Extremum Seeking

Model-Free Real-Time Optimization

Miroslav Krstic
University of California, San Diego
Cymer Center for Control Systems and Dynamics

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Outline

- Basic idea of ES
- Chronology of ES
- ES for multivariable dynamic systems
- Newton-based ES
- Games (Nash equilibrium seeking)
- Infinite-dimensional ES
- Source seeking by fish (hydrofoils)
- Basics of stochastic ES and stochastic averaging
- Stochastic source seeking—bacterial chemotaxis
- Aircraft endurance maximization via stoch ES using atmospheric turbulence

Standard Deterministic **Averaging**

$$\frac{dX_t^{\varepsilon}}{dt} = \varepsilon f(t, X_t^{\varepsilon}, \varepsilon) \qquad \text{(SYS)}$$

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Average system

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Theorem 1 *[Khalil's book, Theorem 10.4] Let $f(t,x,\varepsilon)$ and its partial derivatives with respect to (x,ε) up to the second order be continuous and bounded for $(t,x,\varepsilon) \in [0,\infty) \times D_0 \times [0,\varepsilon_0]$, for every compact set $D_0 \subset D$, where $D \subset \mathbb{R}^n$ is a domain. Suppose f is T-periodic in t for some $T > 0$ and ε is a positive parameter.*

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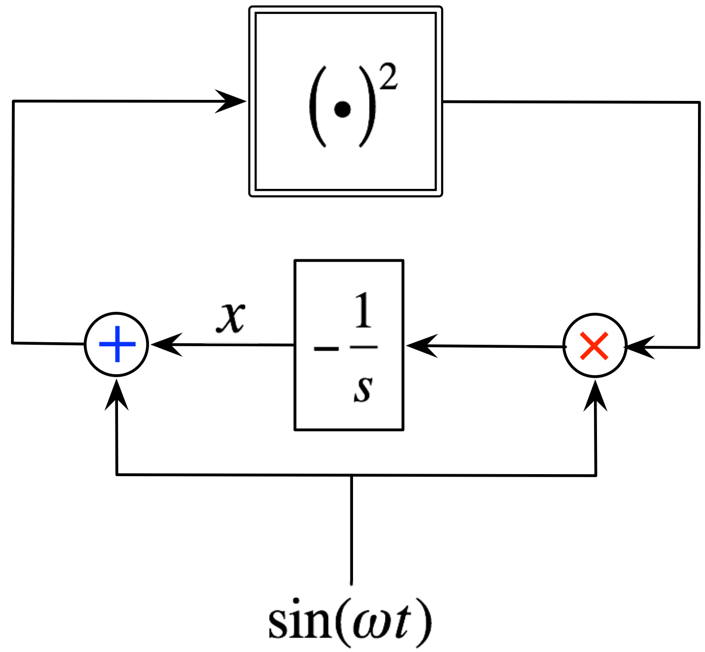
An Averaging Example



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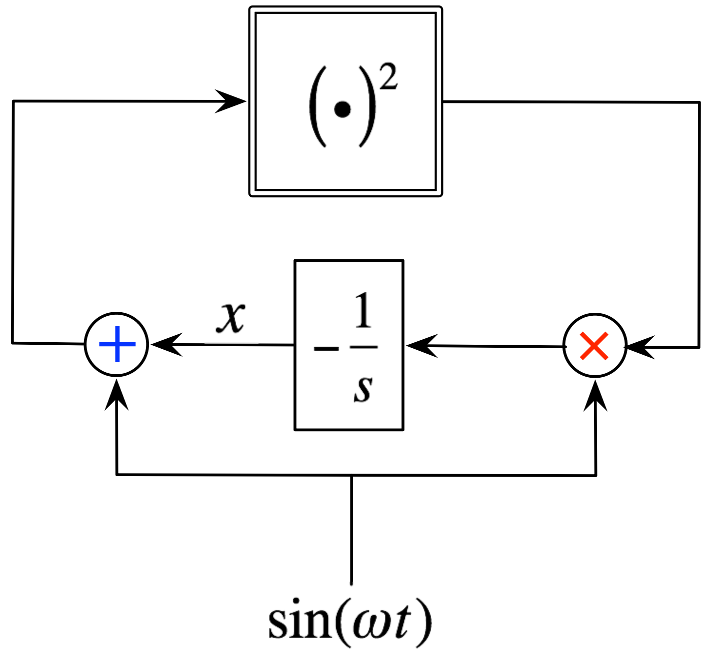

$$\dot{x} \quad = \quad -\sin(\omega t)\,(x+\sin(\omega t))^2$$

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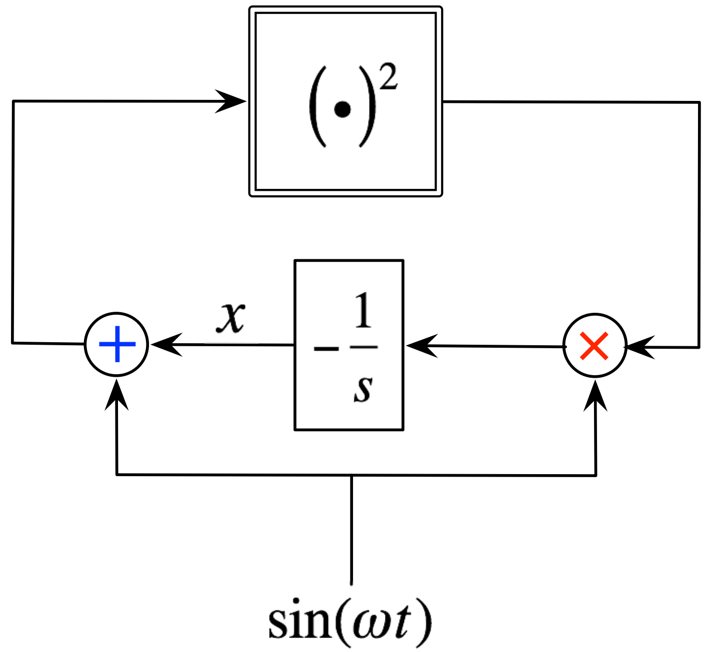
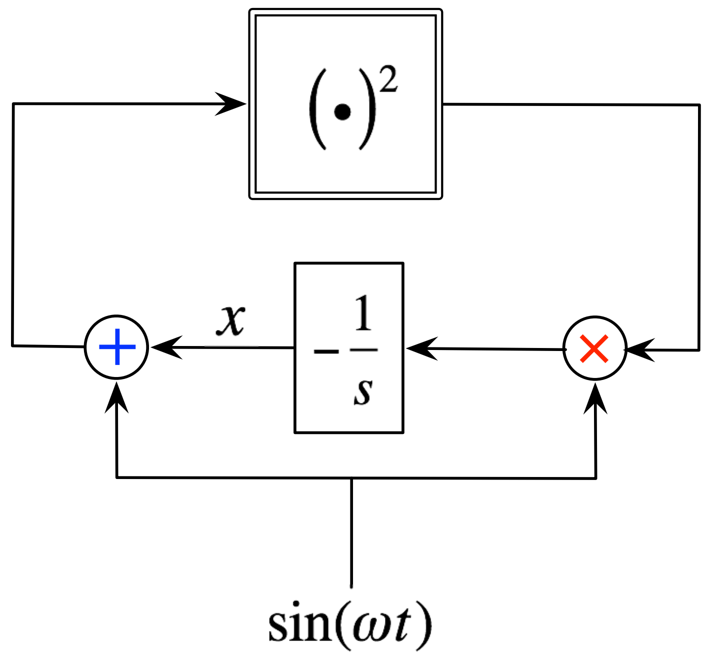


$$
\begin{aligned}
\dot{x} &= -\sin(\omega t)\,(x+\sin(\omega t))^2 \\
&= -x^2 \underbrace{\sin(\omega t)}_{\text{ave} = 0} - 2x \underbrace{\sin^2(\omega t)}_{\text{ave} = \frac{1}{2}} - \underbrace{\sin^3(\omega t)}_{\text{ave} = 0}
\end{aligned}
$$

An Averaging Example



$$\begin{aligned}\dot{x} &= -\sin(\omega t)\,(x+\sin(\omega t))^2 \\ &= -x^2\underbrace{\sin(\omega t)}_{\text{ave = 0}} - 2x\underbrace{\sin^2(\omega t)}_{\text{ave = }\frac{1}{2}} - \underbrace{\sin^3(\omega t)}_{\text{ave = 0}}\end{aligned}$$

$$\dot{x}_{\text{ave}} = -x_{\text{ave}}$$

Theorem 2 *By Theorem 1, For sufficiently large ω, there exists a locally exponentially stable periodic solution $x^{2\pi/\omega}(t)$ such that*

$$\left| x^{2\pi/\omega}(t) \right| \leq O\left(\frac{1}{\omega}\right), \qquad \forall t \geq 0.$$

Theorem 2 *By Theorem 1, For sufficiently large ω, there exists a locally exponentially stable periodic solution $x^{2\pi/\omega}(t)$ such that*

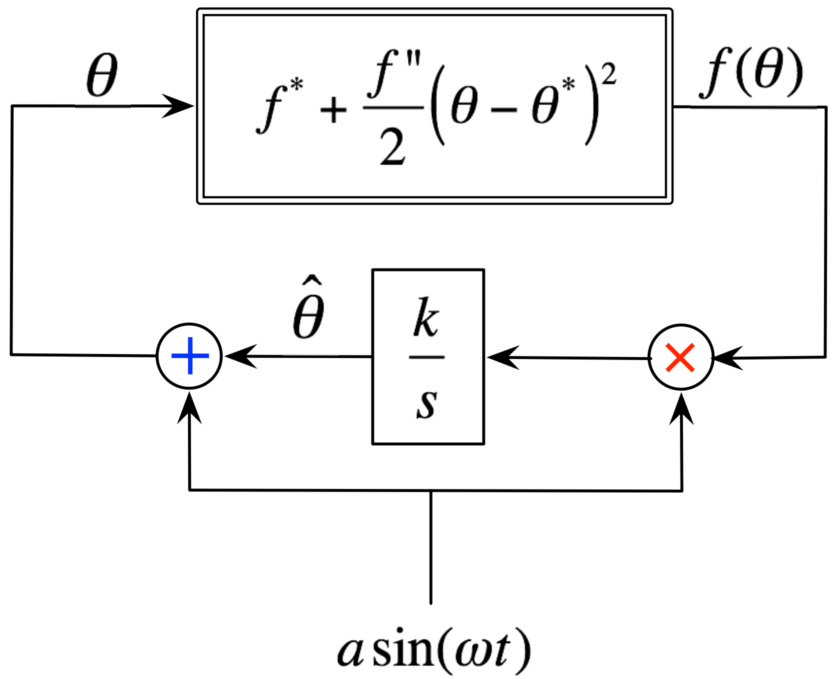
$$\left|x^{2\pi/\omega}(t)\right| \leq O\left(\frac{1}{\omega}\right), \qquad \forall t \geq 0.$$

Corollary 1 *For sufficiently large ω, there exist $M, m > 0$ such that*

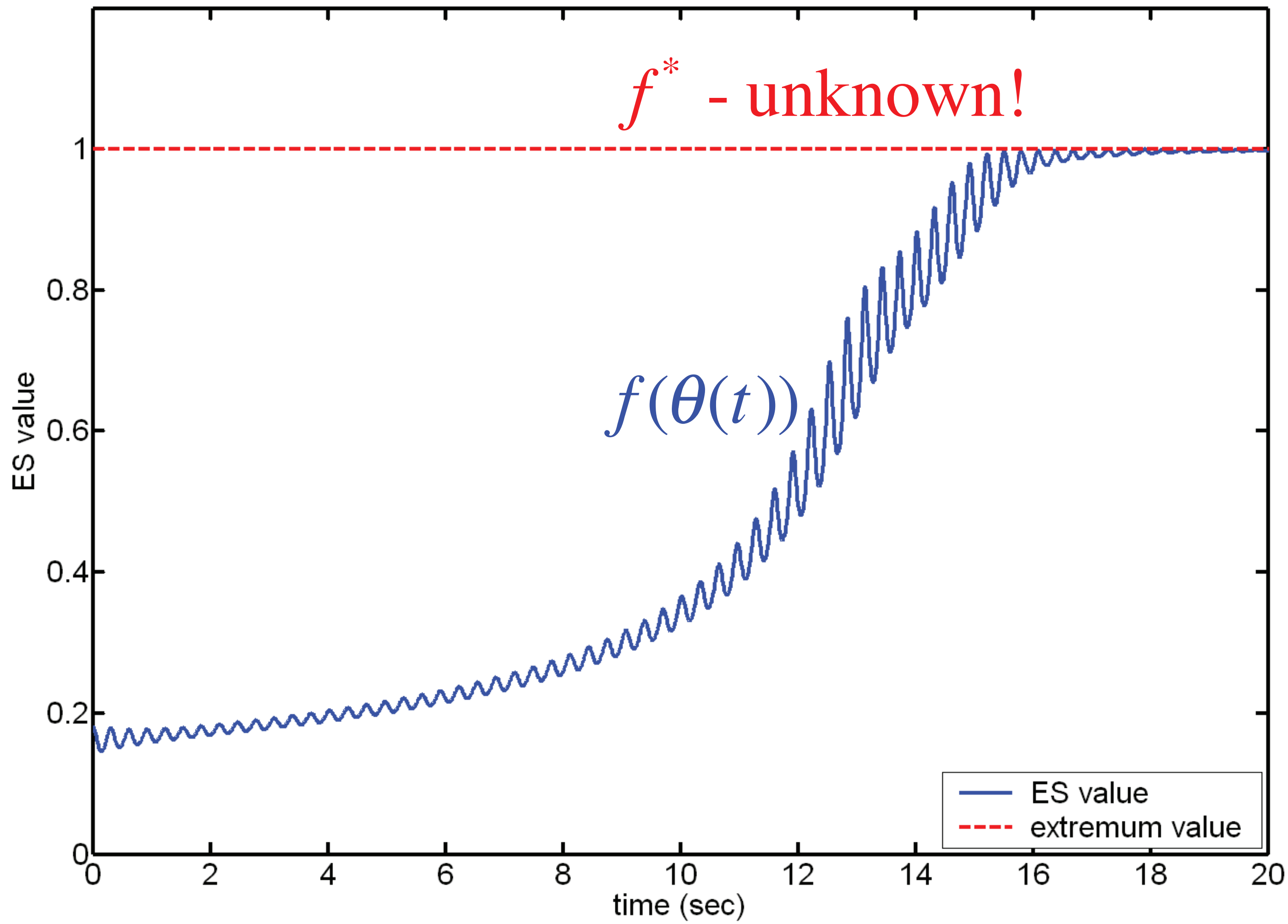
$$|x(t)| \leq M|x(0)|\mathrm{e}^{-mt} + O\left(\frac{1}{\omega}\right), \qquad \forall t \geq 0.$$

Basic Idea of Extremum Seeking

Arbitrary Unknown Quadratic Function

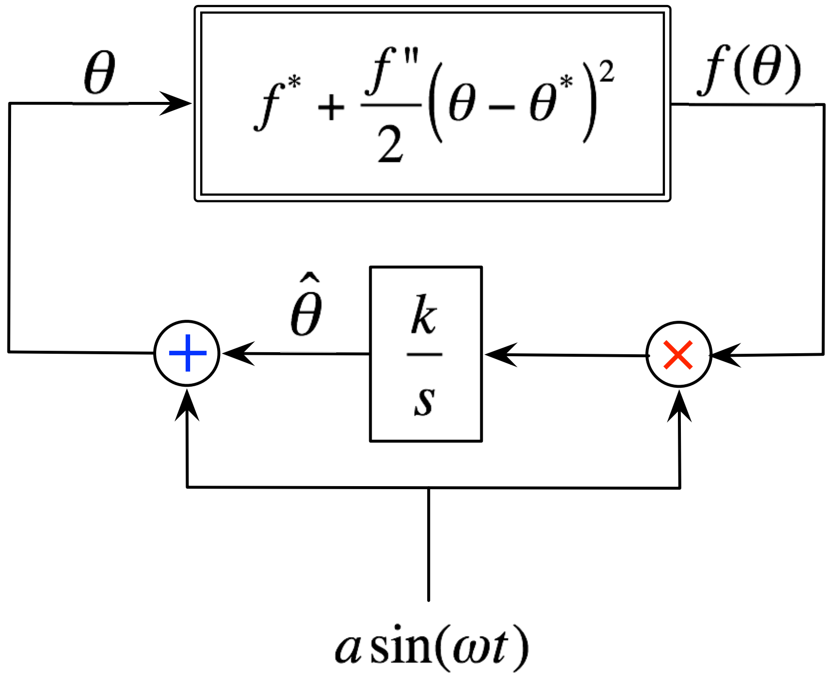


$$\operatorname{sgn} k = -\operatorname{sgn} f''$$



f^* - unknown!
$f(\theta(t))$
ES value
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
20
time (sec)
ES value
extremum value

Arbitrary Unknown Quadratic Function



$$\operatorname{sgn} k = -\operatorname{sgn} f''$$

$$\begin{aligned} \tilde{\theta} &= \hat{\theta} - \theta^* \\ \frac{d\tilde{\theta}}{dt} &= ka\sin(\omega t)\left[f^* + \frac{f''}{2}\left(\tilde{\theta} + a\sin(\omega t)\right)^2\right] \end{aligned}$$

$$\frac{\mathrm{d}\tilde{\theta}_{\mathrm{ave}}}{\mathrm{d}t} = \frac{\overbrace{k f''}^{<0} a^2}{2}\tilde{\theta}_{\mathrm{ave}}$$

$$\frac{\mathrm{d}\tilde{\theta}_{\mathrm{ave}}}{\mathrm{d}t} = \frac{\overbrace{kf''a^2}^{<0}}{2}\tilde{\theta}_{\mathrm{ave}}$$

Theorem 3 *There exists sufficiently large* ω *such that, locally,*

$$|\theta(t) - \theta^*| \leq |\theta(0) - \theta^*| \, \mathrm{e}^{\boxed{\frac{kf''a^2}{2}}t} + O\left(\frac{1}{\omega}\right) + a, \qquad \forall t \geq 0.$$

Chronology of Extremum Seeking

Leblanc, 1922	electric railways
Russia, 1940s	many applications and attempts at theory
Draper & Li, 1951	SI engines (spark timing)
1960s	last wave of efforts towards theory
MK, late 1990s	stability proof and implementation on axial-flow compressors and gas turbine combustors
2000s	numerous applications, including aerodynamic flow control, wind turbines, photovoltaics, fusion
MK and others, late 2000s	mobile robots and UUVs in GPS-denied environments; fish and bacterial locomotion
MK and others, 2010s	non-cooperative games

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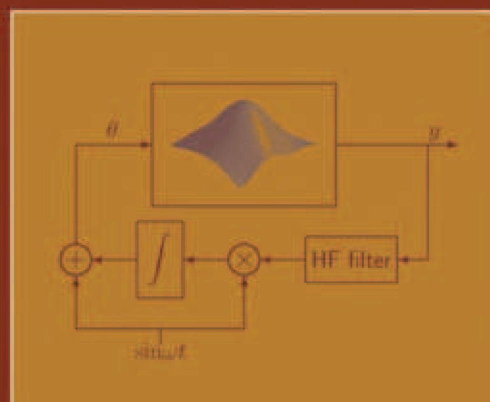
Real-Time Optimization by Extremum-Seeking Control



KARTIK B. ARIYUR
MIROSLAV KRSTIĆ

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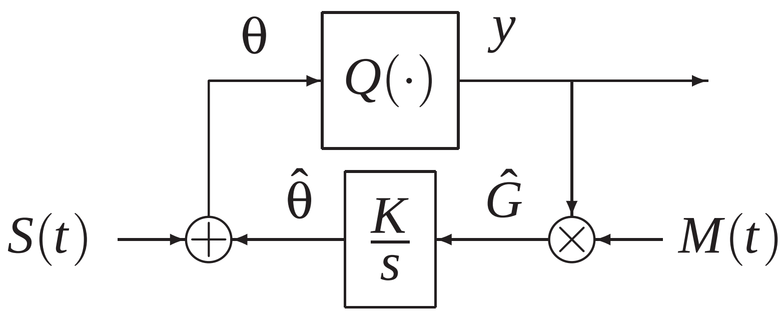
Extremum Seeking for

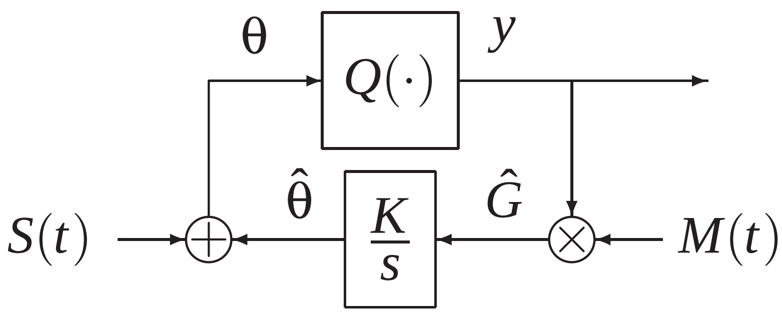
Multivariable Dynamic Systems

ES for multivariable static map



$Q(\cdot)$ = unknown map, y = measurable scalar, $\theta = [\theta_1, \theta_2, \cdots, \theta_n]^T$ = input vector

ES for multivariable static map



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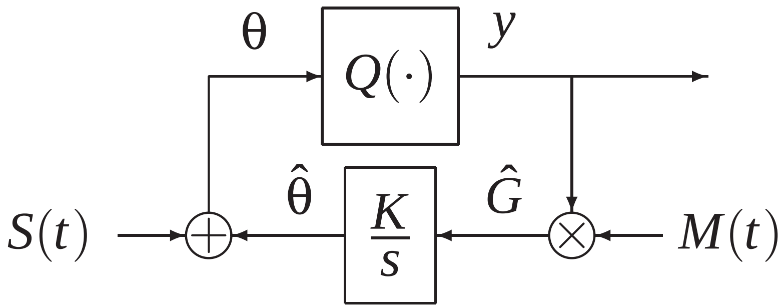
$$S(t) = \begin{bmatrix} a_1 \sin(\omega_1 t) & \cdots & a_n \sin(\omega_n t) \end{bmatrix}^T$$

$$M(t) = \begin{bmatrix} \frac{2}{a_1} \sin(\omega_1 t) & \cdots & \frac{2}{a_n} \sin(\omega_n t) \end{bmatrix}^T$$

ES for multivariable static map



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$a_i \neq 0$, K = positive diagonal matrix

ω_i / ω_j rational, $\omega_i \neq \omega_j$ and $\omega_i + \omega_j \neq \omega_k$ for distinct i, j, and k

ES for multivariable static map

For quadratic map $Q(\theta) = Q^* + \frac{1}{2}(\theta - \theta^*)^T H(\theta - \theta^*)$, the averaged system is

$$\boxed{\dot{\tilde{\theta}} = KH\,\tilde{\theta}} \qquad H = \text{Hessian} < 0$$

ES algorithm for dynamic systems

$$\dot{x} = f(x,u), \qquad u \in \mathbb{R}^n$$
$$y = h(x), \qquad y \in \mathbb{R}$$

Control law $u = \alpha(x, \theta)$ parametrized by $\theta \in \mathbb{R}^n$

Closed-loop system $\dot{x} = f(x, \alpha(x, \theta))$ has equilibria $x = l(\theta)$ parametrized by θ

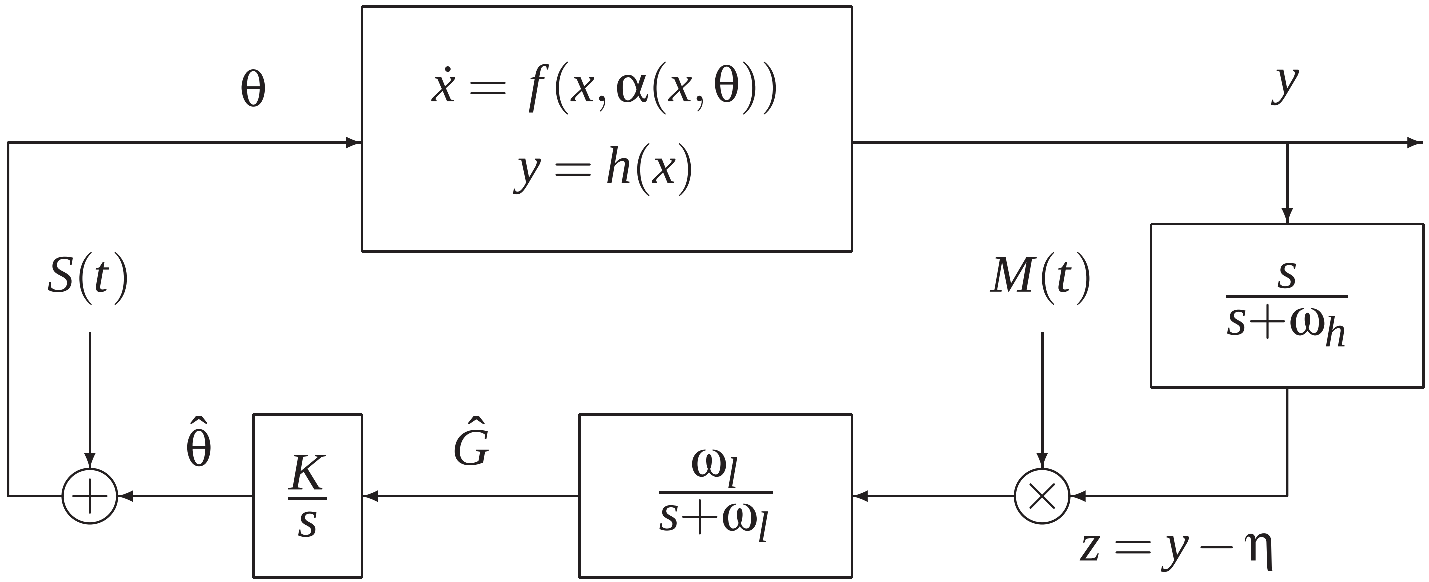
ES algorithm for dynamic systems

Assumption 1 *Equilibria* $x = l(\theta)$ *are loc. exp. stable uniformly in* θ*.*

Assumption 2 $\exists\, \theta^* \in \mathbb{R}^n$ *s.t.*

$$\frac{\partial}{\partial\theta}(h \circ l)(\theta^*) = 0,$$
$$\frac{\partial^2}{\partial\theta^2}(h \circ l)(\theta^*) = H < 0, \qquad H = H^T.$$

ES algorithm for dynamic systems



ES algorithm for dynamic systems

$$\begin{aligned}
\omega_i &= \omega\omega_i' = O(\omega), && \omega_i' \text{ is a rational number}, \quad i \in \{1,2,\ldots,n\}\\
\omega_h &= \omega\omega_H = \omega\delta\omega_H' = O(\omega\delta), && \omega_H' \text{ is } O(1) \text{ positive constant}\\
\omega_l &= \omega\omega_L = \omega\delta\omega_L' = O(\omega\delta), && \omega_L' \text{ is } O(1) \text{ positive constant}\\
\omega_r &= \omega\omega_R = \omega\delta\omega_R' = O(\omega\delta), && \omega_R' \text{ is } O(1) \text{ positive constant}\\
K &= \omega K' = \omega\delta K'' = O(\omega\delta), && K'' > 0 \in \mathbb{R}^{n\times n}
\end{aligned}$$

K'' is diagonal $O(1)$ matrix, ω and δ are small positive constants

$$\omega_i' \notin \left\{\omega_j', \frac{1}{2}(\omega_j'+\omega_k'), \omega_j'+2\omega_k', \omega_j'+\omega_k'\pm\omega_l'\right\}, \quad \text{for all distinct } i, j, k, \text{ and } l$$

Stability of ES algorithm

Closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ \tilde{\theta} \\ \hat{G} \\ \tilde{\eta} \end{bmatrix} = \begin{bmatrix} f(x, \alpha(x, \theta^* + \tilde{\theta} + S(t))) \\ -K\hat{G} \\ -\omega_l \hat{G} + \omega_l \left(y - h \circ l(\theta^*) - \tilde{\eta} \right) M(t) \\ -\omega_h \tilde{\eta} + \omega_h \left(y - h \circ l(\theta^*) \right) \end{bmatrix} \tag{1}$$

Plant, parameter estimator, and two filters

Error variables: $\tilde{\theta} = \hat{\theta} - \theta^*$, $\tilde{\eta} = \eta - h \circ l(\theta^*)$

Main result

Theorem 4 *Consider the feedback system (1) under Assumptions 1 and 2.*

$\exists\, \overline{\omega} > 0$ *and*

$\forall\, \omega \in (0, \overline{\omega}) \quad \exists\, \overline{\delta}(\omega), \overline{a}(\omega) > 0$ *s.t.*

for the given ω *and* $\forall\, |a| \in (0, \overline{a}(\omega))$ *and* $\delta \in (0, \overline{\delta}(\omega))$

$\exists$ *a nbhd of the point* $(x, \hat{\theta}, \hat{G}, \eta) = (l(\theta^*), \theta^*, 0, h \circ l(\theta^*))$ *such that*

any solution of systems (1) from the neighborhood exponentially converges to an $O(\omega + \delta + |a|)$*–neighborhood of that point.*

Furthermore, $y(t)$ *converges to an* $O(\omega + \delta + |a|)$*–neighborhood of* $h \circ l(\theta^*)$*.*

Proof by Singular Perturbation + Averaging (3 time scales!)

Convert to time scale $\tau = \omega t$:

$$\omega \frac{\mathrm{d}x}{\mathrm{d}\tau} = f\left(x, \alpha\left(x, \theta^* + \tilde{\theta} + \overline{S}(\tau)\right)\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{bmatrix} \tilde{\theta} \\ \hat{G} \\ \tilde{\eta} \end{bmatrix} = \delta \begin{bmatrix} -K''\hat{G} \\ -\omega_L'\hat{G} + \omega_L'\left(y - h\circ l(\theta^*) - \tilde{\eta}\right)\overline{M}(\tau) \\ -\omega_H'\tilde{\eta} + \omega_H'\left(y - h\circ l(\theta^*)\right) \end{bmatrix}$$

$\overline{S}(\tau) = S(t/\omega), \overline{M}(\tau) = M(t/\omega)$

First study reduced/slow system ($\omega = 0$) by averaging.

(Boundary layer model e.s. because plant is e.s.)

Averaging analysis

Theorem 5 *Consider reduced system under Assumption 2.* $\exists\ \bar{\delta}, \bar{a} > 0$ *s.t.* $\forall\ \delta \in (0, \bar{\delta})$ *and* $|a| \in (0, \bar{a})$ *the reduced system has a unique exponentially stable periodic solution* $\left(\tilde{\theta}_r^{\Pi}(\tau), \hat{G}_r^{\Pi}(\tau), \tilde{\eta}_r^{\Pi}(\tau)\right)$ *of period* Π *and this solution satisfies*

$$\left|\tilde{\theta}_{r,i}^{\Pi}(\tau) - \sum_{j=1}^{n} c_{j,j}^{i} a_j^2\right| \leq O(\delta + |a|^3)$$

$$\left|\hat{G}_r^{\Pi}(\tau)\right| \leq O(\delta)$$

$$\left|\tilde{\eta}_r^{\Pi}(\tau) - \frac{1}{4}\sum_{i=1}^{n} H_{i,i} a_i^2\right| \leq O(\delta + |a|^4)$$

for all $\tau \geq 0$*,*

where

$$\begin{bmatrix} c_{j,j}^{1} \\ \vdots \\ c_{j,j}^{i-1} \\ \\ c_{j,j}^{i} \\ \\ c_{j,j}^{i+1} \\ \vdots \\ c_{j,j}^{n} \end{bmatrix} = -\frac{1}{12}H^{-1} \begin{bmatrix} \frac{\partial^3 (h \circ l)}{\partial z_j \partial z_1^2}(\theta^*) \\ \vdots \\ \frac{\partial^3 (h \circ l)}{\partial z_j \partial z_{j-1}^2}(\theta^*) \\ \\ \frac{3}{2}\frac{\partial^3 (h \circ l)}{\partial z_j^3}(\theta^*) \\ \\ \frac{\partial^3 (h \circ l)}{\partial z_j \partial z_{j+1}^2}(\theta^*) \\ \vdots \\ \frac{\partial^3 (h \circ l)}{\partial z_j \partial z_n^2}(\theta^*) \end{bmatrix}$$

Newton-Based Extremum Seeking

Weakness of Gradient Algorithm

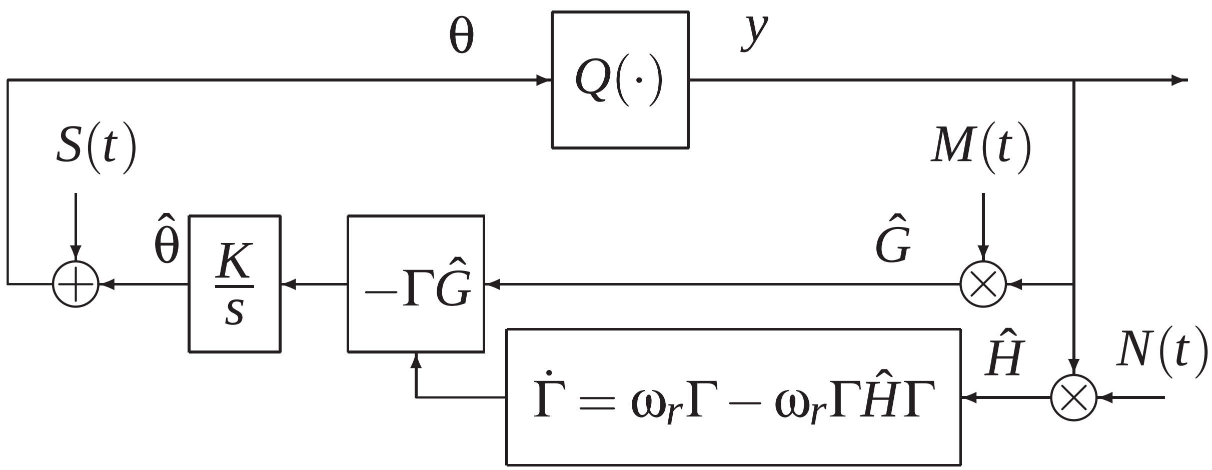
Convergence rate depends on **unknown** Hessian

Newton algorithm removes this weakness.

Pitfall of Newton approach for multivariable maps:

Requires an inverse of the Hessian matrix estimates—not necessarily invertible!

Newton algorithm for static map



1. Multiplic. excitation $N(t)$: generate estimate of Hessian $\dfrac{\partial^2 Q(\theta)}{\partial\theta^2}$ as $\hat{H}(t) = N(t)y(t)$

2. Riccati martrix diff eq $\Gamma(t)$: generate estimate of Hessian's *inverse* matrix

Estimate of the Hessian matrix

Taylor expansion

$$
\begin{aligned}
y &= Q(\theta^* + \tilde{\theta} + S(t)) \\
&= \underbrace{Q(\theta^*) + \frac{1}{2}\left(\tilde{\theta} + S(t)\right)^T \boxed{H} \left(\tilde{\theta} + S(t)\right)}_{\text{quadratic in } \tilde{\theta} + S(t)} + \underbrace{R(\tilde{\theta} + S(t))}_{\text{H.O.T}}
\end{aligned}
$$

$$
H := \frac{\partial^2 Q(\boxed{\theta^*})}{\partial \theta^2} < 0
$$

Task: design $N(t)$ so that $\mathrm{Ave}\{N(t)y - \text{H.O.T}\} = H$

Estimate of the Hessian matrix

After lengthy averaging calculations, we find

$$\boxed{N_{ii}(t) = \frac{16}{a_i^2}\left(\sin^2(\omega_i t) - \frac{1}{2}\right)}$$

$$\boxed{N_{ij}(t) = \frac{4}{a_i a_j}\sin(\omega_i t)\sin(\omega_j t)}$$

Computing the estimate of the inverse of the Hessian matrix

Matrix inversion of $\hat{H}(t)$ = bad

Consider low-pass filter of Hessian estimate:

$$\boxed{\dot{\mathcal{H}} = -\omega_r \mathcal{H} + \omega_r \hat{H}}$$

$\mathcal{H}(t) - \hat{H}(t) \to 0$

Denote $\Gamma = \mathcal{H}^{-1}$. Ricatti equation:

$$\boxed{\dot{\Gamma} = \omega_r \Gamma - \omega_r \Gamma \hat{H} \Gamma}$$

$\Gamma(t) - \hat{H}(t)^{-1} \to 0$

Computing the estimate of the inverse of the Hessian matrix

Equilibria:

$\Gamma^* = 0_{n\times n}$ unstable

$\Gamma^* = \hat{H}^{-1}$ loc. exp. stable (provided $\hat{H}$ settles)

Computing the estimate of the inverse of the Hessian matrix

For a quadratic map, the averaged system in error variables $\tilde{\theta} = \hat{\theta} - \theta^*$, $\tilde{\Gamma} = \Gamma - H^{-1}$ is

$$\begin{aligned}
\frac{d\tilde{\theta}^{\mathrm{ave}}}{dt} &= -K\tilde{\theta}^{\mathrm{ave}} - K\tilde{\Gamma}^{\mathrm{ave}}H\tilde{\theta}^{\mathrm{ave}} \\
\frac{d\tilde{\Gamma}^{\mathrm{ave}}}{dt} &= -\omega_r\tilde{\Gamma}^{\mathrm{ave}} - \omega_r\tilde{\Gamma}^{\mathrm{ave}}H\tilde{\Gamma}^{\mathrm{ave}}
\end{aligned}$$

(local) convergence rate user-assignable!

Simulation results

Static quadratic map: $y = Q(\theta) = Q^* + \frac{1}{2}(\theta - \theta^*)^T H(\theta - \theta^*)$

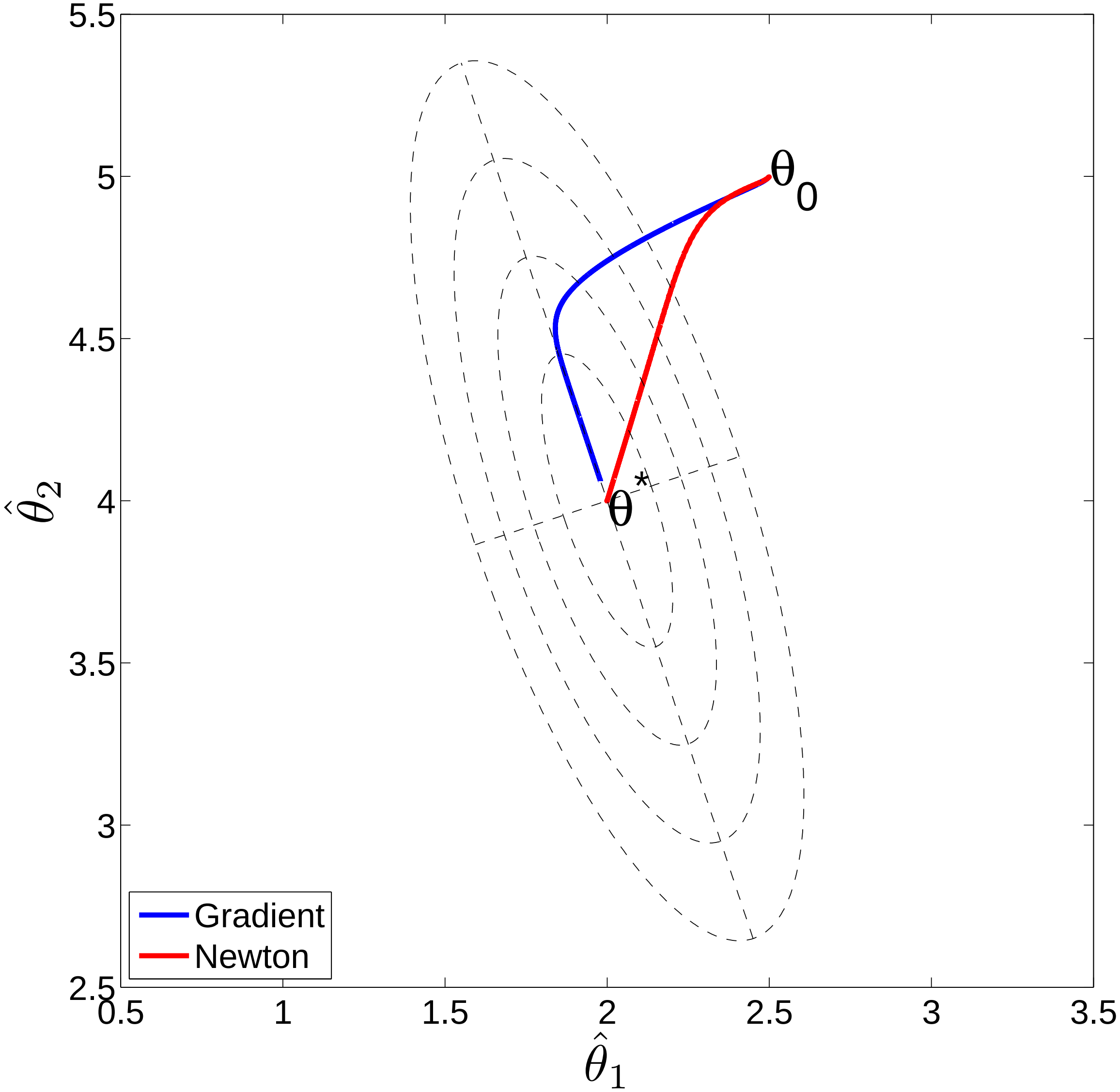
All ES parameters chosen the same except gain matrices.

Gradient convergence: $K_g H$. Newton convergence: $-K_n \Gamma(t) H$.

We select $\boxed{\Gamma(0) = -K_n^{-1} K_g}$ (fair)

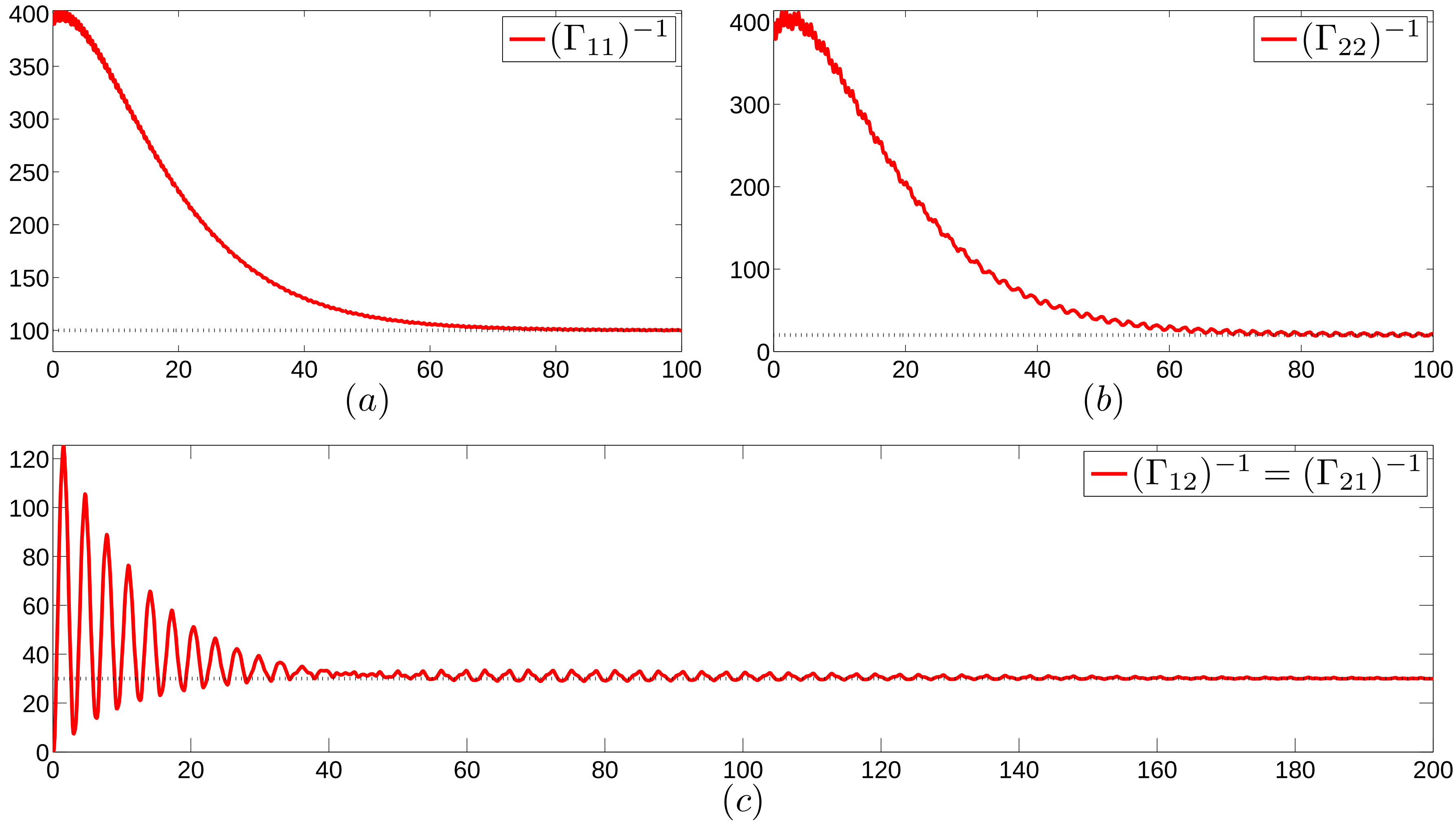
$\delta = 0.1$, $\omega = 0.1\mathrm{rad/s}$, $\omega_1 = 70\omega$, $\omega_2 = 50\omega$, $\omega_L' = 10$, $\omega_H' = 8$, $\omega_R' = 10$, $a = [0.1 \;\; 0.1]^T$, $K_g'' = 10^{-4}\mathrm{diag}([-25 \;\; -25])$, $K_n'' = \mathrm{diag}([1 \;\; 1])$, $\Gamma_0^{-1} = 400\mathrm{diag}([1 \;\; 1])$, $\hat{\theta}_0 = [2.5 \;\; 5]^T$, $Q^* = 100$, $\theta^* = [2 \;\; 4]^T$, $H_{11} = 100$, $H_{12} = H_{21} = 30$, and $H_{22} = 20$.

Simulation results



Phase portrait and level sets

Convergence of the estimate of the Hessian inverse matrix, $\Gamma(t)$



The "straight" transient in the phase space starts after $\Gamma(t)$ has converged.

Attempt at non-local stability analysis

For the scalar case we can prove semiglobal stability on the set $(\hat{\theta}, \gamma) \in \mathbb{R} \times \mathbb{R}_+$.

Error system

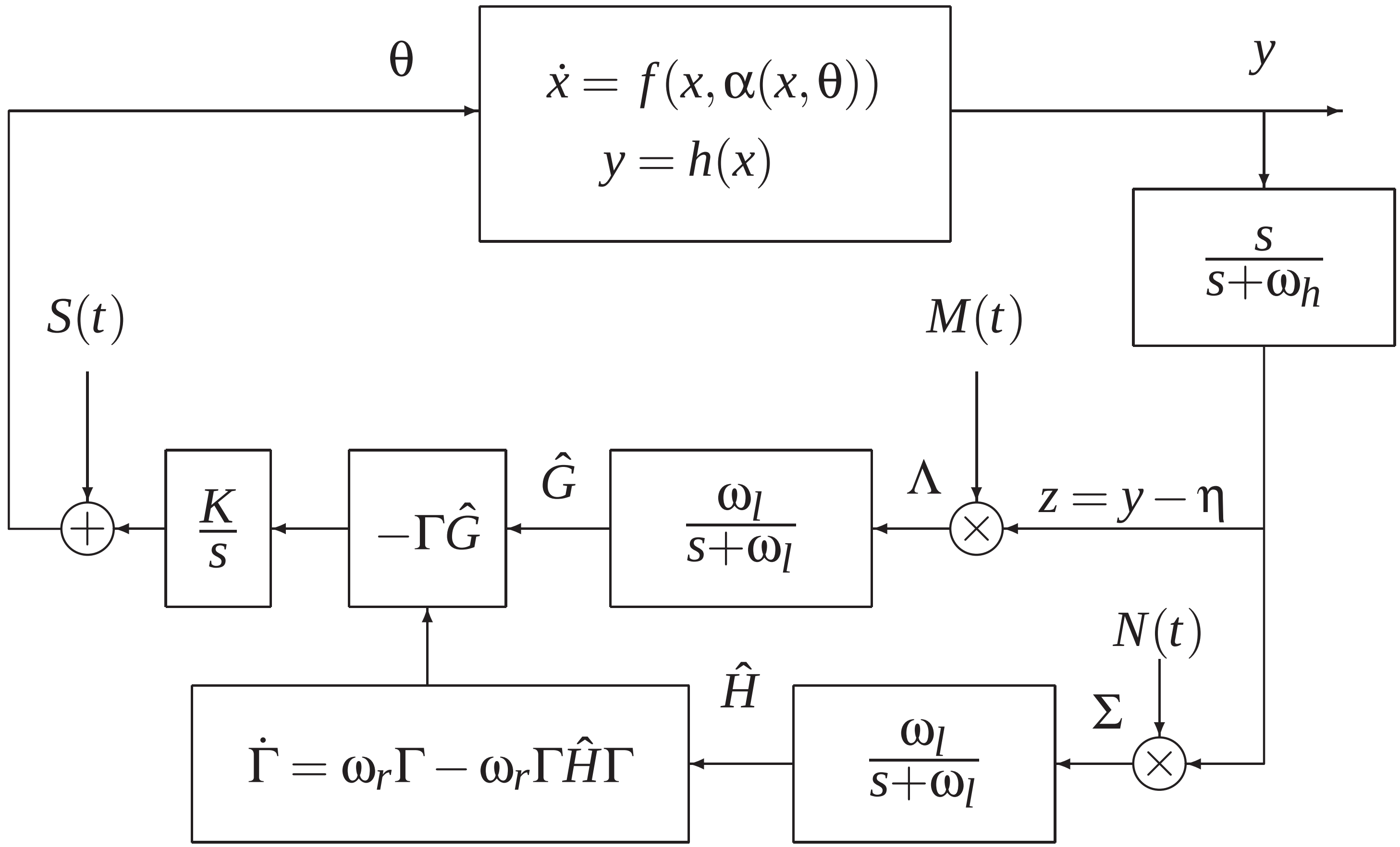
$$\begin{aligned} \dot{\tilde{\theta}} &= -\tilde{\theta} - \tilde{\theta}\tilde{\gamma} \\ \dot{\tilde{\gamma}} &= -\tilde{\gamma} - \tilde{\gamma}^2 \end{aligned}$$

Global Lyapunov fcn on $\mathbb{R} \times (-1, \infty)$:

$$V = \frac{1}{2}\ln\left(1 + \tilde{\theta}^2\right) + \tilde{\gamma} - \ln(1 + \tilde{\gamma})$$

(Hessian = 1)

Newton algorithm for dynamic systems



Nash Equilibrium Seeking

(non-cooperative games)

Non-Cooperative Games

Multiple players, multiple cost functions.

Team optimization — ‘easy’ mulitvariable problems.

Selfish optimization — harder, because overall convexity is lost.

Non-Cooperative Games

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Simplest case: two players, zero-sum game (H_∞ control). One saddle surface, equilibrium at the saddle point.

Harder: two players, non-zero sum. Two saddle surfaces, equilibrium at the intersection of their “ridges.”

Even harder: 3 or more players, Nash game

Two Players — Duopoly

Coca-Cola vs. Pepsi

Boeing vs. Airbus

Two Players — Duopoly

Coca-Cola vs. Pepsi

Boeing vs. Airbus

Let f_A and f_B be two firms that produce the same good and compete for profit by setting their respective prices, v_A and v_B.

Profit model:

$$\begin{aligned} J_A(t) &= i_A(t)\,(v_A(t) - m_A)\,, \\ J_B(t) &= i_B(t)\,(v_B(t) - m_B)\,, \end{aligned}$$

where i_A and i_B are the number of sales and m_A and m_B are the marginal costs.

Two Players — Duopoly

Coca-Cola vs. Pepsi
Boeing vs. Airbus

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where i_A and i_B are the number of sales and m_A and m_B are the marginal costs.

Sales model where the consumer prefers f_A:

$$i_A(t) = I - i_B(t), \qquad i_B(t) = \frac{v_A(t) - v_B(t)}{p},$$

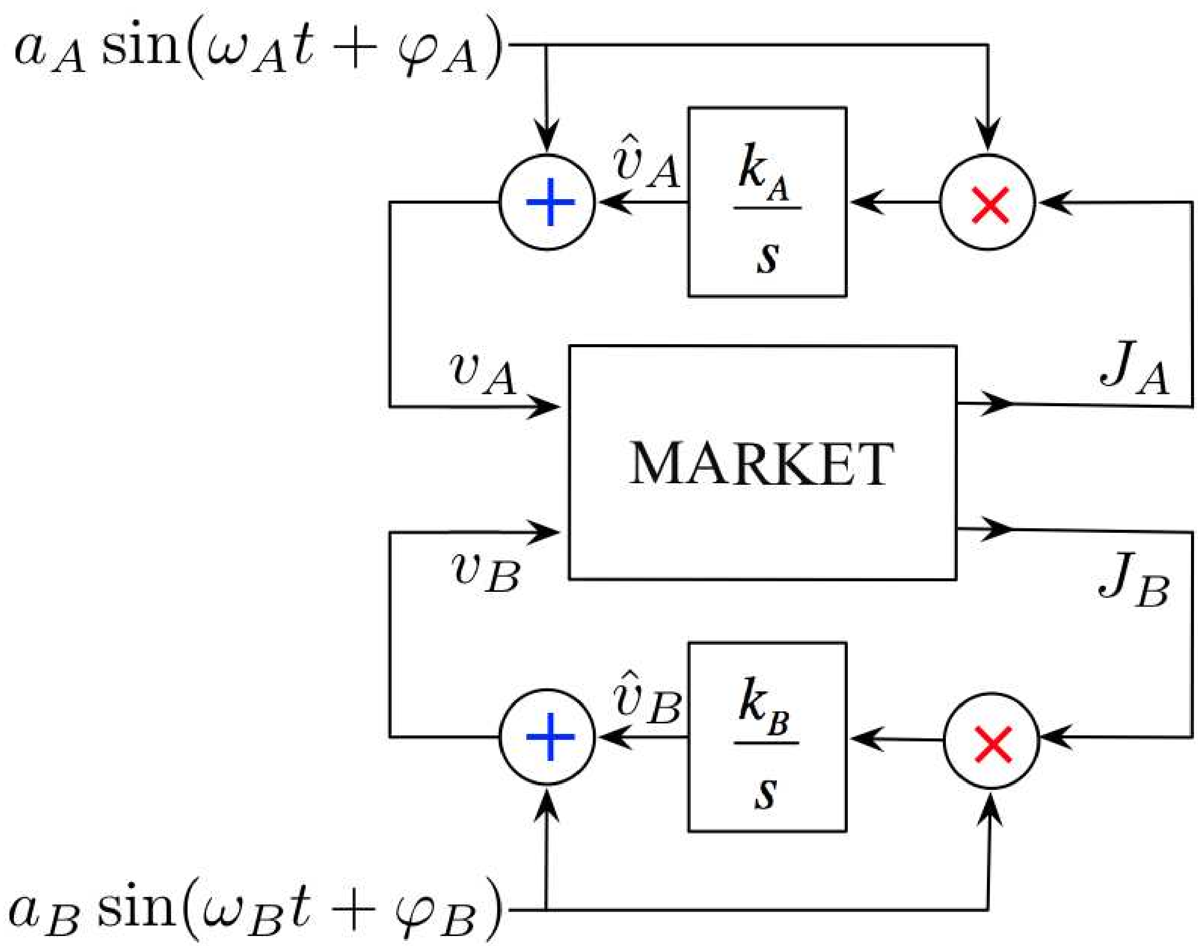
where I are the total sales and $p > 0$ quantifies the preference of the consumer for f_A.

The profit functions $J_A(v_A, v_B)$ and $J_B(v_A, v_B)$ are both quadratic functions of the prices v_A and v_B.

The Nash strategies are

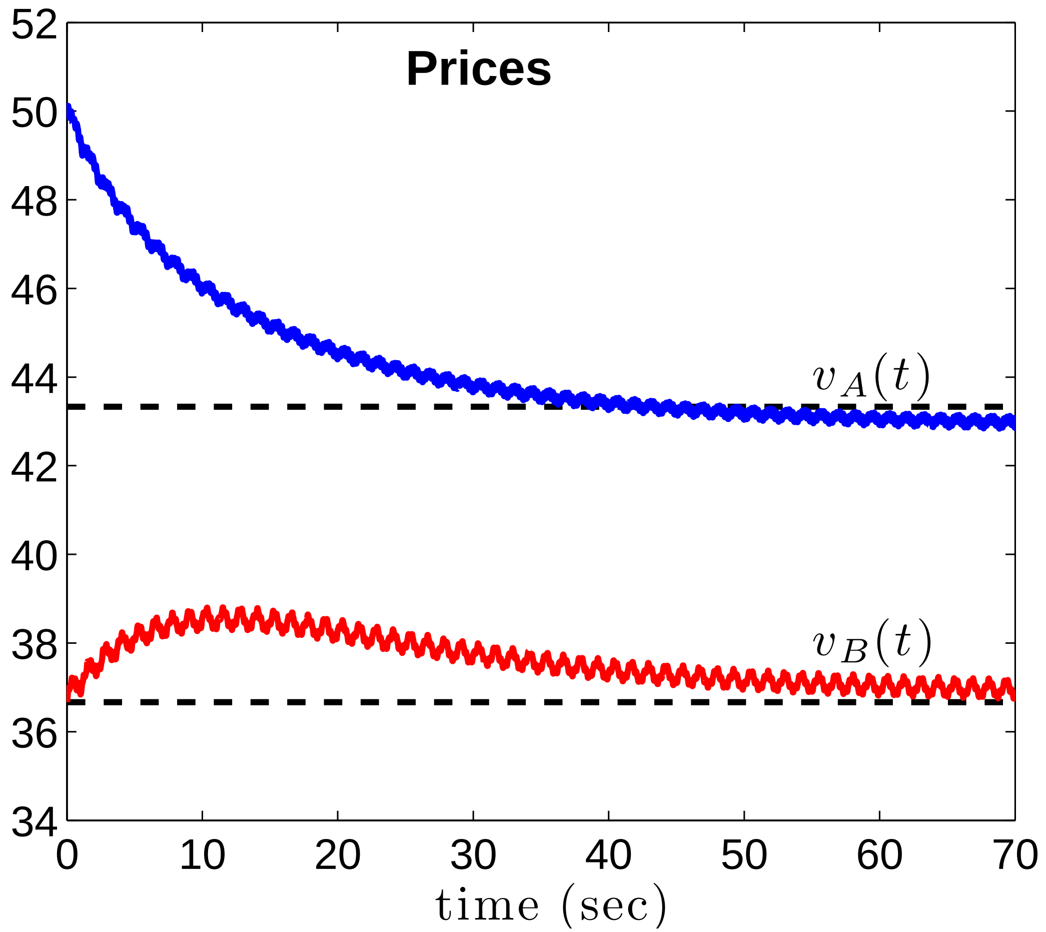
$$v_A^* = \frac{2m_A + m_B + 2Ip}{3}, \qquad v_B^* = \frac{m_A + 2m_B + Ip}{3}.$$

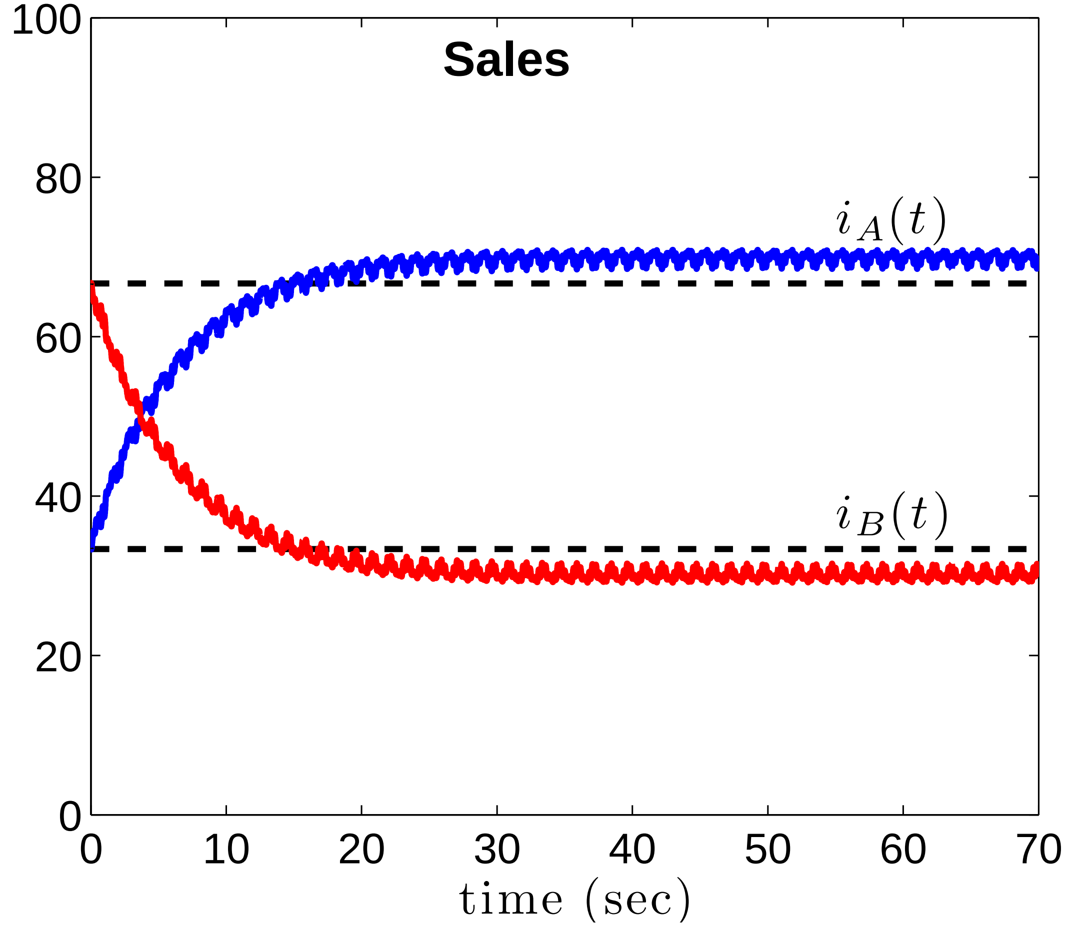
How can the players ever know each other's marginal costs, the customers preference, or the overall market demand?

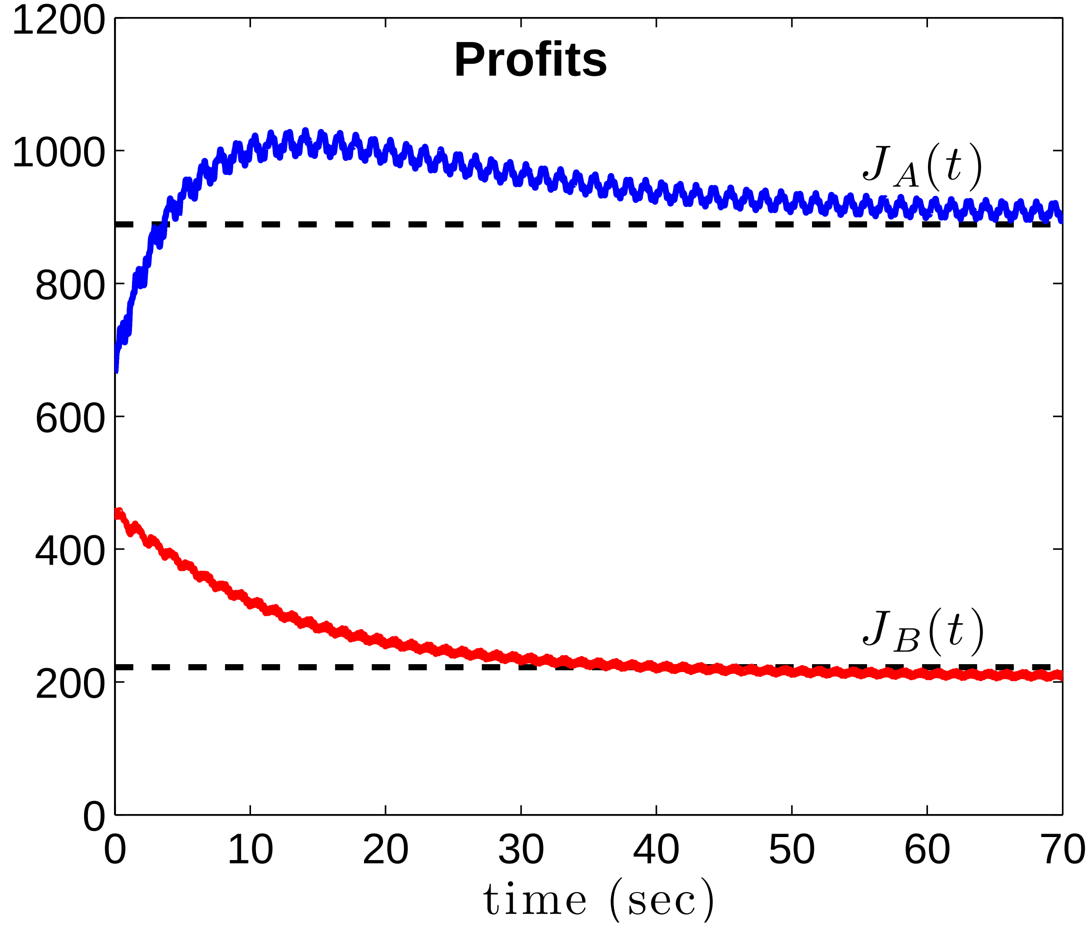


Extremum seeking applied by firms f_A and f_B in a duopoly

Simulation with $m_A = m_B = 30$, $I = 100$, $p = 0.2$.




Sales
100
80
60
40
20
0
0
10
20
30
40
50
60
70
time (sec)
$i_A(t)$
$i_B(t)$


Profits
$J_A(t)$
$J_B(t)$
1200
1000
800
600
400
200
0
0
10
20
30
40
50
60
70
time (sec)

Theorem 6 *Let $\omega_A \neq \omega_B$, $2\omega_A \neq \omega_B$, and $\omega_A \neq 2\omega_B$. There exists ω^* such that, for all $\omega_A, \omega_B > \omega^*$, if $|\Delta(0)|$ is sufficiently small, then for all $t \geq 0$,*

$$|\Delta(t)| \leq Me^{-mt}|\Delta(0)| + O\left(\frac{1}{\min(\omega_A, \omega_B)} + \max(a_A, a_B)\right),$$

where

$$\begin{aligned}
\Delta(t) &= \begin{pmatrix} v_A(t) - v_A^*, & v_B(t) - v_B^* \end{pmatrix}^T \\
M &= \sqrt{\frac{\max(k_A a_A^2, k_B a_B^2)}{\min(k_A a_A^2, k_B a_B^2)}} \\
m &= \frac{1}{2p}\min(k_A a_A^2, k_B a_B^2)
\end{aligned}$$

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\end{aligned}$$

Proof. Let $\tau = \underline{\omega} t$ and $\underline{\omega} = \min(\omega_A, \omega_B)$. The average system is

$$\frac{d}{d\tau}\begin{pmatrix} \tilde{v}_A^{\text{ave}} \\ \tilde{v}_B^{\text{ave}} \end{pmatrix} = \frac{1}{2\underline{\omega} p}\begin{pmatrix} -2k_A a_A^2 & k_A a_A^2 \\ k_B a_B^2 & -2k_B a_B^2 \end{pmatrix}\begin{pmatrix} \tilde{v}_A^{\text{ave}} \\ \tilde{v}_B^{\text{ave}} \end{pmatrix}.$$

Q.E.D.

General Nonquadratic Games with N Players

Consider the payoff function of player i:

$$J_i = h_i(u_i, u_{-i})$$

where $u_i \in \mathbb{R}$ is player i's action and $u_{-i} = [u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N]$ represents the actions of the other players.

General Nonquadratic Games with N Players

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ES strategy:

$$\begin{aligned} \dot{\hat{u}}_i(t) &= k_i \mu_i(t) J_i(t) \\ \mu_i(t) &= a_i \sin(\omega_i t + \varphi_i) \\ u_i(t) &= \hat{u}_i(t) + \mu_i(t) \end{aligned}$$

Assumption 3 *There exists at least one (possibly multiple) isolated Nash equilibrium* $u^* = [u_1^*, \ldots, u_N^*]$ *such that*

$$\frac{\partial h_i}{\partial u_i}(u^*) = 0\,, \qquad \frac{\partial^2 h_i}{\partial u_i^2}(u^*) < 0,$$

for all $i \in \{1, \ldots, N\}$.

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for all $i \in \{1, \dots, N\}$.

Assumption 4 *The matrix*

$$\Lambda = \begin{bmatrix} \frac{\partial^2 h_1(u^*)}{\partial u_1^2} & \frac{\partial^2 h_1(u^*)}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 h_1(u^*)}{\partial u_1 \partial u_N} \\ \frac{\partial^2 h_2(u^*)}{\partial u_1 \partial u_2} & \frac{\partial^2 h_2(u^*)}{\partial u_2^2} & & \\ \vdots & & \ddots & \\ \frac{\partial^2 h_N(u^*)}{\partial u_1 \partial u_N} & & & \frac{\partial^2 h_N(u^*)}{\partial u_N^2} \end{bmatrix}$$

is diagonally dominant and hence, nonsingular.

Theorem 7 *Let $\omega_i \neq \omega_j$, $\omega_i \neq \omega_j + \omega_k$, $2\omega_i \neq \omega_j + \omega_k$, and $\omega_i \neq 2\omega_j + \omega_k$ for all i, j, $k \in \{1, \ldots, N\}$. Then there exists ω^*, $\overline{a}$ and M, $m > 0$ such that, for all $\min_i \omega_i > \omega^*$ and $a_i \in (0, \overline{a})$, if $|\Delta(0)|$ is sufficiently small, then for all $t \geq 0$,*

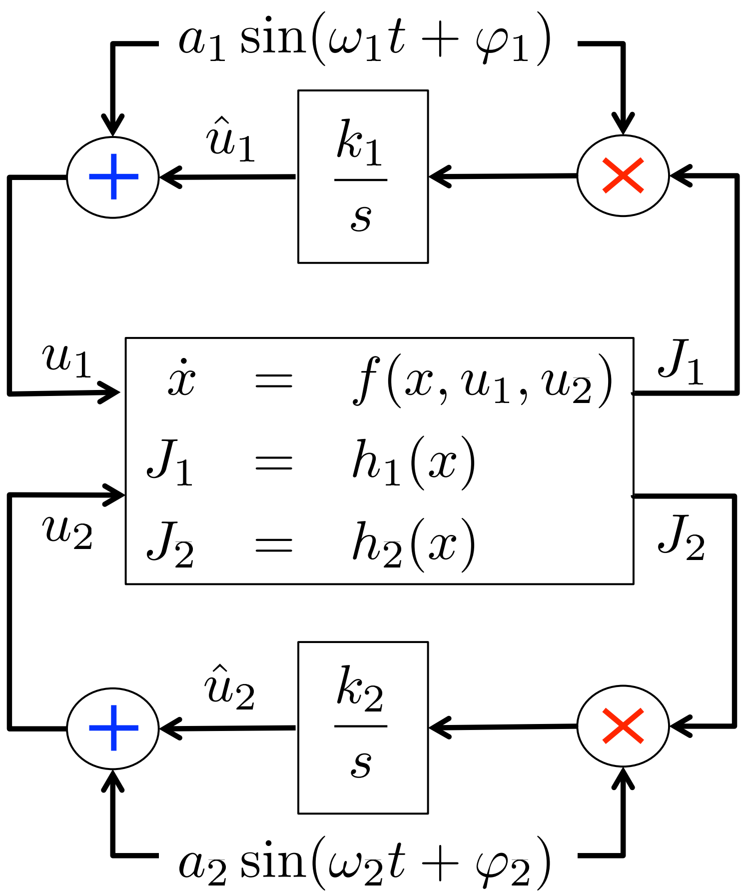
$$\left|\Delta(t)\right| \leq M e^{-mt} \left|\Delta(0)\right| + O\left(\max_i a_i^3\right),$$

where

$$\Delta(t) = \left[\hat{u}_1(t) - u_1^* - \textstyle\sum_{j=1}^N c_{jj}^1 a_j^2, \ \ldots, \ \hat{u}_N(t) - u_N^* - \sum_{j=1}^N c_{jj}^N a_j^2\right]$$

$$\begin{bmatrix} c_{ii}^1 \\ \vdots \\ c_{ii}^{i-1} \\ c_{ii}^i \\ c_{ii}^{i+1} \\ \vdots \\ c_{ii}^N \end{bmatrix} = -\frac{1}{4}\Lambda^{-1} \begin{bmatrix} \frac{\partial^3 h_1}{\partial u_1 \partial u_i^2}(u^*) \\ \vdots \\ \frac{\partial^3 h_{i-1}}{\partial u_{i-1} \partial u_i^2}(u^*) \\ \frac{1}{2}\frac{\partial^3 h_i}{\partial u_i^3}(u^*) \\ \frac{\partial^3 h_{i+1}}{\partial u_i^2 \partial u_{i+1}}(u^*) \\ \vdots \\ \frac{\partial^3 h_N}{\partial u_i^2 \partial u_N}(u^*) \end{bmatrix}$$

Numerical Example with Dynamics and Non-Quadratic Payoffs



$$\dot{x}_1 = -4x_1 + x_1 x_2 + u_1$$
$$\dot{x}_2 = -4x_2 + u_2$$
$$J_1 = -16x_1^2 + 8x_1^2 x_2 - x_1^2 x_2^2 - 4x_1 x_2^2 + 15x_1 x_2 + 4x_1$$
$$J_2 = -64x_2^3 + 48x_1 x_2 - 12x_1 x_2^2$$

$$
\begin{aligned}
\dot{x}_1 &= -4x_1 + x_1x_2 + u_1 \\
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J_2 &= -64x_2^3 + 48x_1x_2 - 12x_1x_2^2
\end{aligned}
$$

Steady-state payoffs

$$
\begin{aligned}
J_1 &= -u_1^2 + u_1u_2 + u_1 \\
J_2 &= -u_2^3 + 3u_1u_2
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_1 &= -4x_1 + x_1x_2 + u_1 \\
\dot{x}_2 &= -4x_2 + u_2 \\
J_1 &= -16x_1^2 + 8x_1^2x_2 - x_1^2x_2^2 - 4x_1x_2^2 + 15x_1x_2 + 4x_1 \\
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\end{aligned}
$$

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$$
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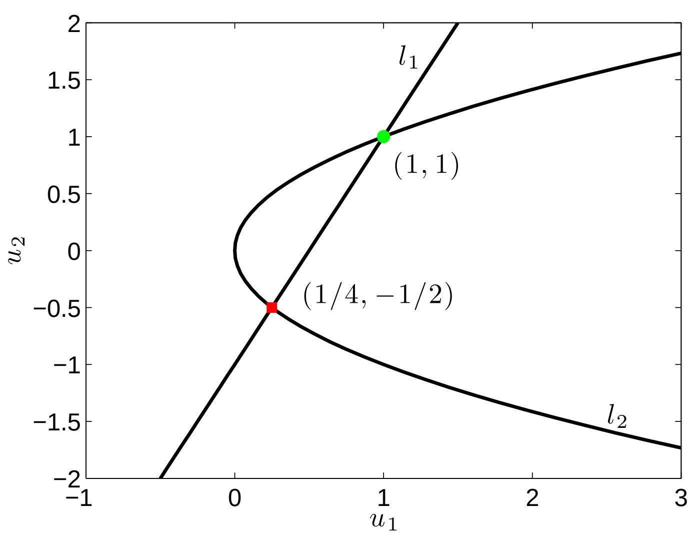
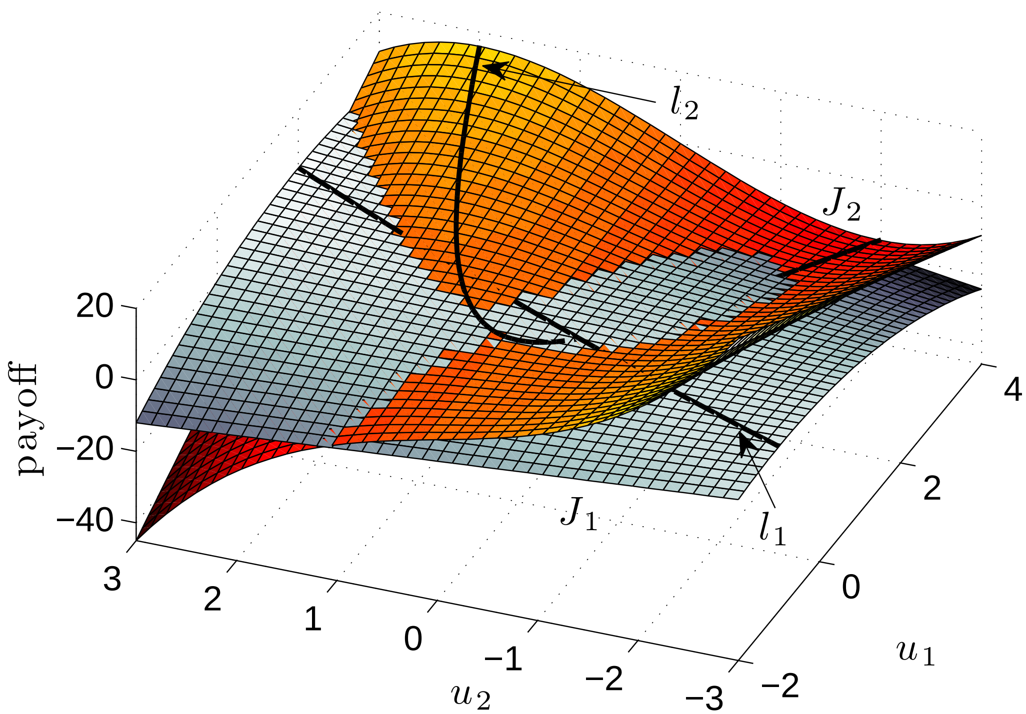
J_1 &= -u_1^2 + u_1u_2 + u_1 \\
J_2 &= -u_2^3 + 3u_1u_2
\end{aligned}
$$

Reaction curves

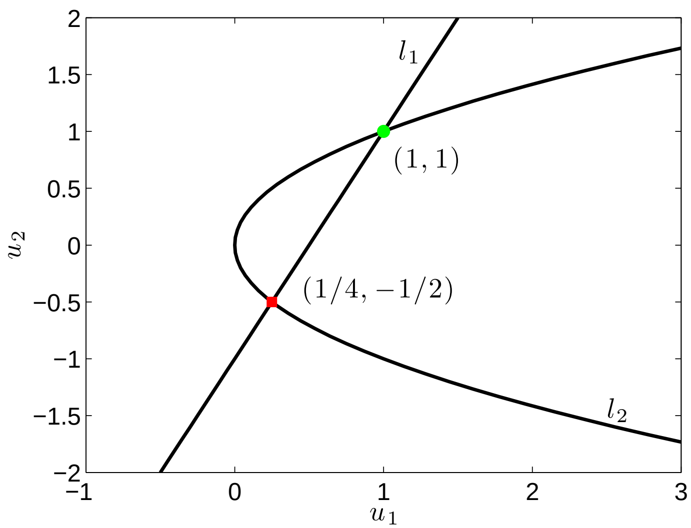
$$
\begin{aligned}
l_1 &\triangleq \left\{ u_1 = \frac{1}{2}(u_2 + 1) \right\} \\
l_2 &\triangleq \left\{ u_2^2 = u_1 \right\}
\end{aligned}
$$



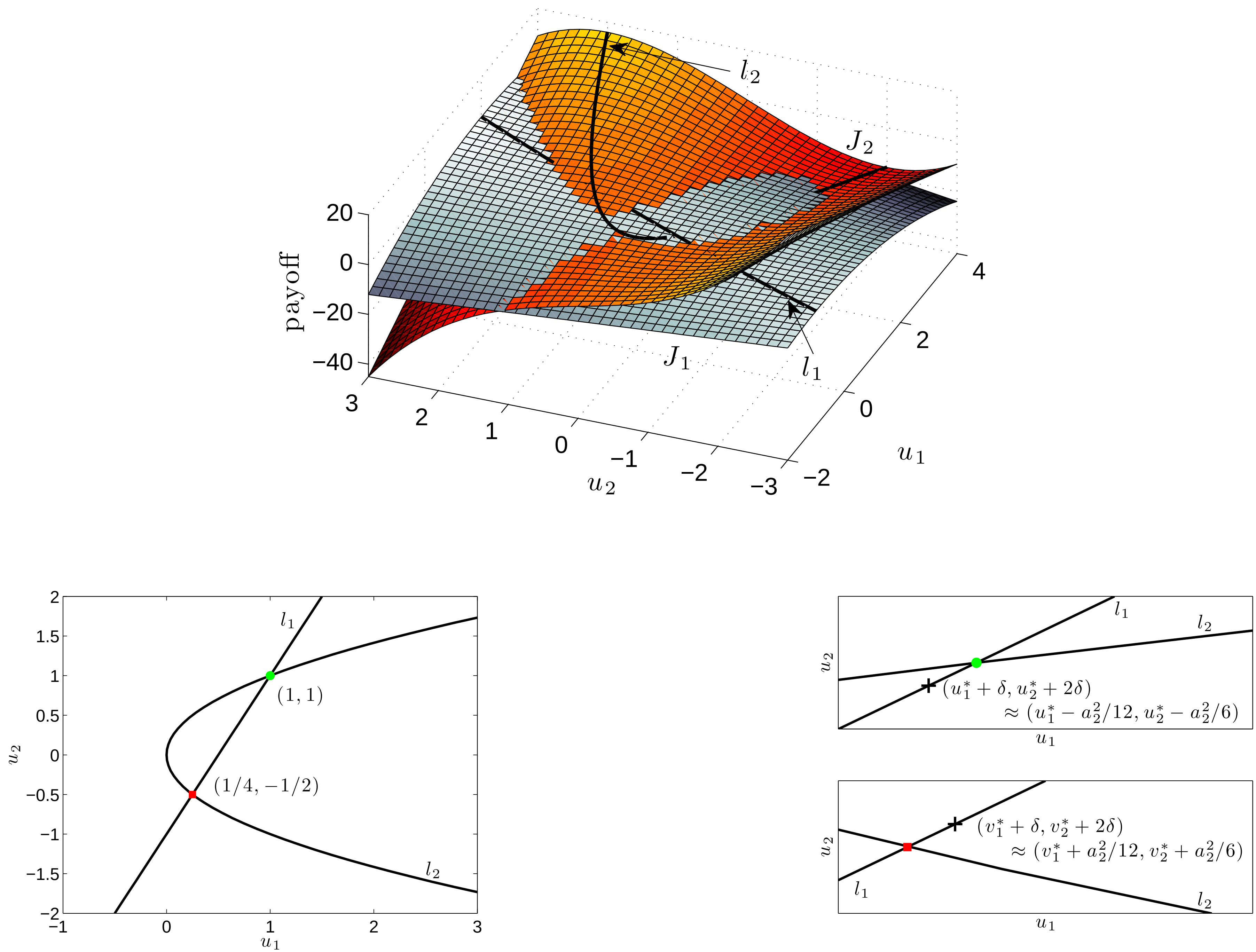
l_1
l_2
$(1, 1)$
$(1/4, -1/2)$
u_1
u_2
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−1
0
1
2
3



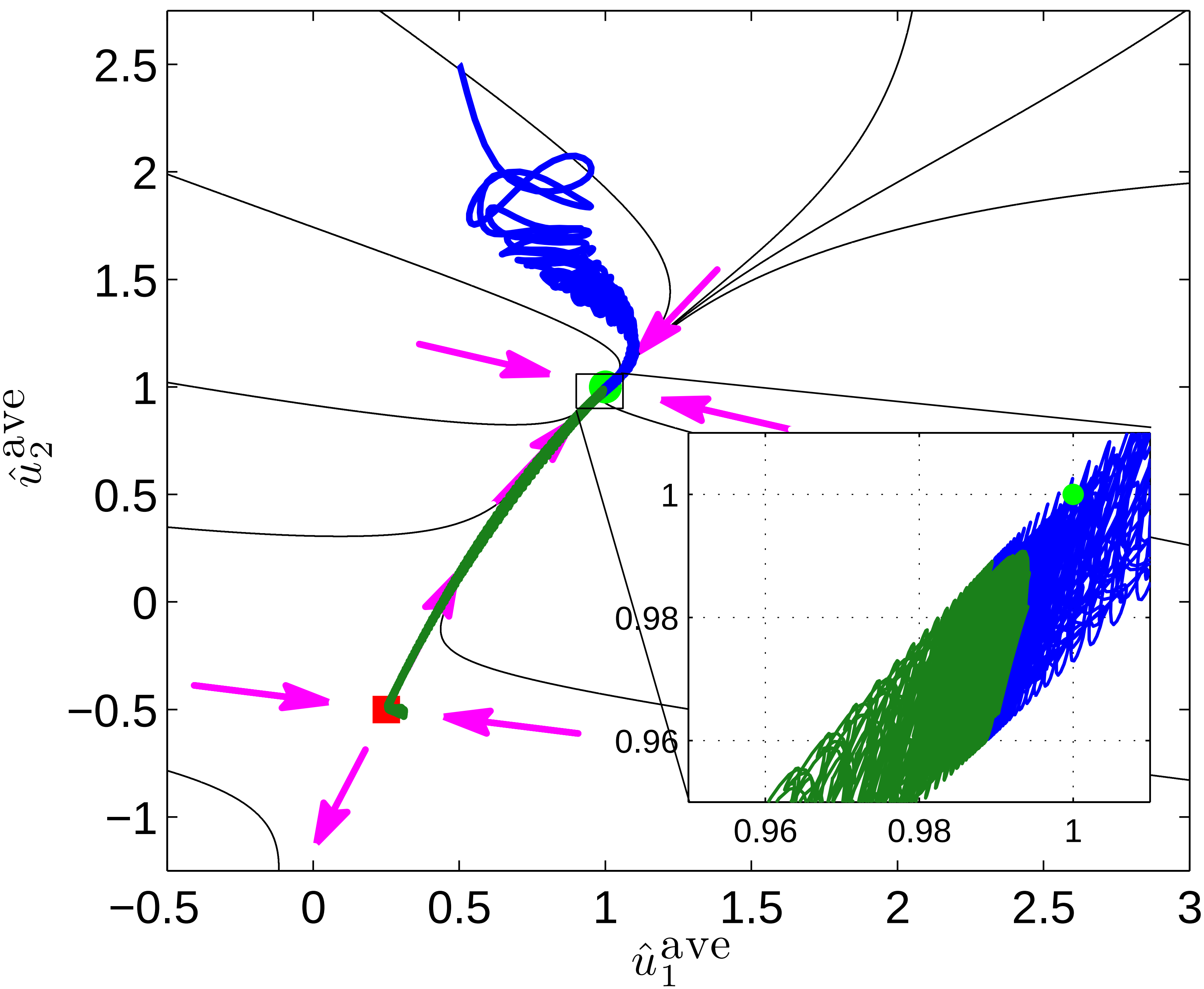
l_2
J_2
payoff
20
0
−20
−40
J_1
l_1
3
2
1
0
−1
−2
−3
u_2
4
2
0
−2
u_1



l_1
$(1, 1)$
$(1/4, -1/2)$
l_2
u_2
u_1
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−1
0
1
2
3



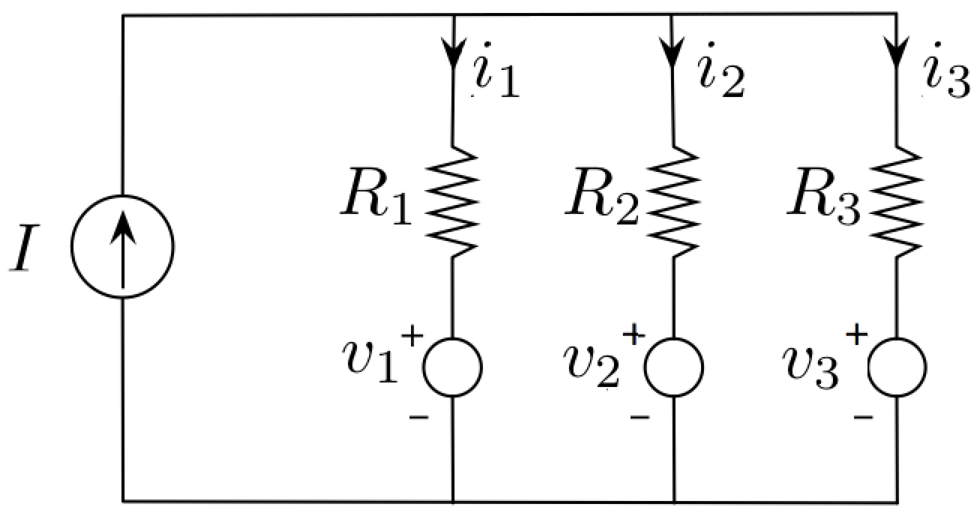
payoff
J_1
J_2
l_1
l_2
u_1
u_2
$(1, 1)$
$(1/4, -1/2)$
$(u_1^* + \delta, u_2^* + 2\delta)$
$\approx (u_1^* - a_2^2/12, u_2^* - a_2^2/6)$
$(v_1^* + \delta, v_2^* + 2\delta)$
$\approx (v_1^* + a_2^2/12, v_2^* + a_2^2/6)$



2.5
2
1.5
1
0.5
0
−0.5
−1
−0.5
0
0.5
1
1.5
2
2.5
3
$\hat{u}_1^{\mathrm{ave}}$
$\hat{u}_2^{\mathrm{ave}}$
1
0.98
0.96
0.96
0.98
1

Oligopoly (N competing firms)

$$\begin{aligned} v_i &= \text{price of firm } i \\ m_i &= \text{marginal cost of firm } i \\ i_i &= \text{sales volume of firm } i \\ 1/R_i &= \text{preference for (conductance of sales towards) firm } i \end{aligned}$$



$\boxed{J_i(t) = i_i(t)(v_i(t) - m_i)}$ $=$ profit of firm i $=$ power absorbed by generator i

For N players, the sales volume is obtained as

$$
\begin{aligned}
i_i(t) &= \frac{R_{||}}{R_i}\left(I - \frac{v_i(t)}{\overline{R}_i} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{v_j(t)}{R_j} \right), \\
R_{||} &= \left(\sum_{k=1}^{N} \frac{1}{R_k} \right)^{-1}, \quad \overline{R}_i = \left(\sum_{\substack{k=1 \\ k\neq i}}^{N} \frac{1}{R_k} \right)^{-1}
\end{aligned}
$$

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$$
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R_{||} &= \left(\sum_{k=1}^{N} \frac{1}{R_k}\right)^{-1}, \quad \overline{R}_i = \left(\sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{1}{R_k}\right)^{-1}
\end{aligned}
$$

Nash prices:

$$
v_i^* = \frac{\Lambda R_i}{2R_i + \overline{R}_i}\left(\overline{R}_i I + m_i + \sum_{j=1}^{N} \frac{m_j \overline{R}_i - m_i \overline{R}_j}{2R_j + \overline{R}_j}\right),
$$

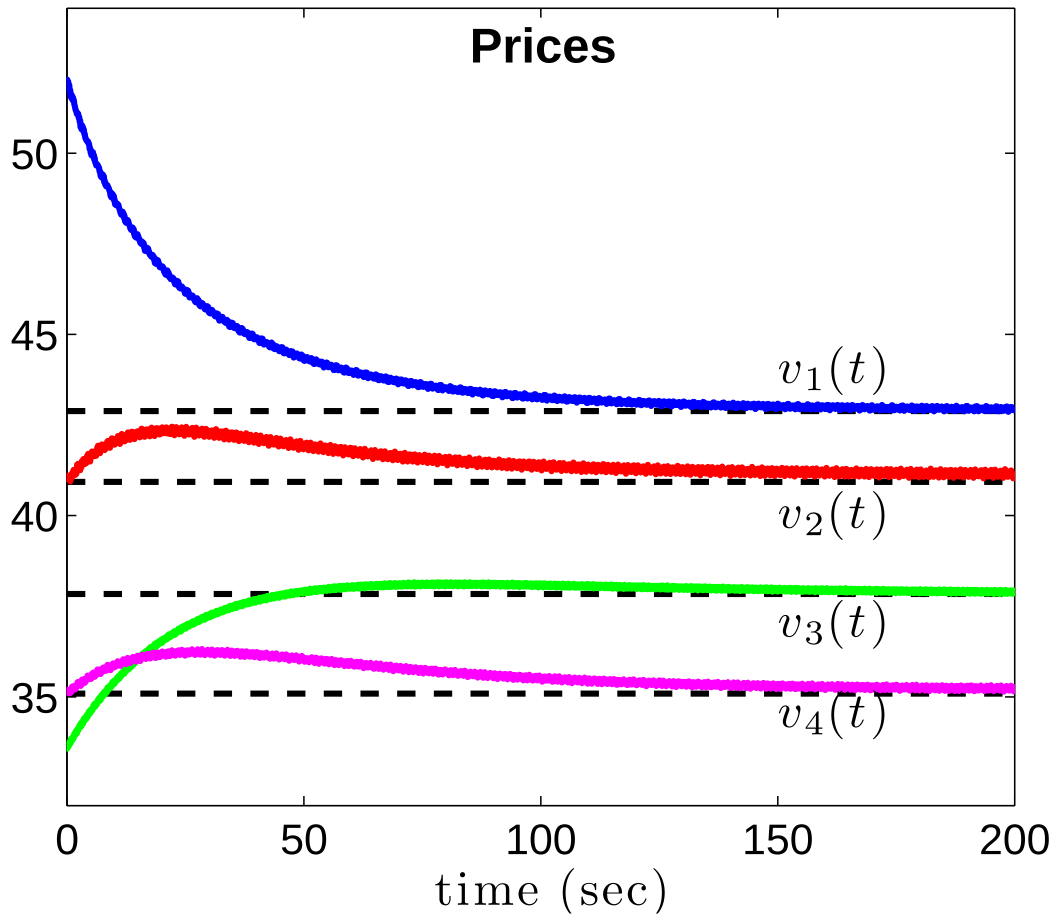
where

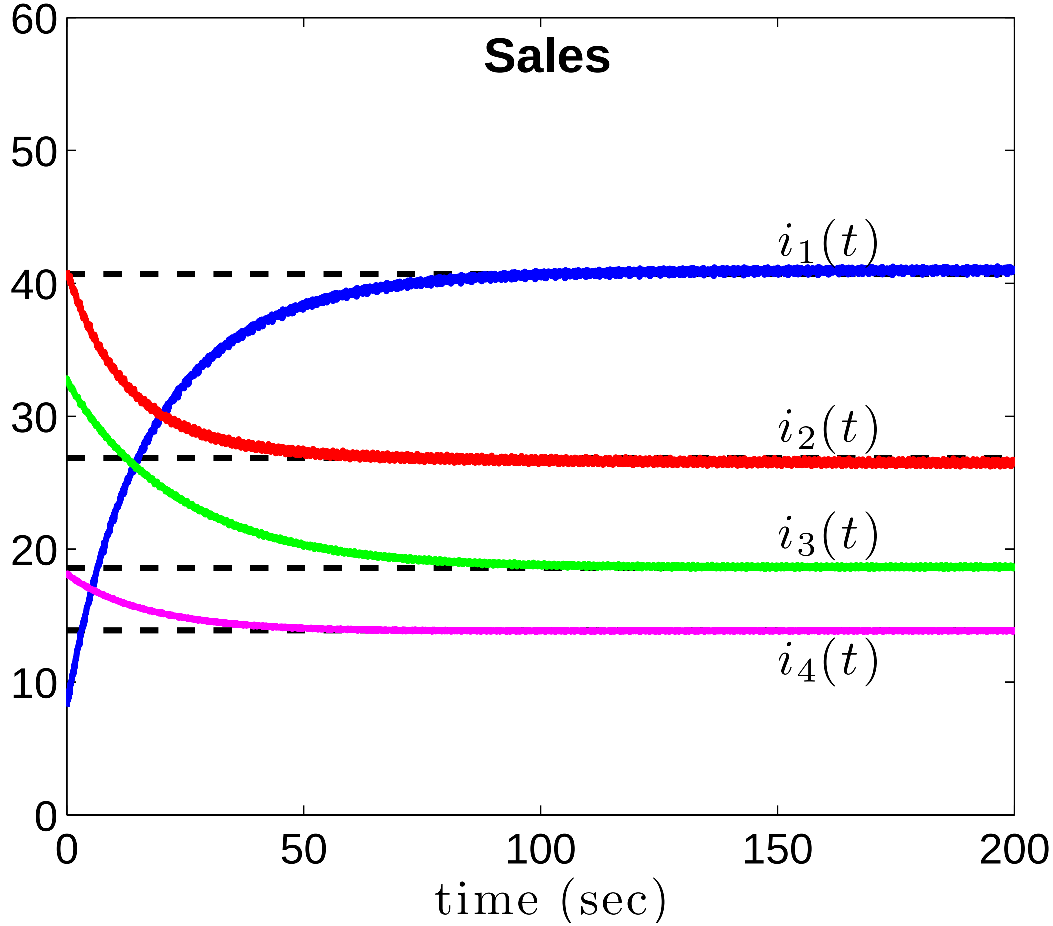
$$
\Lambda = \left(1 - \sum_{j=1}^{N} \frac{\overline{R}_j}{2R_j + \overline{R}_j}\right)^{-1} > 0
$$

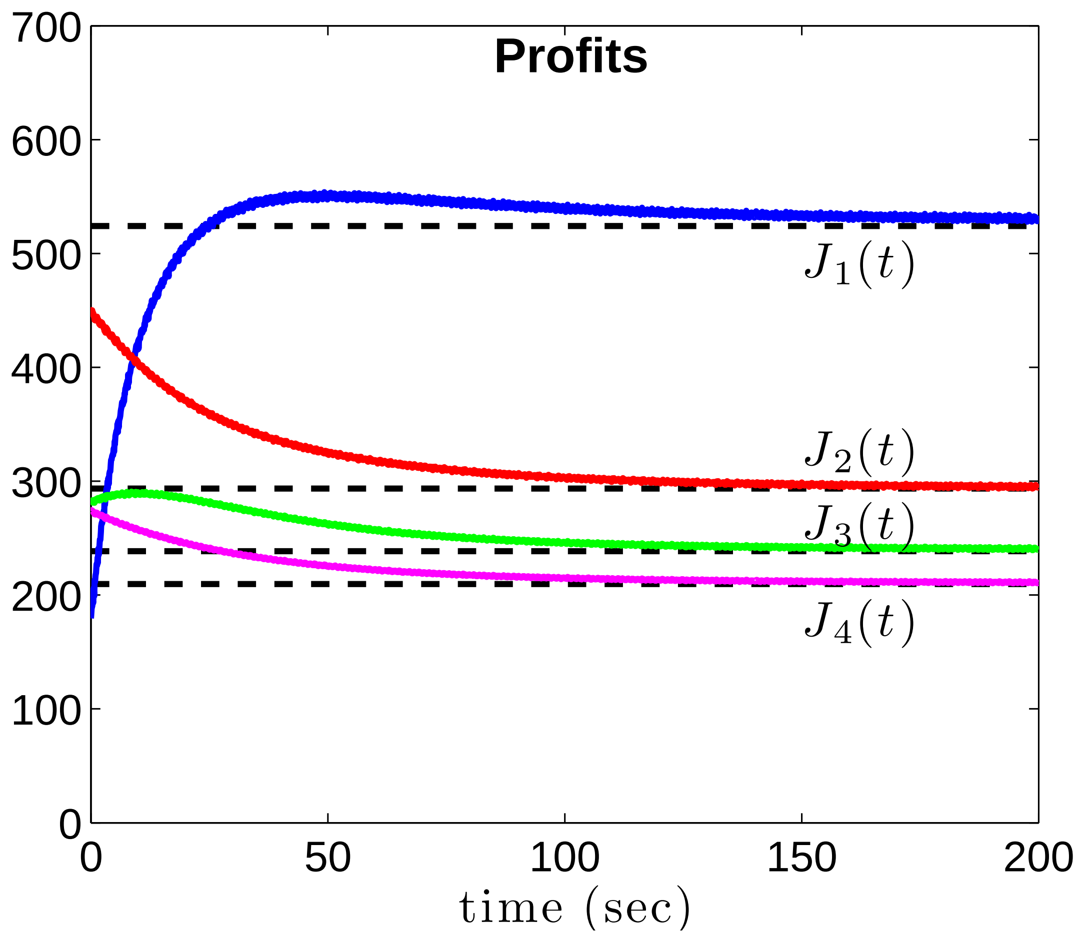
Extremum seeking strategy:

$$
\begin{aligned}
\frac{\mathrm{d}\hat{v}_i(t)}{\mathrm{d}t} &= k_i \mu_i(t) J_i(t)\,, \qquad \mu_i(t) = a_i \sin(\omega_i t + \varphi_i) \\
v_i(t) &= \hat{v}_i(t) + \mu_i(t)
\end{aligned}
$$

Simulation with $m_1 = 22$, $m_2 = 20$, $m_3 = 26$, $m_4 = 20$, $I = 100$, $R_1 = 0.25$, $R_2 = 0.78$, $R_3 = 1.10$, and $R_4 = 0.40$.




Sales
60
50
40
30
20
10
0
0
50
100
150
200
time (sec)
$i_1(t)$
$i_2(t)$
$i_3(t)$
$i_4(t)$


Profits
700
600
500
400
300
200
100
0
0
50
100
150
200
time (sec)
$J_1(t)$
$J_2(t)$
$J_3(t)$
$J_4(t)$

Theorem 8 *Let* $\omega_i \neq \omega_j$, $2\omega_i \neq \omega_j$ *for all* $i \neq j$, $i, j = 1, \ldots, N$. *There exists* ω^* *such that, for all* $\min_i \omega_i > \omega^*$, *if* $|\Delta(0)|$ *is sufficiently small, then for all* $t \geq 0$,

$$|\Delta(t)| \leq \Xi e^{-\xi t} |\Delta(0)| + O\left(\frac{1}{\min_i \omega_i} + \max_i a_i\right),$$

where

$$\begin{aligned}
\Delta(t) &= \begin{pmatrix} v_1(t) - v_1^*, & \ldots, & v_N(t) - v_N^* \end{pmatrix}^T \\
\Xi &= \sqrt{\frac{\max_i\{k_i a_i^2\}}{\min_i\{k_i a_i^2\}}} \\
\xi &= \frac{R_{||} \min_i\{k_i a_i^2\}}{2 \max_i\{R_i \Gamma_i\}} \\
\Gamma_i &= \min_{j \in \{1,\ldots,N\}, j \neq i} R_j
\end{aligned}$$

Proof. Let $\tau = \underline{\omega} t$ where $\underline{\omega} = \min_i \omega_i$. The average system is obtained as $\frac{d}{d\tau}\tilde{v}^{\text{ave}} = A\tilde{v}^{\text{ave}}$ where

$$A = \frac{R_{||}}{2\underline{\omega}} \begin{pmatrix} -\frac{2k_1 a_1^2}{R_1 \overline{R}_1} & \frac{k_1 a_1^2}{R_1 R_2} & \cdots & \frac{k_1 a_1^2}{R_1 R_N} \\ \frac{k_2 a_2^2}{R_2 R_1} & -\frac{2k_2 a_2^2}{R_2 \overline{R}_2} & & \\ \vdots & & \ddots & \\ \frac{k_N a_N^2}{R_N R_1} & & & -\frac{2k_N a_N^2}{R_N \overline{R}_N} \end{pmatrix}$$

is diagonally dominant no matter what the R_i's.

Let $V = (\tilde{v}^{\text{ave}})^T P \tilde{v}^{\text{ave}}$ be a Lyapunov function, where $P = \frac{\underline{\omega}}{R_{||}} \text{diag}\left(\frac{1}{k_1 a_1^2}, \ldots, \frac{1}{k_N a_N^2}\right)$ and satisfies the Lyapunov equation $PA + A^T P = -Q$,

$$Q = \begin{pmatrix} \frac{2}{R_1 \overline{R}_1} & -\frac{1}{R_1 R_2} & \cdots & -\frac{1}{R_1 R_N} \\ -\frac{1}{R_2 R_1} & \frac{2}{R_2 \overline{R}_2} & & \\ \vdots & & \ddots & \\ -\frac{1}{R_N R_1} & & & \frac{2}{R_N \overline{R}_N} \end{pmatrix}.$$

The matrix Q is positive definite symmetric and diagonally dominant, namely,

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} \left| q_{i,j} \right| = \frac{1}{R_i \overline{R}_i} < \frac{2}{R_i \overline{R}_i} = \left| q_{i,i} \right| .$$

From the Gershgorin Theorem, $\lambda_i(Q) \in \frac{1}{R_i \overline{R}_i}[1,3]$, which implies that

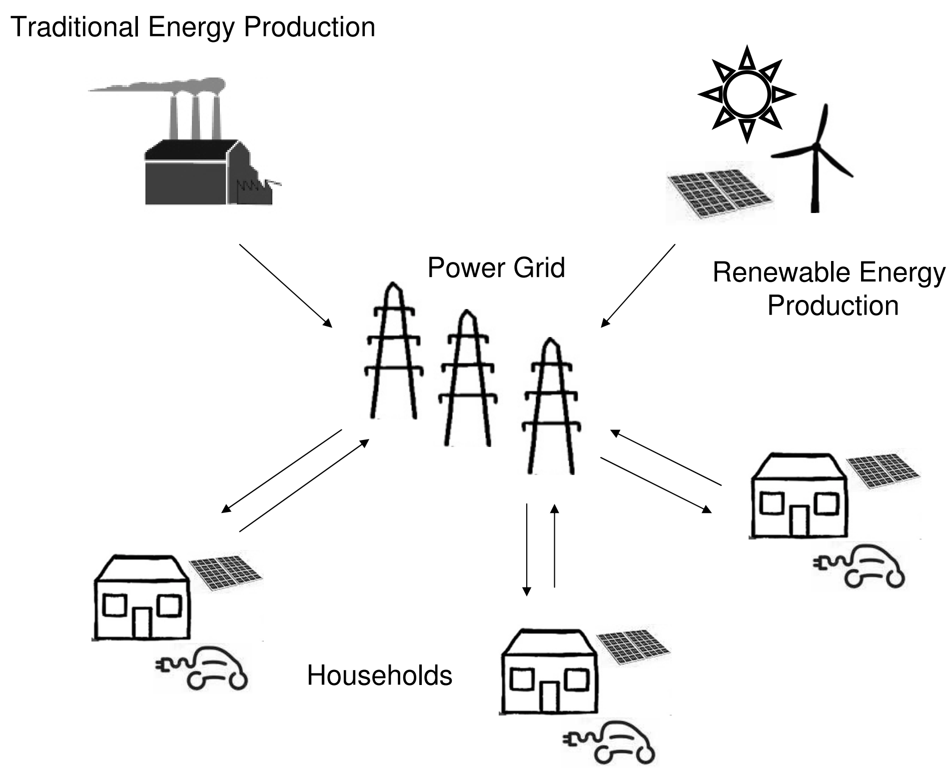
$$\lambda_{\min}(Q) > \frac{1}{\max_i \{R_i \overline{R}_i\}} > \frac{1}{\max\{R_i \Gamma_i\}}.$$

Q.E.D.

Continuum of Players

Stock market (Robert Aumann)

Battery-equipped ("plug-in electric") vehicles connected to the power grid and trading power with utilities and other households



Oligopoly w/ uncountably many non-atomic players, indexed by continuum index $x \in [0,1]$.

The profit of firm $f(x)$:

$$J(x,t) = i(x,t)\left(v(x,t) - m(x)\right),$$

with the sales modeled as

$$\begin{aligned} i(x,t) &= \frac{R_{||}}{R(x)}\left(I - \frac{v(x,t)}{R_{||}} + \int_0^1 \frac{v(y,t)}{R(y)}\,dy\right), \\ R_{||} &= \left(\int_0^1 \frac{dy}{R(y)}\right)^{-1}. \end{aligned}$$

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The Nash equilibrium values of the prices and the corresponding sales are

$$\begin{aligned} v^*(x) &= R_{||}\left(I + \frac{1}{2}\frac{m(x)}{R_{||}} + \frac{1}{2}\int_0^1 \frac{m(y)}{R(y)}\,dy\right), \\ i^*(x) &= \frac{R_{||}}{R(x)}\left(I - \frac{1}{2}\frac{m(x)}{R_{||}} + \frac{1}{2}\int_0^1 \frac{m(y)}{R(y)}\,dy\right). \end{aligned}$$

Extremum seeking algorithm:

$$
\begin{aligned}
\frac{\partial}{\partial t}\hat{v}(x,t) &= k(x)\mu(x,t)J(x,t) \\
\mu(x,t) &= a(x)\sin(\omega(x)t+\varphi(x)) \\
v(x,t) &= \hat{v}(x,t)+\mu(x,t)
\end{aligned}
$$

where $a(x), k(x) > 0$, for all $x \in [0,1]$.

Not a PDE but an ODE with a continuum state.

Integro(x)-differential(t) equation

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Satisfied if no frequency is used by more than a countable number of players.

(The set $\Omega_{\underline{\omega}}$ contains all functions that are either strictly increasing or strictly decreasing, as well as all bounded $C^1[0,1]$ positive functions whose derivative is zero on a set of measure zero.)

Theorem 9 *There exists* $\underline{\omega}^*$ *such that, for all functions* $\omega \in \Omega_{\underline{\omega}^*}$*, if the* $L_2[0,1]$ *norm of* $\Delta(x,0)$ *is sufficiently small, then for all* $t \geq 0$*,*

$$\int_0^1 \Delta^2(x,t)dx \leq \Sigma e^{-\sigma t} \int_0^1 \Delta^2(x,0)dx + O\left(\frac{1}{\min_x \omega^2(x)} + \max_x a^2(x)\right),$$

where

$$\begin{aligned}
\Delta(x,t) &= v(x,t) - v^*(x) \\
\Sigma &= \frac{\max_x\{k(x)a^2(x)\}}{\min_x\{k(x)a^2(x)\}} \\
\sigma &= \frac{\min_x\{k(x)a^2(x)\}}{\max_x\{R(x)\}}
\end{aligned}$$

$\Omega_{\underline{\omega}}$: set of bounded positive measurable functions $\omega : [0,1] \to \mathbb{R}_+$ bounded from below by $\underline{\omega}$ such that no element in the union of the image of $\omega(\cdot)$ and $2\omega(\cdot)$ has a level set of positive measure.

Proof. Error system

$$\frac{\partial}{\partial t}\tilde{v}(x,t) = \frac{k(x)}{R(x)}G[\tilde{v},R,i^*,\mu](x,t),$$

with the operator G defined as

$$\begin{aligned} G[\tilde{v},R,i^*,\mu](x,t) \quad &\triangleq \quad \mu(x,t)\Bigg[\left(R(x)i^*(x)-\tilde{v}(x,t)+\left\langle \frac{R_{||}}{R},\tilde{v}\right\rangle(t)\right)(R(x)i^*(x)+\tilde{v}(x,t)) \\ &+\mu(x,t)\left(-2\tilde{v}(x,t)+\left\langle \frac{R_{||}}{R},\tilde{v}\right\rangle(t)\right) \\ &+\left\langle \frac{R_{||}}{R},\mu\right\rangle(t)\,(R(x)i^*(x)+\tilde{v}(x,t))+\mu(x,t)\left\langle \frac{R_{||}}{R},\mu\right\rangle(t)-\mu^2(x,t)\Bigg], \end{aligned}$$

where $\langle a,b\rangle(t) \triangleq \int_0^1 a(y,t)b(y,t)dy$.

Recall: $\mu(x,t) = a(x)\sin(\omega(x)t+\varphi(x))$

To apply infinite-time averaging ("general averaging") to the infinite dimensional system, we have to compute integrals in both x and time and verify the conditions of the **dominated convergence theorem** for their integrands, to justify swapping the order of integrals in x and limits in τ.

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Let $\underline{\omega} = \min_x\{\omega(x)\}$, $\gamma(x) = \omega(x)/\underline{\omega}$, and $\tau = \underline{\omega}t$.

We obtain the average system

$$\underbrace{\frac{\partial}{\partial\tau}\tilde{v}^{\text{ave}}(x,\tau)}_{\text{derivative in time}} = -\frac{k(x)a^2(x)}{\underline{\omega}R(x)}\tilde{v}^{\text{ave}}(x,\tau) + \frac{R_{||}}{2}\frac{k(x)a^2(x)}{\underline{\omega}R(x)}\underbrace{\int_0^1 \frac{\tilde{v}^{\text{ave}}(y,\tau)}{R(y)}dy}_{\text{integral in } x}$$

Let $V(\tau)$ be a Lyapunov functional defined as

$$V(\tau) = \frac{\omega}{2}\int_0^1 \frac{1}{k(x)a^2(x)}\left(\tilde{v}^{\text{ave}}\right)^2(x,\tau)\,dx$$

and bounded by

$$\frac{\omega\int_0^1 \left(\tilde{v}^{\text{ave}}\right)^2(x,\tau)\,dx}{2\max_x\{k(x)a^2(x)\}} \leq V(\tau) \leq \frac{\omega\int_0^1 \left(\tilde{v}^{\text{ave}}\right)^2(x,\tau)\,dx}{2\min_x\{k(x)a^2(x)\}}$$

Taking the time derivative and applying the Cauchy-Schwarz inequality, we obtain

$$\dot{V} \leq -\frac{1}{2}\int_0^1 \frac{\left(\tilde{v}^{\text{ave}}\right)^2(x,\tau)}{R(x)}\,dx$$

From the infinite-dimensional averaging theory in [Hale and Verduyn Lunel, 1990], the result of the theorem follows.

Q.E.D.

Proposition *The spectrum of the average system is*

$$\underbrace{\left\{-\frac{k(x)a^2(x)}{R(x)}, \quad x \in [0,1]\right\}}_{\text{continuous stable spectrum}} \cup \underbrace{\left\{\text{all } \lambda \in \mathbb{C} \text{ that satisfy } \int_0^1 \frac{\lambda R_{||}}{\lambda R(x) + k(x)a^2(x)} dx = -1\right\}}_{\text{a stable discrete eigenvalue}}$$

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Example Let $k(x)=a(x)\equiv 1$ and $R(x)=\frac{1}{2}+x$ (linearly growing resistance).

The spectrum is

$$\left[-2,-\frac{2}{3}\right]\cup\left\{-\frac{1}{2}\right\}$$

GPS-Denied Source Seeking

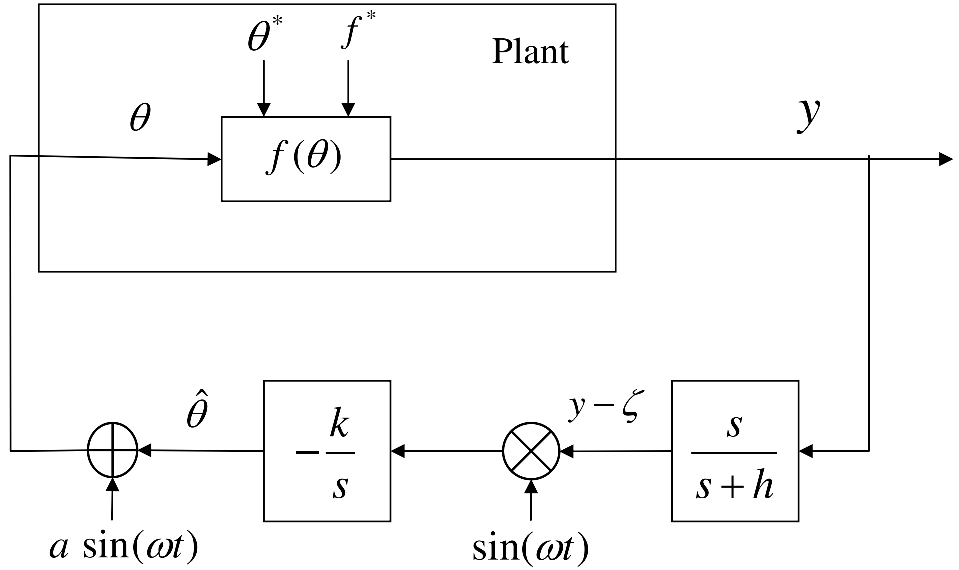
Fish seeking food in vortex flow

$$\text{Jukowski foil curvature}(t) = \cos(\omega t) + k \sin(\omega t)\ \underbrace{H(s)[J(t)]}_{\substack{\text{high-pass} \\ \text{filtered} \\ \text{concentration}}}$$

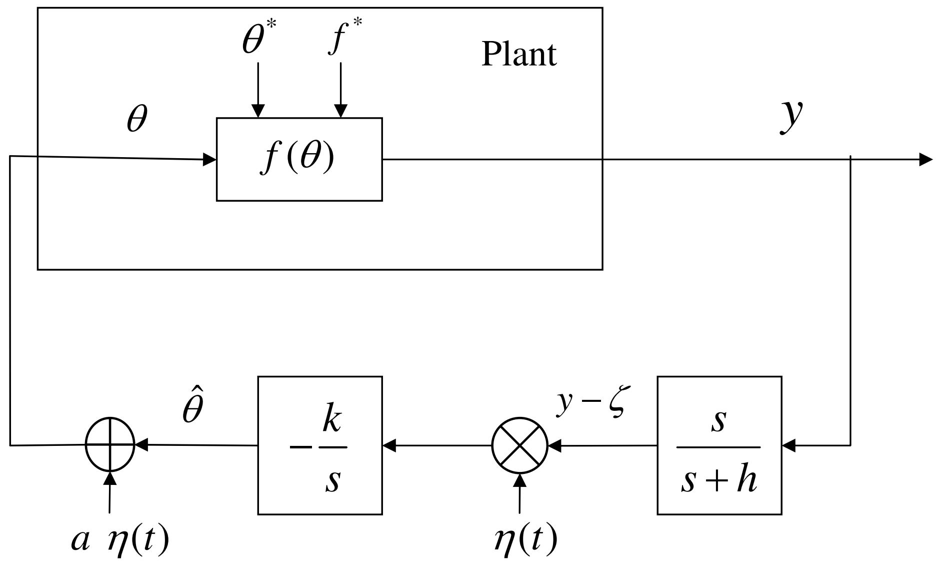
MOVIE

Stochastic Extremum Seeking

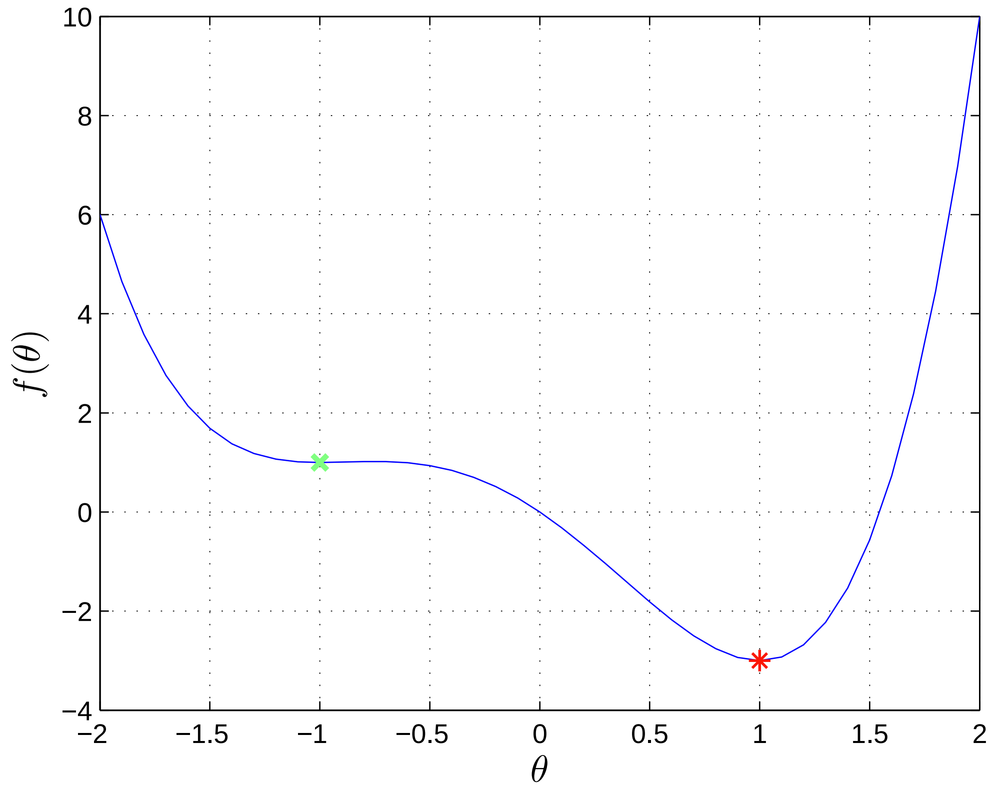
Deterministic ES



Stochastic ES



$$\left(\eta = \frac{\sqrt{\varepsilon q}}{\varepsilon s+1}[\dot{W}] \quad \text{or} \quad \varepsilon d\eta = -\eta dt + \sqrt{\varepsilon q}\, dW\ \right)$$

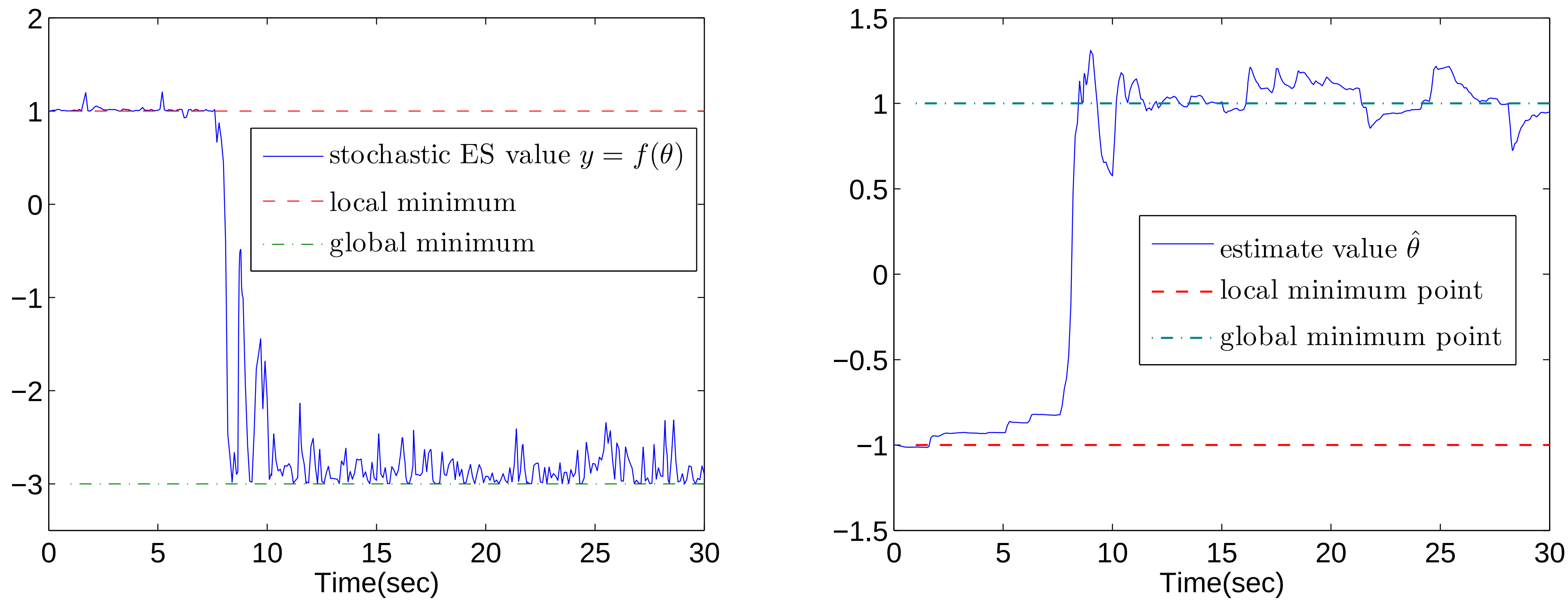


A quartic static map

$$f(\theta) = \theta^4 + \theta^3 - 2\theta^2 - 3\theta\,,$$

with local minimum $f(-1) = 1$ and global minimum $f(1) = -3$.

2nd derivatives at the minima are $f''(-1) = 2 < 14 = f''(1)$, which is consistent with the global min at $\theta = 1$ being much "deeper" and "sharper" than the local min at $\theta = -1$.



Time response of a discrete-time version of the stochastic ES algorithm, starting from the local minimum, $\hat{\theta}(0) = -1$. The parameters are chosen as $q = 1, \varepsilon = 0.25$, $a = 0.8, k = 10$.

A heuristic analysis of a simple stochastic ES algorithm

To simplify our analysis, we eliminate the washout filter.

Perturbation signal (colored noise)

$$\varepsilon d\eta = -\eta dt + \sqrt{\varepsilon} q dW \tag{2}$$

Input

$$\theta(t) = \hat{\theta}(t) + a\eta(t) \tag{3}$$

Estimation error

$$\tilde{\theta}(t) = \theta^* - \hat{\theta}(t) \tag{4}$$

Estimation error governed by

$$\begin{aligned} \dot{\tilde{\theta}}(t) &= -\dot{\hat{\theta}}(t) \\ &= k\eta(t) f(\theta(t)) \,. \end{aligned} \tag{5}$$

Applying the Taylor expansion to $f(\theta)$ around θ^* up to second order we get

$$f(\theta) \approx f(\theta^*) + \underbrace{f'(\theta^*)}_{=0 \text{ by assmpn}} \left(a\eta - \tilde{\theta}\right) + \frac{1}{2} f''(\theta^*) \left(a\eta - \tilde{\theta}\right)^2 . \tag{6}$$

Substituting (6) into (5) and grouping the terms in powers of η we obtain

$$\begin{aligned} \dot{\tilde{\theta}}(t) \approx\ & k \Big\{ \eta(t) \left[f(\theta^*) + \frac{1}{2} f''(\theta^*) \tilde{\theta}^2(t) \right] \\ & - \eta^2(t) a f''(\theta^*) \tilde{\theta}(t) \\ & + \eta^3(t) \frac{a^2}{2} f''(\theta^*) \Big\} . \end{aligned} \tag{7}$$

The signal $\eta(t)$ is governed by $\varepsilon d\eta = -\eta dt + \sqrt{\varepsilon} q dW$, where $W(t)$ is the Wiener process. With small ε, the signal η is a close approximation of white noise $\dot{W}(t)$.

Using elementary Ito calculus,

$$\lim_{t\to\infty} E\{\eta(t)\} = 0 \tag{8}$$

$$\lim_{t\to\infty} E\left\{\eta^2(t)\right\} = \frac{q^2}{2} \tag{9}$$

$$\lim_{t\to\infty} E\left\{\eta^3(t)\right\} = 0\,. \tag{10}$$

To illustrate how these relations are obtained, we consider the case of η^2, namely, (9), which is obtained by applying Ito's differentiation rule to η^2, which yields the ODE

$$\frac{\varepsilon}{2}\frac{dE\{\eta^2\}}{dt} = -E\{\eta^2\} + \frac{q^2}{2}$$

The solution of this linear ODE is

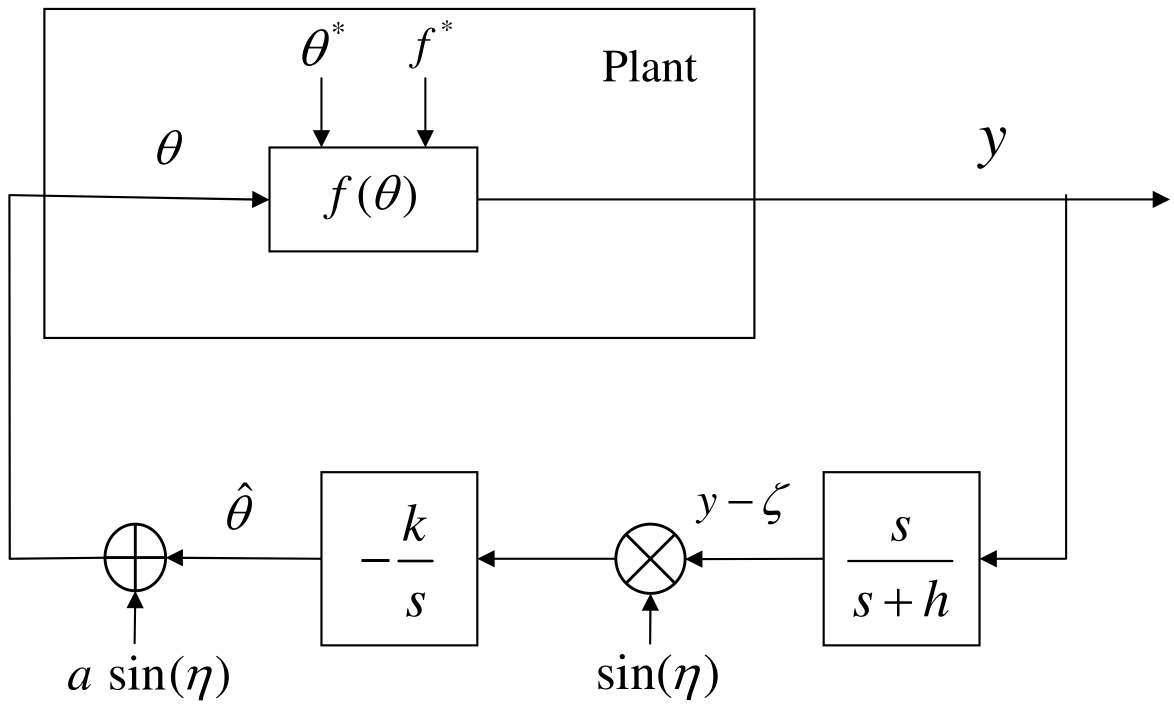
$$\begin{aligned} E\{\eta^2(t)\} &= e^{-2t/\varepsilon}E\{\eta^2(0)\} + \frac{q^2}{2}\left(1 - e^{-2t/\varepsilon}\right) \\ &\to \frac{q^2}{2} \qquad \text{as } t \to \infty. \end{aligned}$$

When ε is small, it is clear that the convergence in time t is very fast. This is the case with the convergence rates of all three expectations given in (8), (9), and (10).

Approximating now the η-terms in (7) by their respective expectations, after a short transient whose length is $O(\varepsilon)$, the estimation error is governed by

$$\boxed{\dot{\tilde{\theta}}(t) \approx -\frac{kaq^2}{2} f''(\theta^*)\tilde{\theta}(t)}$$

Unfortunately, the scheme with the unbounded stochastic perturbation $\eta(t)$ is not amenable to rigorous analysis. To make analysis feasible, using stochastic averaging theory, we replace η by a bounded stochastic perturbation $\sin(\eta)$.



$$\left(\eta = \frac{\sqrt{\varepsilon}q}{\varepsilon s+1}[\dot{W}] \quad \text{or} \quad \varepsilon d\eta = -\eta dt + \sqrt{\varepsilon}q dW\right)$$

Convergence speeds of the two algorithms are related as $\dfrac{\text{speed}_{\sin(\eta)}}{\text{speed}_{\eta}} = \dfrac{\left(1-e^{-q^2}\right)}{q^2}$.

Stochastic Averaging

Theorem [Liu and Krstic, TAC 2010]

Removes the following restrictions in the stochastic averaging theorems by Blankenship-Papanicolou, Freidlin, Khasminskii, Korolyuk, Kushner, Skorokhod:

- system's right-hand side linearly bounded
- average system globally exponentially stable
- 'vanishing' (equilibrium-preserving) stochastic perturbation
- time interval finite

Consider the system

$$\frac{dX_t^\varepsilon}{dt} = a(X_t^\varepsilon, Y_{t/\varepsilon}), \quad X_0^\varepsilon = x,$$

where $X_t^\varepsilon \in \mathbb{R}^n$; $Y_t \in \mathbb{R}^m$ is a time-homogeneous continuous Markov process defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the σ-field, and P is the probability measure. Let $D \subset \mathbb{R}^n$ be a domain (open connected set) of $\mathbb{R}^n$ and S_Y be the living space of the perturbation process $(Y_t, t \geq 0)$.

Suppose that the vector field $a(x,y)$ is a continuous function of (x,y), and for any $x \in D$, it is a bounded function of y. Further, suppose that it satisfies the locally Lipschitz condition in $x \in D$ uniformly in $y \in S_Y$, i.e., for any compact subset $D_0 \subset D$, there is a constant k_{D_0} such that for all $x', x'' \in D_0$ and all $y \in S_Y$, $|a(x',y) - a(x'',y)| \leq k_{D_0} |x' - x''|$. Assume that the perturbation process $(Y_t, t \geq 0)$ is ergodic with invariant distribution μ.

If the equilibrium $\overline{X}_t \equiv \overline{x} \in D$ of the average system

$$\frac{d\overline{X}_t}{dt} = \overline{a}(\overline{X}_t), \qquad \overline{X}_0 = x,$$

where

$$\overline{a}(x) = \int_{S_Y} a(x,y)\mu(dy),$$

is exponentially stable, then there exist constants $r > 0$, $c > 0$, $\gamma > 0$ and a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ such that for any initial condition $x \in \{x' \in D : |x' - \overline{x}| < r\}$, and any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \inf \left\{ t \geq 0 : |X_t^\varepsilon - \overline{x}| > c|x|e^{-\gamma t} + \delta \right\} = +\infty, \ a.s..$$

and

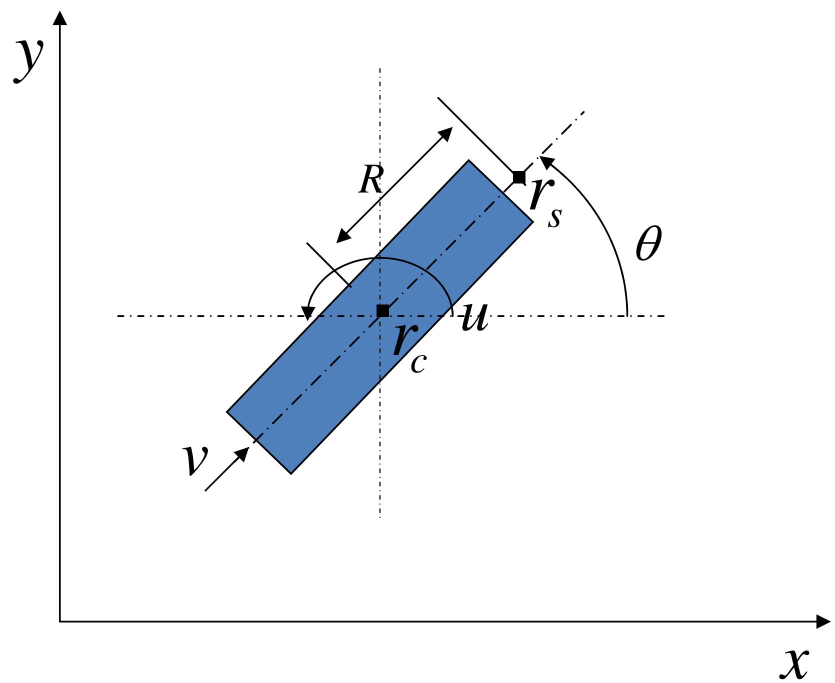
$$\lim_{\varepsilon \to 0} P\left\{ |X_t^\varepsilon - \overline{x}| \leq c|x|e^{-\gamma t} + \delta, \forall t \in [0, T(\varepsilon)] \right\} = 1$$

with $\lim_{\varepsilon \to 0} T(\varepsilon) = +\infty$.

Stochastic Nonholonomic Source Seeking

(autonomous vehicles and bacterial locomotion)

Vehicle model



Eqns of motion for vehicle center:

$$
\begin{aligned}
\dot{r}_c &= ve^{j\theta} \\
\dot{\theta} &= u
\end{aligned}
$$

Sensor is located at $r_s = r_c + Re^{j\theta}$.

Problem statement

Task: seek a source that emits a spatially distributed signal $J = f(r(x,y))$, which has an isolated local maximum $f(r^*)$ at r^*.

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Assumption (in the analysis, but not in simulations): Spatial distribution is quadratic, with circular level sets, i.e.,

$$f(r) = f^* - q_r|r - r^*|^2.$$

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$$f(r) = f^* - q_r|r - r^*|^2 .$$

Result of the paper: A stochastic seeking algorithm and a proof of local convergence to r^*, in a particular probabilistic sense, without the knowledge of q_r, r^*, f^* and without the measurement of $r_c(t)$, using only the measurement of $J(t)$ at the vehicle sensor.

Two control approaches

- tuning of angular velocity [forward velocity = const.]
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Angular velocity controller:

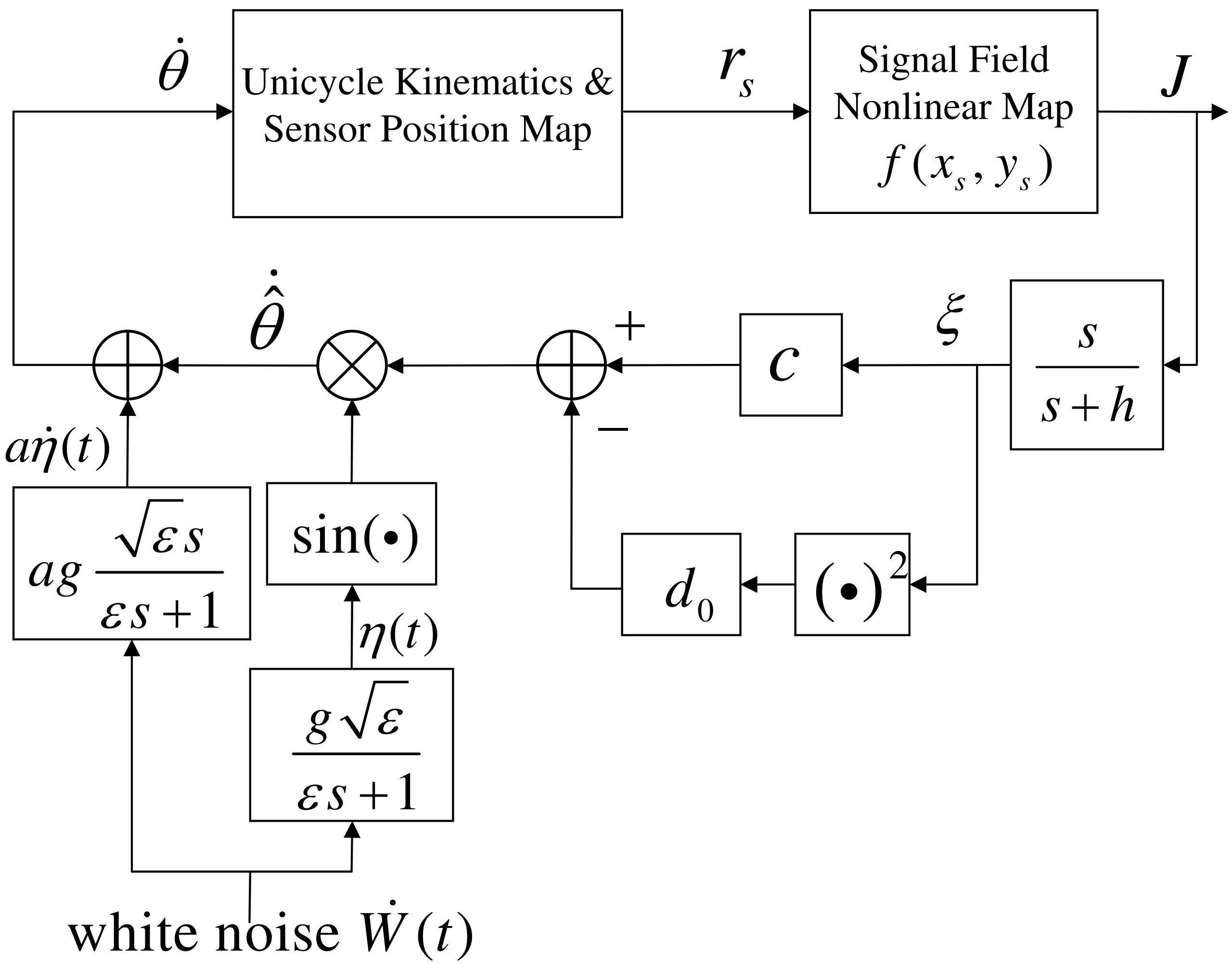
$$u = cJ\sin(\eta) + a\underbrace{\frac{1}{\varepsilon}\left(-\eta + g\sqrt{\varepsilon}\dot{W}\right)}_{d\eta/dt}$$

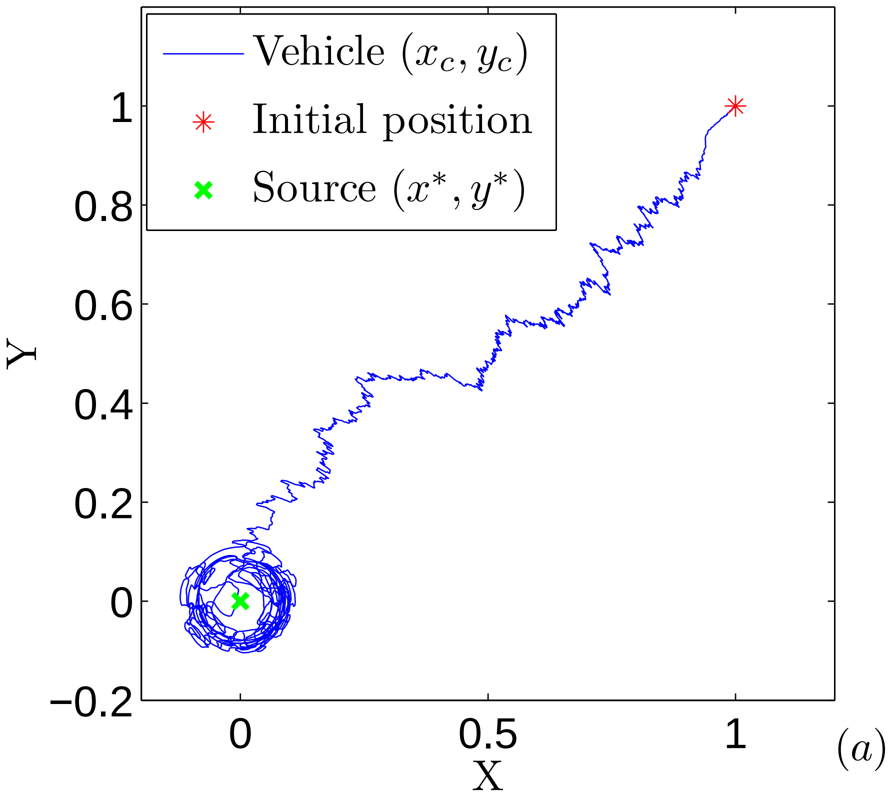
Forward velocity controller:

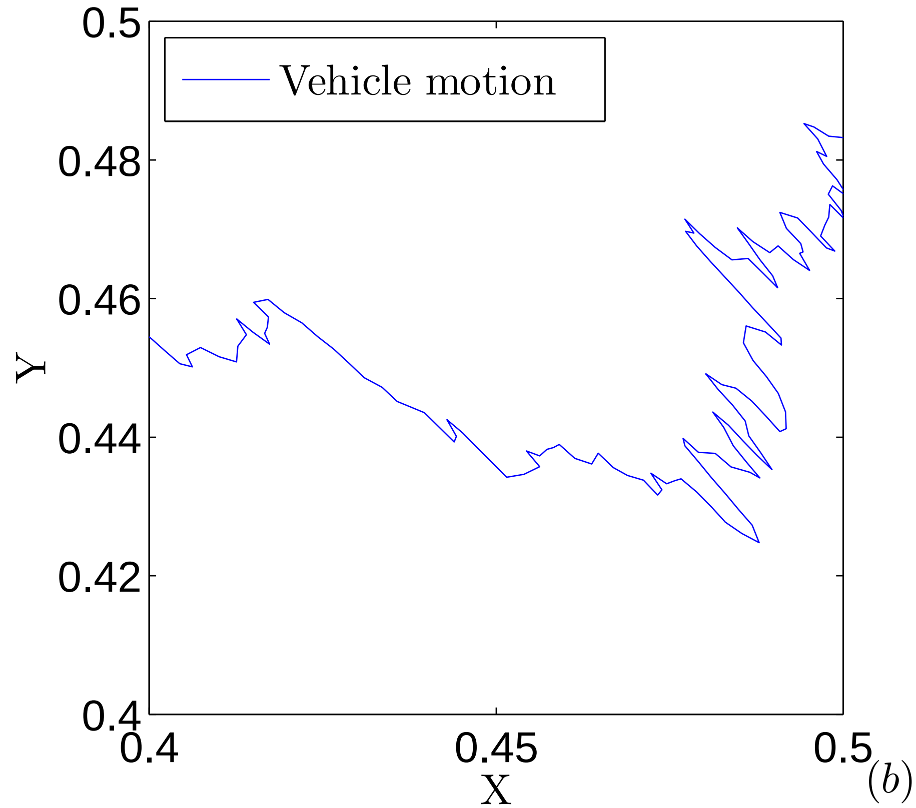
$$v = cJ\sin(\eta) + a\underbrace{\frac{1}{\varepsilon}\left[-\Big(\eta\cos(\eta) + g^2\sin(\eta)\Big) + g\sqrt{\varepsilon}\cos(\eta)\dot{W}\right]}_{d\sin(\eta)/dt}$$

where $\eta = \dfrac{g\sqrt{\varepsilon}}{\varepsilon s + 1}[\dot{W}]$

Controller with $v(t) = V_c \equiv \text{const}$ and tuning of angular velocity







(WLOG we place the source at the origin, $r^* = (0, 0)$.)

Theorem Consider the stochastic system in the block diagram (5 states):

$$\frac{d}{dt}\begin{bmatrix} r_c \\ e \\ \theta \\ \eta \end{bmatrix} = \begin{bmatrix} V_c e^{j\theta} \\ h\xi \\ -\frac{a}{\varepsilon}\eta + (c\xi - d_0\xi^2)\sin(\eta) \\ -\frac{1}{\varepsilon}\eta \end{bmatrix} dt + \frac{g}{\sqrt{\varepsilon}}\begin{bmatrix} 0 \\ 0 \\ a \\ 1 \end{bmatrix} dW$$

$$\xi = -(q_r\left|r_c + Re^{j\theta} - r^*\right|^2 + e)$$

and let the parameters $h, V_c, a, g > 0$ be chosen such that

$$\frac{1}{h} > \frac{R}{2V_c}\left(2 - \frac{I_2(2a,g)}{I_1(a,g)I_2(a,g)}\right),$$

where $I_1(a,g) = e^{-\frac{a^2g^2}{4}}$, $I_2(a,g) = \frac{1}{2}\left[e^{-\frac{(a-1)^2g^2}{4}} - e^{-\frac{(a+1)^2g^2}{4}}\right]$.

Denote

$$\rho = \sqrt{\frac{V_c I_1(a,g)}{2q_r cRI_2(a,g)}}.$$

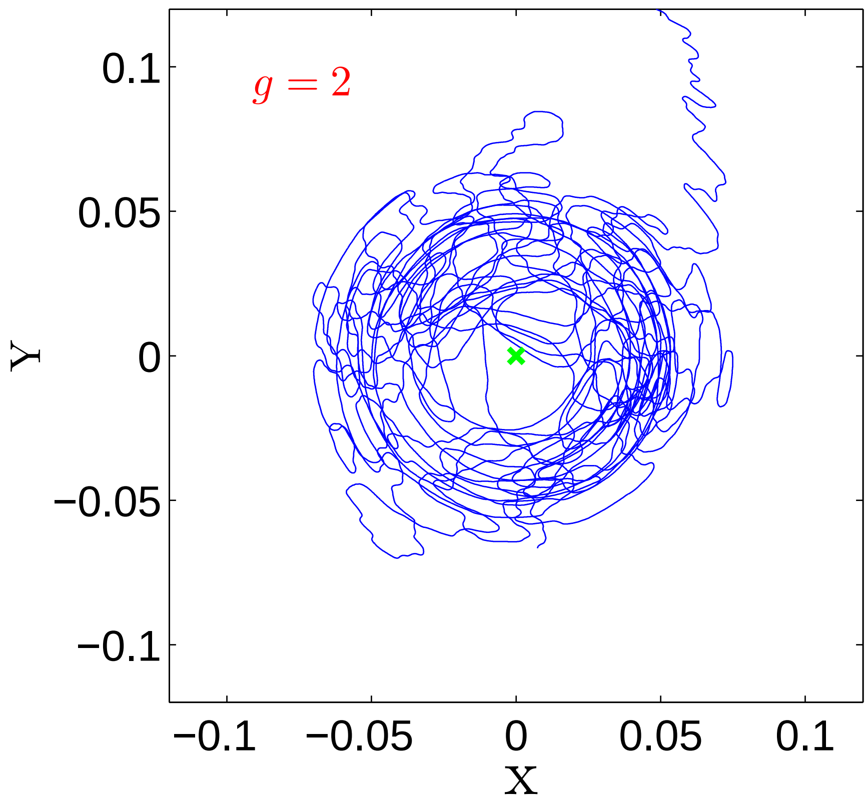
If the initial conditions $r_c(0)$, $\theta(0)$, $e(0)$ are such that the quantities

$$\big||r_c(0)-r^*|-\rho\big|, \quad \left|e(0)+q_r(R^2+\rho^2)\right|,$$

$$\text{either } \left|\theta(0)-\arg(r^*-r_c(0))+\frac{\pi}{2}\right| \text{ or } \left|\theta(0)-\arg(r^*-r_c(0))-\frac{\pi}{2}\right|$$

are sufficiently small,

i.e., if the vehicle starts close to the annulus and not pointing too far away from the annulus



(b)

then there exists a constant $C_0 > 0$ dependent on the initial condition $(r_c(0), \theta(0), e(0))$ and on the parameters $a, c, d_0, h, R, V_c, q_r, g$, a constant $\gamma_0 > 0$ dependent only on the parameters $a, c, d_0, h, R, V_c, q_r, g$, and a function $T(\varepsilon) : (0, \varepsilon_0) \to \mathbb{N}$ with the property

$$\lim_{\varepsilon \to 0} T(\varepsilon) = \infty,$$

such that for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \quad \inf\left\{t \geq 0 : ||r_c(t) - r^*| - \rho| > C_0 e^{-\gamma_0 t} + \delta\right\} = \infty, \quad \text{a.s.}$$

and

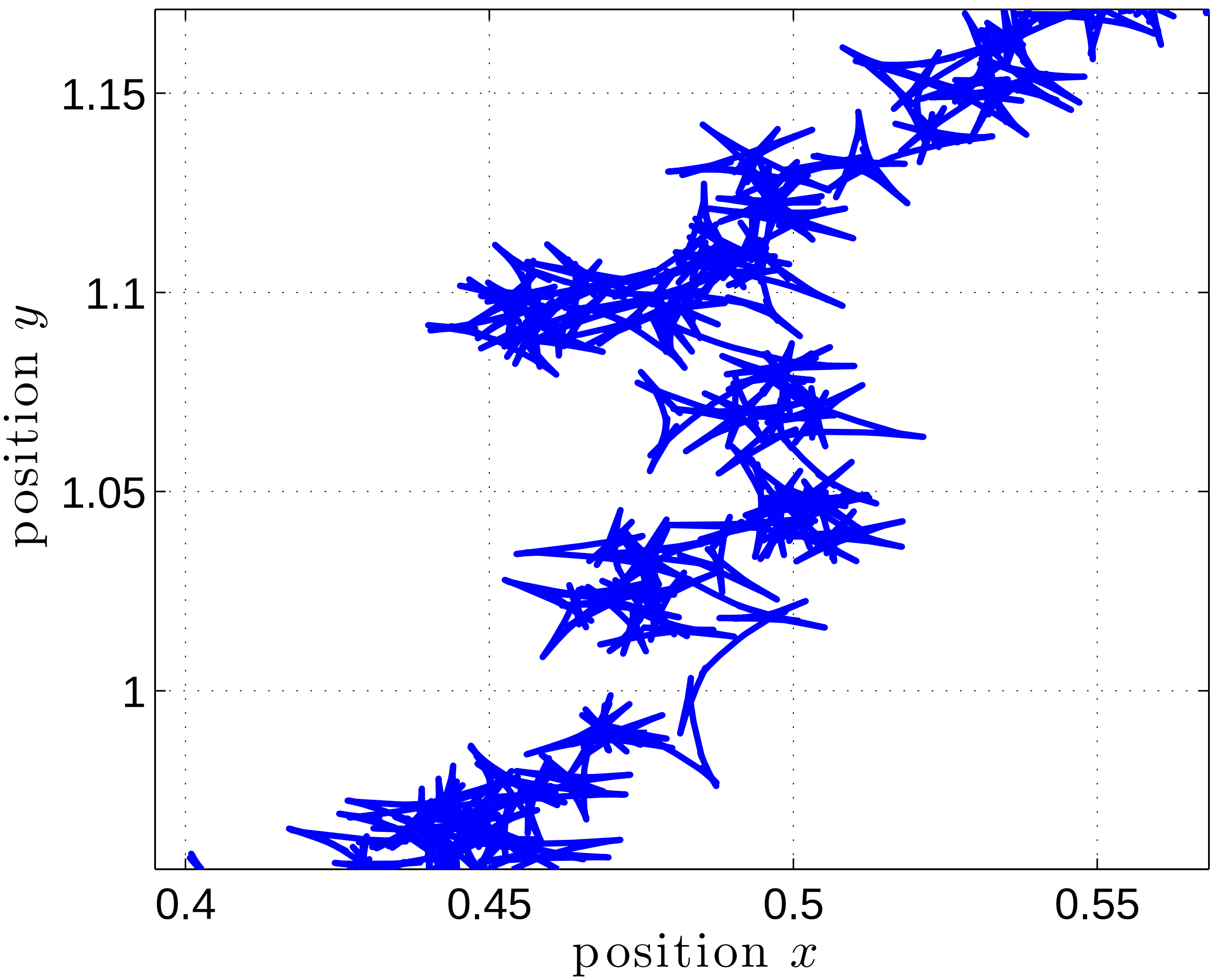
$$\lim_{\varepsilon \to 0} \quad P\left\{||r_c(t) - r^*| - \rho| \leq C_0 e^{-\gamma_0 t} + \delta,\ \forall t \in [0, T(\varepsilon)]\right\} = 1\,.$$

Built mobile robots and a ‘plume/wind’ tunnel; tested the algorithms.

(UCSD-led ONR MURI on olfactory sensing/localization, 2007-2012.)

MOVIE

Controller with $\dot{\theta} = \mu = \text{const}$ and tuning of forward velocity



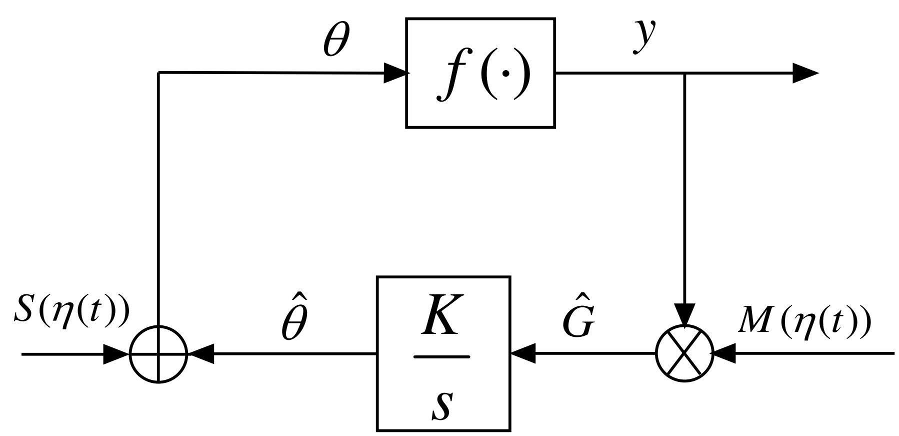
Source at $(0, 0)$.

Theorem

$$\lim_{\varepsilon\to 0} P\left\{ \left\| \begin{bmatrix} x^{\varepsilon}(t) - x^{*} \\ y^{\varepsilon}(t) - x^{*} \\ \theta^{\varepsilon}(t) - \theta_0 - \mu t \end{bmatrix} \right\| \leq c \left\| \begin{bmatrix} x_0 - x^{*} \\ y_0 - y^{*} \end{bmatrix} \right\| e^{-\gamma t} + \delta + O(a),\ \forall t \in [0, T(\varepsilon)] \right\} = 1$$

Newton-Based Stochastic ES

Multivariable Gradient ES



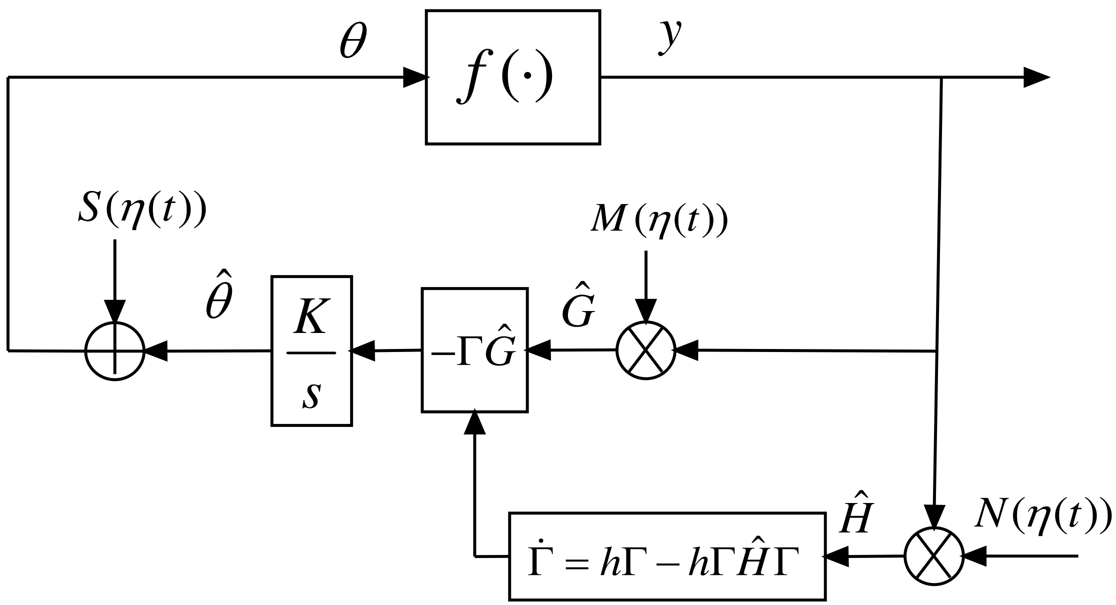
$$
\begin{aligned}
S(\eta(t)) &= [a_1 \sin(\eta_1(t)), \ldots, a_n \sin(\eta_n(t))]^T, \\
M(\eta(t)) &= \left[\frac{1}{a_1 G_0(q_1)} \sin(\eta_1(t)), \ldots, \frac{1}{a_n G_0(q_n)} \sin(\eta_n(t))\right]^T
\end{aligned}
$$

where $\eta_i = \frac{q_i\sqrt{\varepsilon_i}}{\varepsilon_i s + 1}[\dot{W}_i]$ and $G_0(q) = \frac{1}{2}(1 - e^{-q^2})$

Convergence rate governed by the unknown Hessian at the extremum:

$$
\frac{d\tilde{\theta}^{\text{ave}}(t)}{dt} = KH\tilde{\theta}^{\text{ave}}(t)
$$

Newton-based ES



$$
\begin{aligned}
N_{ii} &= \frac{4}{a_i^2 G_0^2(\sqrt{2}q_i)}\left[\sin^2(\eta_i) - G_0(q_i)\right] \\
N_{ij} &= \frac{1}{a_i a_j G_0(q_i) G_0(q_j)} \sin(\eta_i)\sin(\eta_j), \quad i \neq j
\end{aligned}
$$

Γ: matrix Riccati diff. eq. that estimates the inverse of Hessian matrix

Convergence rate user-assignable!

$$
\begin{aligned}
\frac{d\tilde{\theta}^{\text{ave}}}{dt} &= -\boxed{K}\,\tilde{\theta}^{\text{ave}} - K\tilde{\Gamma}H\tilde{\theta}^{\text{ave}} \\
\frac{d\tilde{\Gamma}^{\text{ave}}}{dt} &= -\boxed{h}\,\tilde{\Gamma}^{\text{ave}} - h\tilde{\Gamma}^{\text{ave}}H\tilde{\Gamma}^{\text{ave}}
\end{aligned}
$$

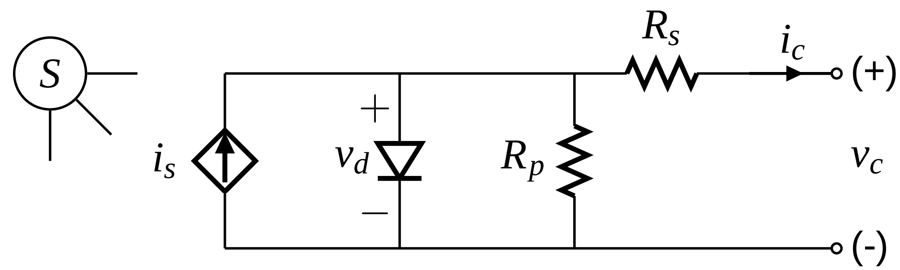
Photovoltaic Arrays

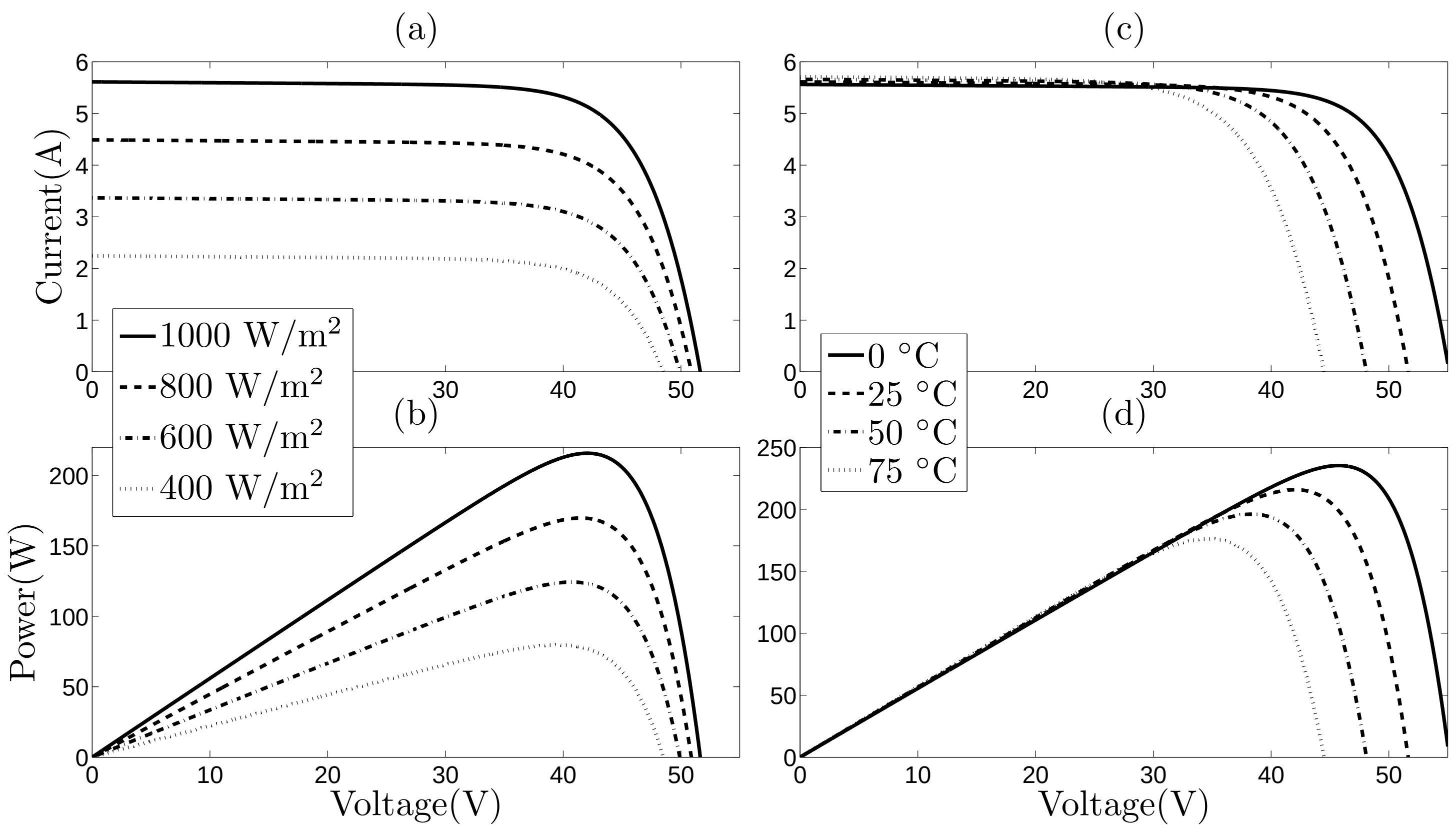
maximum power point tracking

Azad Ghaffari

Maximum power point tracking

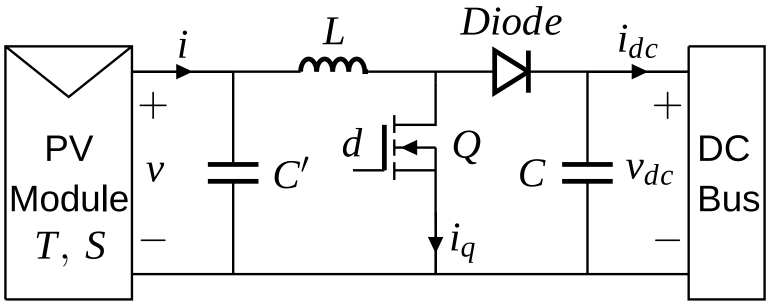
PV cell





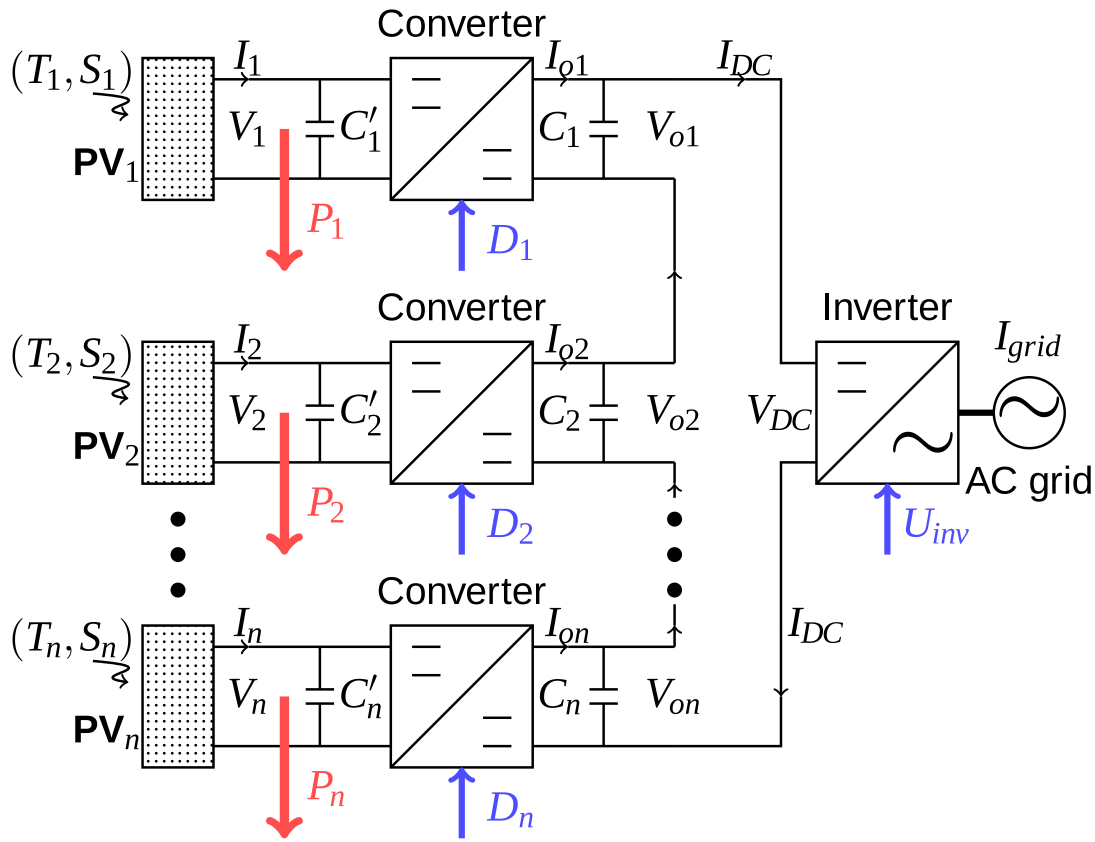
(a) and (b) varying irradiance, $T=25°$C (c) and (d) varying temperature, $S=1000$W/m^2

Maximum power point tracking

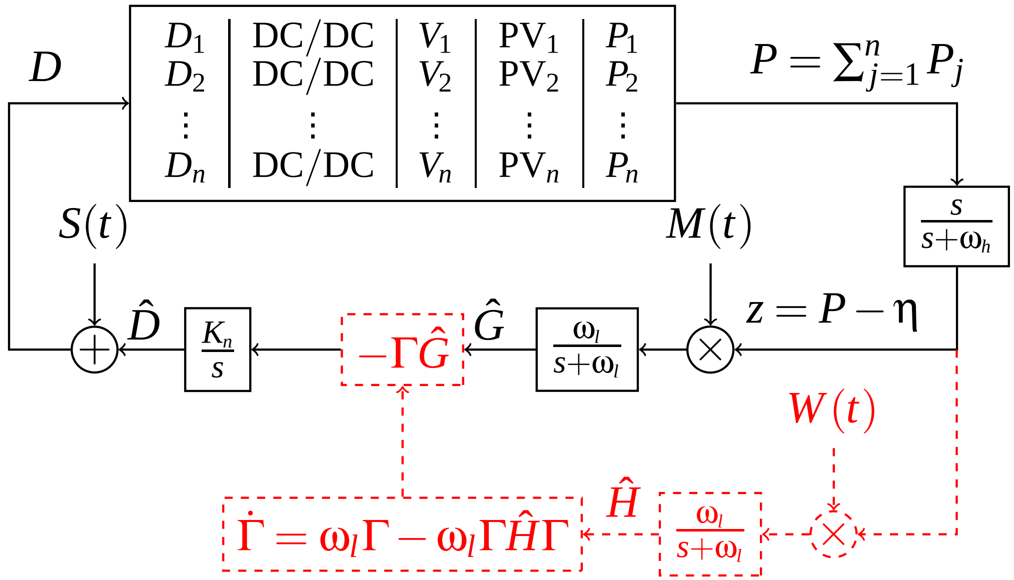


DC/DC boost converter

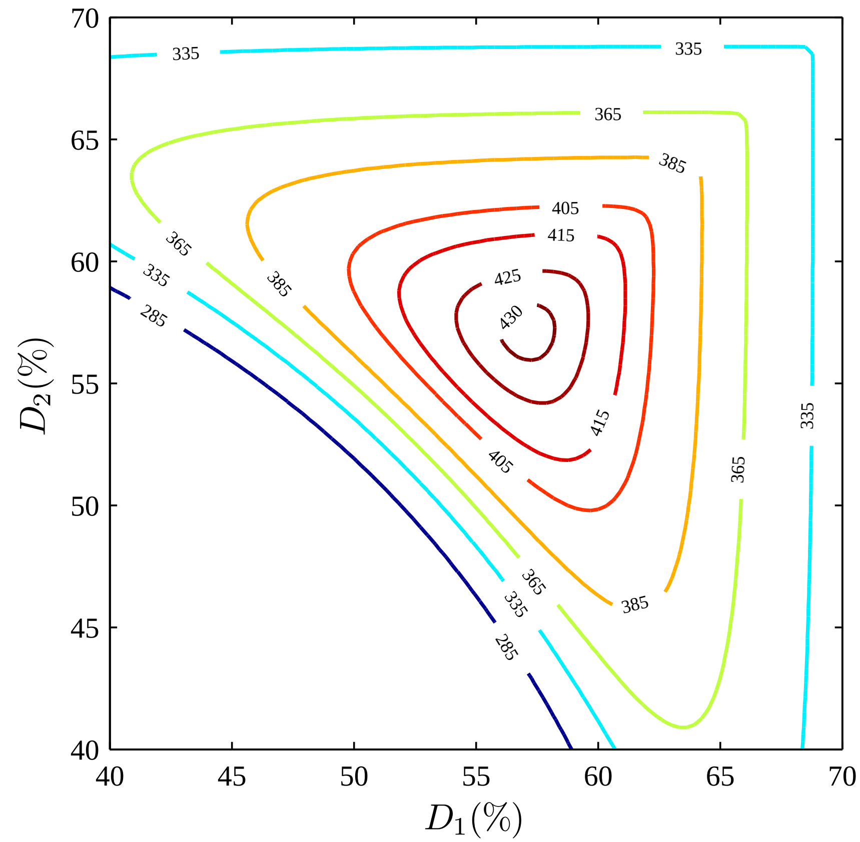
(Voltage v_{dc} controlled via pulsewidth d on transistor Q)



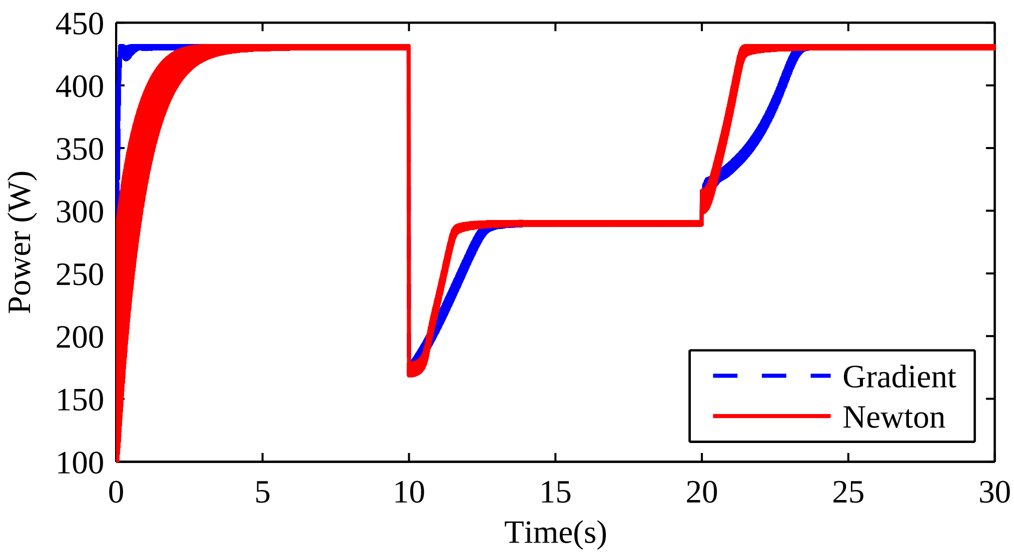
Cascade PV system including n PV modules



Newton ES algorithm for tuning pulsewidths



Power (Watt) versus pulsewidths



Generated power under **partial shading**

Extremum Seeking Based on

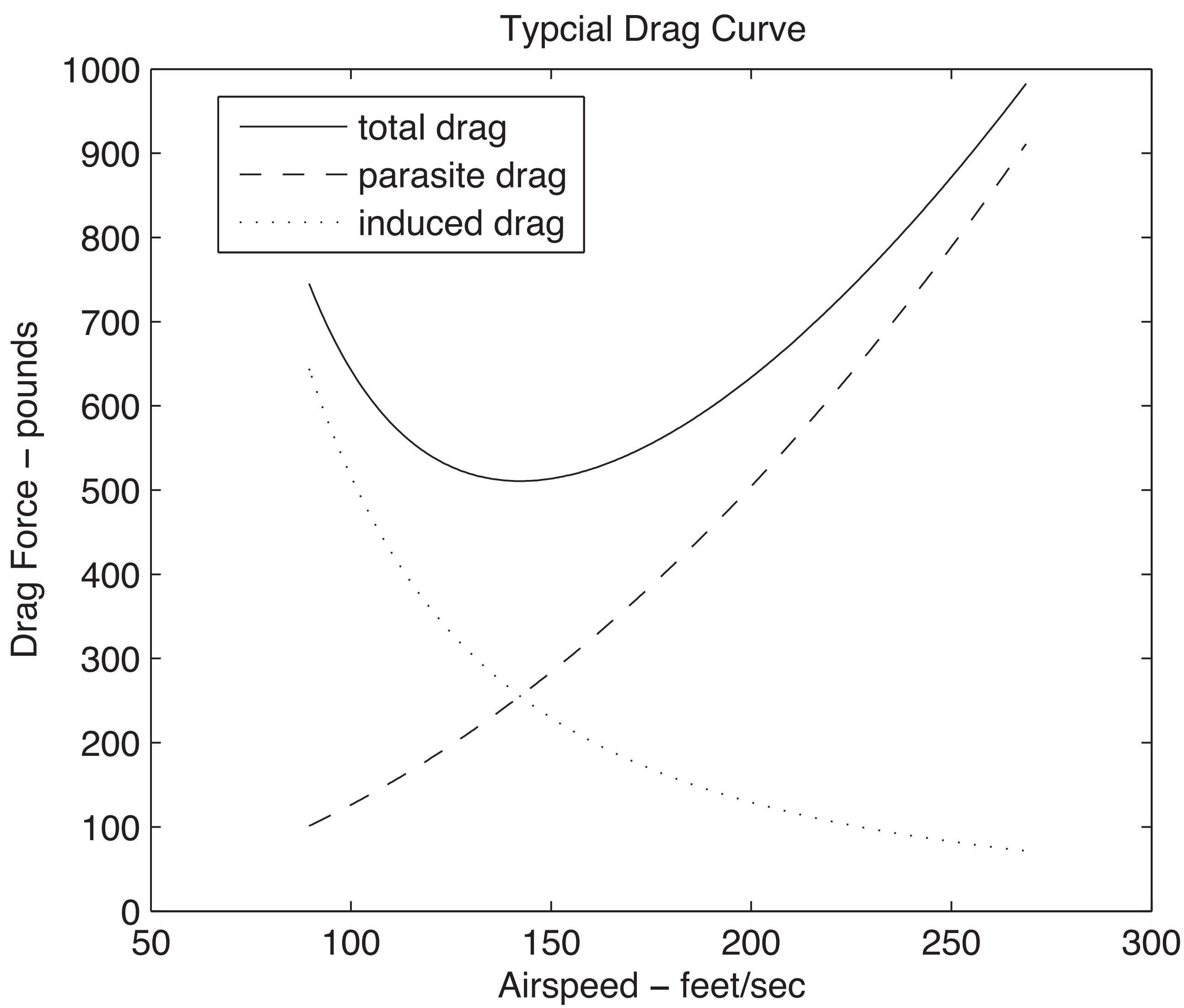
Atmospheric Turbulence for

Aircraft Endurance

(ES without injecting a perturbation)

Endurance = the length of time an aircraft can remain airborne

Goal: Find the airspeed for optimal endurance



Jet aircraft (Global Hawk)

- Min. **fuel** $\propto$ thrust $=$ drag(**speed**)
- Speed controlled through throttle
- Altitude controlled through elevator

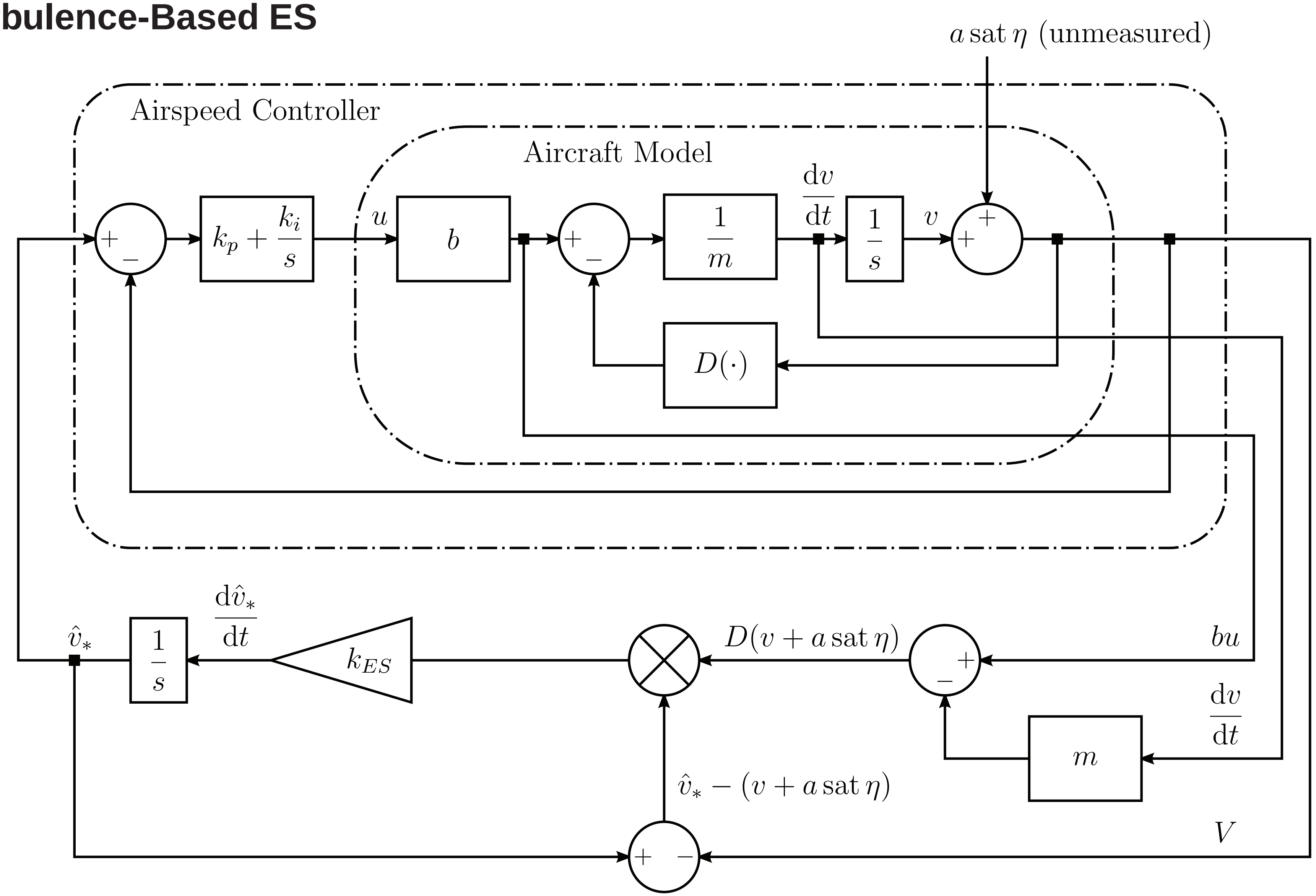
Propeller aircraft (Predator)

- Min. **fuel** $\propto$ power $=$ thurst$\times$speed $=$ drag(**speed**)$\times$**speed**
- Speed controlled through elevator
- Altitude controlled through throttle

- The optimal speed is different for each individual aircraft

- Perturbing the airspeed may burn more fuel and annoys air traffic control

Turbulence-Based ES



V = airspeed v = ground speed a sat η = turbulence

u = throttle D = drag

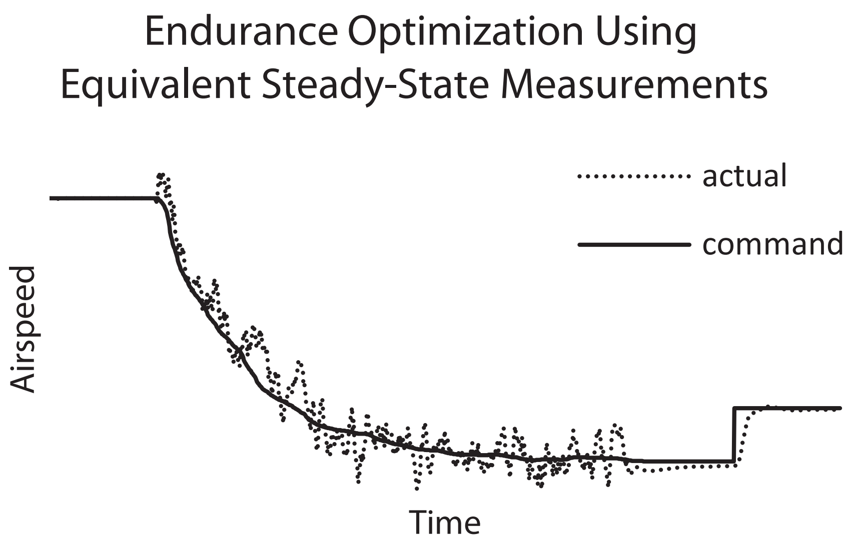
measured: airspeed, accel, & throttle setting

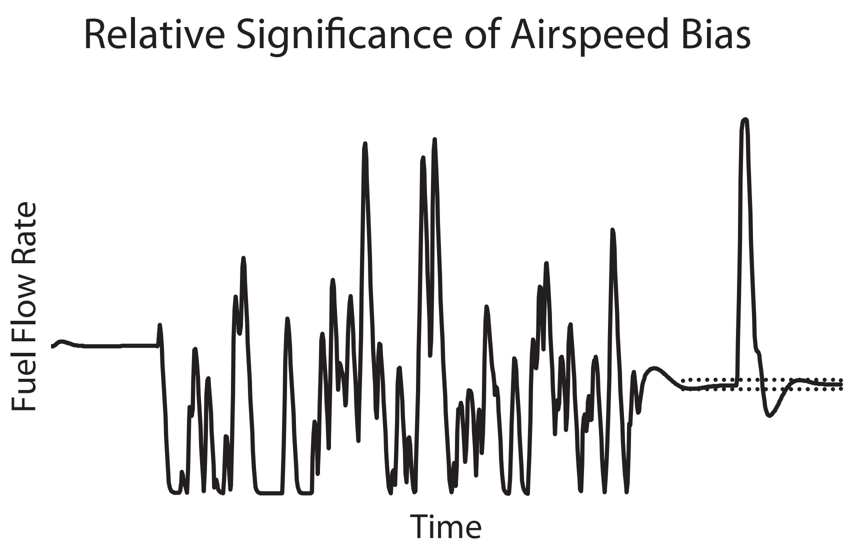
Theorem

If the adaptation gain is chosen small (in inverse proportion to upper bound on minimum drag) then an airspeed near the value that minimizes drag (with bias $\propto$ variance of airspeed and 3rd derivative of drag curve) is weakly exponentially stable (wp1 as turbulence time constant $\rightarrow 0$).

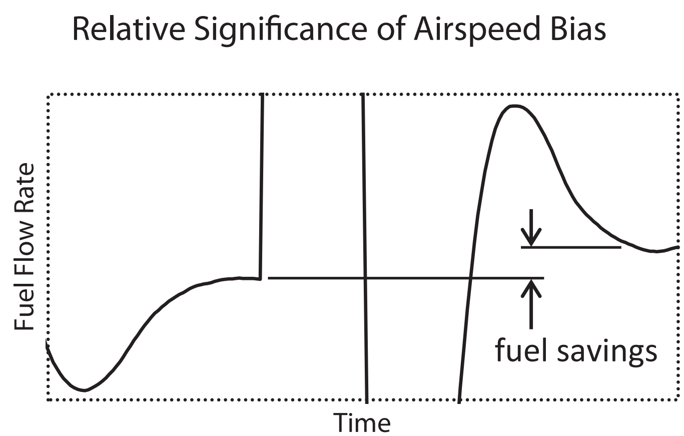
Theorem	**Simulation** on Northrop Grumman proprietary software
turbulence saturated filtered gaussian	Dryden turbulence
no vertical turbulence	
altitude = pitch = const	6 DOF model

Proprietary aircraft performance data (result presented w/o units)





1. Cruising in calm air
2. Turbulence encounter begins
   - Fuel flow rate fluctuates
   - Airspeed optimization begins
3. Turbulence encounter ends
   - Fuel flow rate stabilizes
4. Go to nominal loiter speed
   - Fuel ↑ slightly after switch from ES to loiter speed



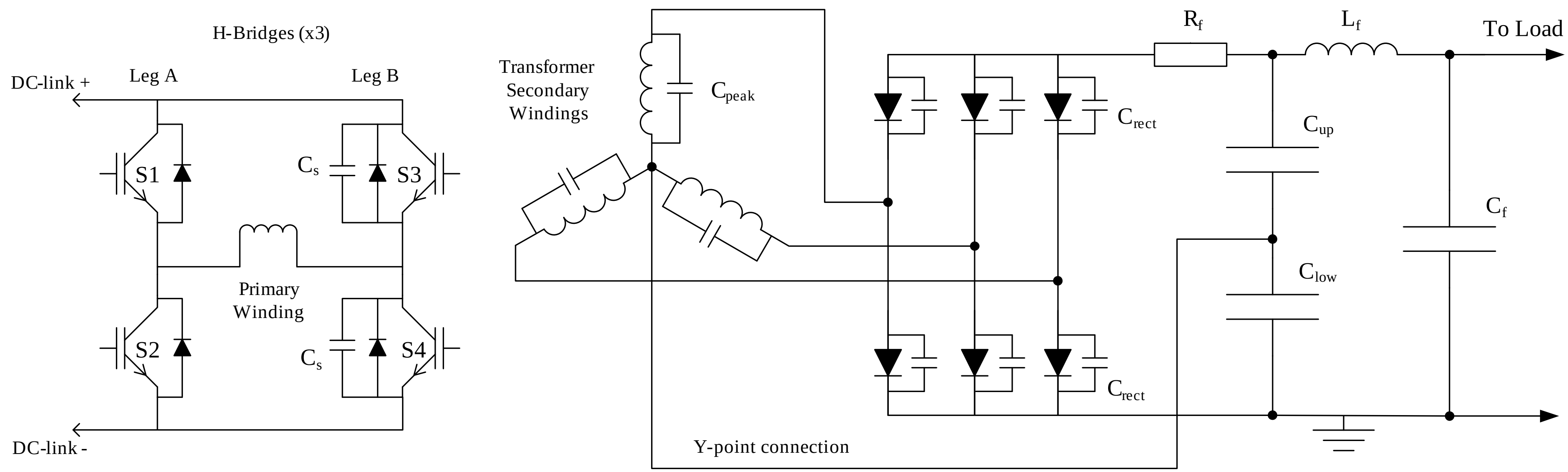
Some more applications of ES

High Voltage Converter Modulators in Accelerators (LANL)

Alex Scheinker

High Voltage Converter Modulator for Klystron RF source

(at Los Alamos and Oak Ridge national labs)

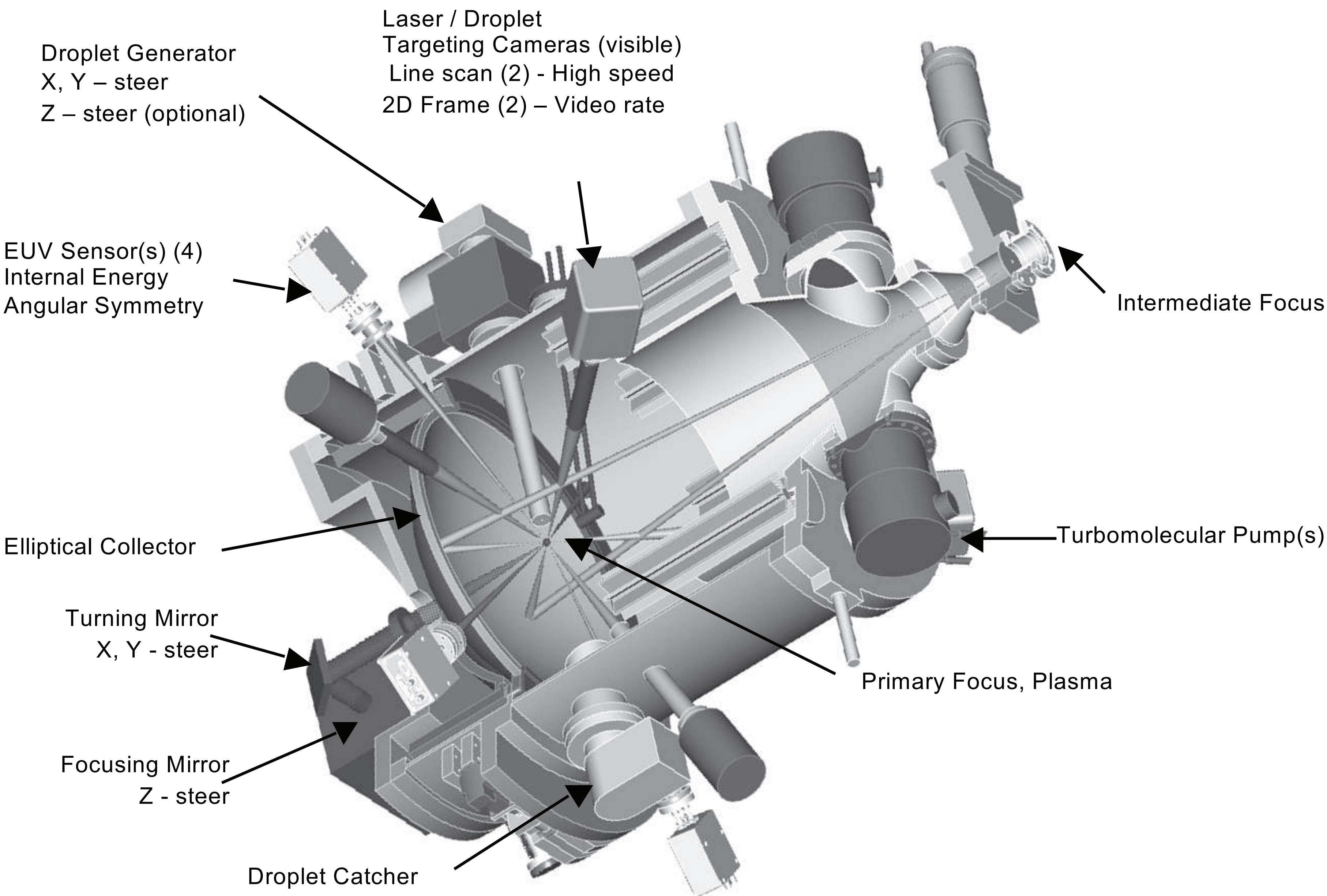


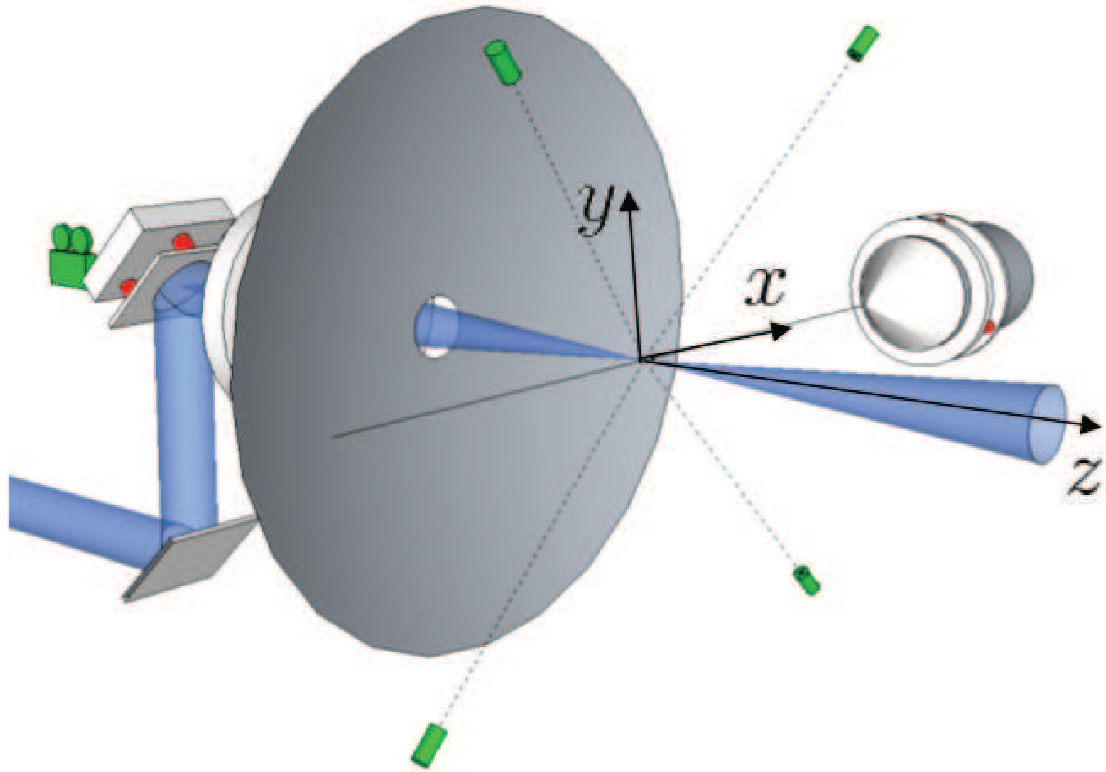
Voltage step response to -100kV in 50μs ($\sim 10^9$ V/s)

Beam Alignment Control
to Maximize EUV Production

Paul Frihauf and Matt Graham Cymer

EUV (extreme ultra-violet) system

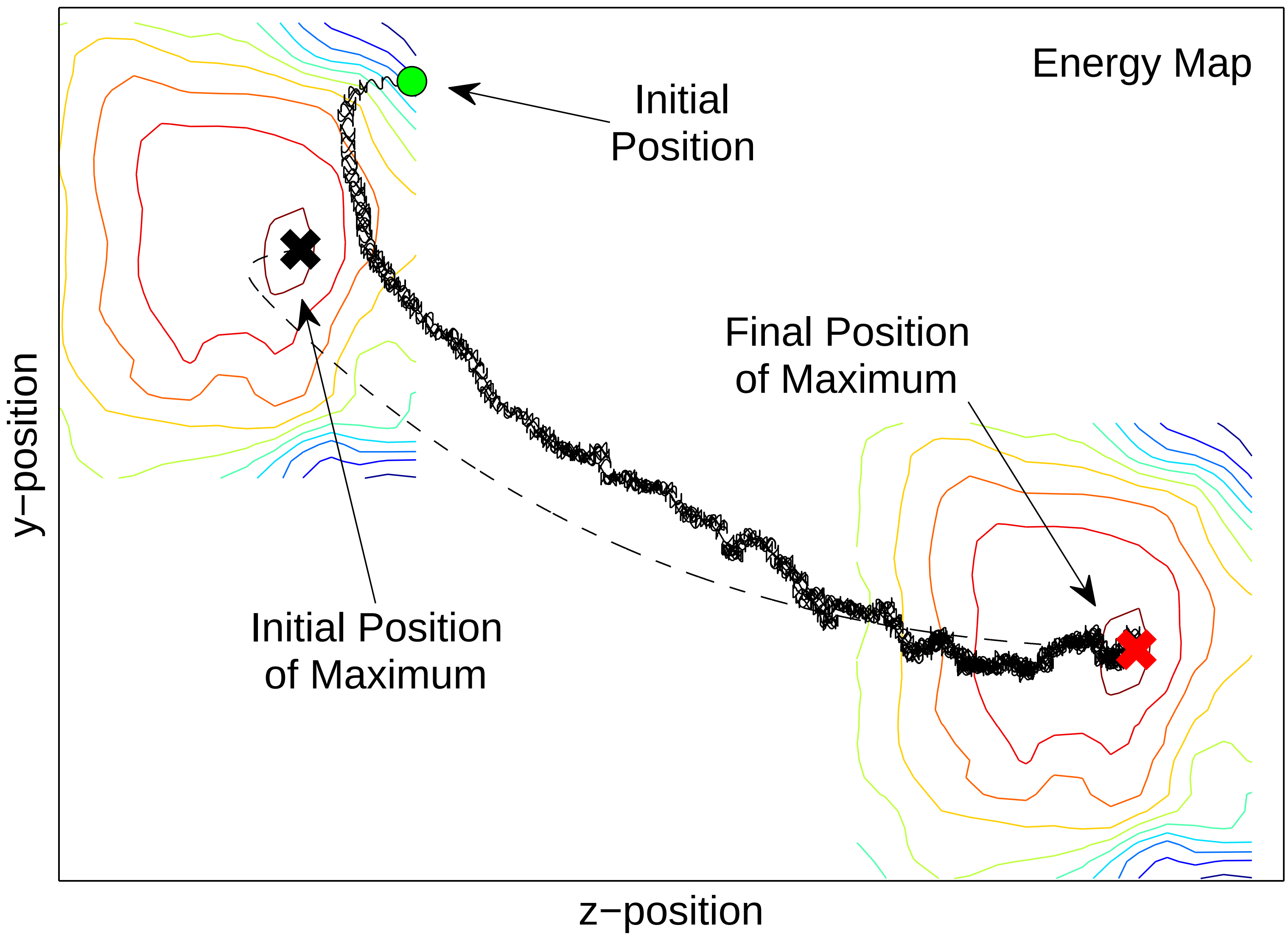




Z: lens moved w/ stepper motors

Y: mirror moved w/ piezos

Energy maximization in EUV (under temperature-induced drift)



Dank u wel

Shu-Jun Liu • Miroslav Krstic

Stochastic Averaging and Stochastic Extremum Seeking

Springer