the functions

(a) (3 points) $y_k = \left(\frac{1}{7}\right)^{k-2} 1_k$.

(b) (3 points) $y_k = \left(\frac{1}{5}\right)^{k+1} 1_{k-1}$.

(c) (3 points) $y_k = k\left(-\frac{1}{3}\right)^{k} 1_{k-1}$.

Solution:

(a)

$$y_k = \left(\frac{1}{7}\right)^{k-2} 1_k = 49\left(\frac{1}{7}\right)^{k} 1_k$$

$$Y(z) = 49\frac{z}{z-\frac{1}{7}}$$

(b)

$$y_k = \left(\frac{1}{5}\right)^{k+1} 1_{k-1} = \frac{1}{25}\left(\frac{1}{5}\right)^{k-1} 1_{k-1}$$

$$Y(z) = \frac{1}{25}z^{-1}\frac{z}{z-\frac{1}{5}} = \frac{1}{25z-5}$$

(c)

$$y_k = k\left(-\frac{1}{3}\right)^{k} 1_{k-1} = k\left(-\frac{1}{3}\right)^{k} 1_k$$

$$Y(z) = \frac{-3z}{(3z+1)^2}$$

Problem 8. (8 points)

Let the persistent forcing signal $u_k = \sin\left(\frac{\pi}{2}k\right) 1_k$ drive the discrete system

$$Y(z) = \frac{2(z^2+1)}{2z^2 - z - 1}\, U(z).$$

Does this system, despite persistent forcing, reach a steady state? If so, what is $\lim_{k\to\infty} y_k$?

Hint: Use the Final Value Theorem for the $\mathcal{Z}$ transform:
If all the poles of $(z-1)Y(z)$ are inside the unit circle, then $y(\infty) = \lim_{z\to 1}(z-1)Y(z)$.

Solution:

$$U(z) = \frac{z}{z^2+1}$$

$$Y(z) = \frac{2(z^2+1)}{2z^2 - z - 1}\frac{z}{z^2+1)} = \frac{2z}{2z^2 - z - 1}$$

Note that $2z^2 - z - 1 = (z-1)(2z+1)$. The pole of $(z-1)Y(z)$ is $z = -1/2$ and lies inside the unit circle and therefore FVT applies.

$$y(\infty) = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}\frac{2z(z-1)}{(z-1)(2z+1} = \lim_{z\to 1}\frac{2z}{2z+1} = \frac{2}{3}\,.$$

Problem 9. (9 points)

Solve the difference equation

$$6y_{k+2} + y_{k+1} - y_k = 0, \qquad y_0 = 1,\ y_1 = \frac{1}{2}\,.$$

Solution:

$$6(z^2Y(z) - zy_1 - z^2y_0) + zY(z) - zy_0 - Y(z) = 0$$

$$(6z^2 + z - 1)Y(z) - 6z\frac{1}{2} - 6z^2 - z = 0$$

$$Y(z) = \frac{6z^2 + 4z}{6z^2 + z - 1} = \frac{z^2 + \frac{2}{3}z}{(z + \frac{1}{2})(z - \frac{1}{3})} = z\left(\frac{C_1}{z + \frac{1}{2}} + \frac{C_2}{z - \frac{1}{3}}\right)$$

$$C_1 = \left(z + \frac{1}{2}\right)\frac{1}{z}Y(z)\big|_{z=-\frac{1}{2}} = \frac{-\frac{1}{2} + \frac{2}{3}}{-\frac{1}{2} - \frac{1}{3}} = -\frac{1}{5}$$

$$C_2 = \left(z - \frac{1}{3}\right)\frac{1}{z}Y(z)\big|_{z=\frac{1}{3}} = \frac{\frac{1}{3} + \frac{2}{3}}{\frac{1}{3} + \frac{1}{2}} = \frac{6}{5}$$

$$Y(z) = -\frac{1}{5}\frac{z}{z + \frac{1}{2}} + \frac{6}{5}\frac{z}{z - \frac{1}{3}}$$

$$y_k = -\frac{1}{5}\left(-\frac{1}{2}\right)^k 1_k + \frac{6}{5}\left(\frac{1}{3}\right)^k 1_k$$