Boundary Control of PDEs:

A Short Course on Backstepping Designs

Miroslav Krstic

Andrey Smyshlyaev, Rafael Vazquez, Jennie Cochran, Antranik Siranosian

http://flyingv.ucsd.edu/pde.pdf

UCSB, 2006

Outline

- Introduction (Miroslav)
- Parabolic PDEs—Lyapunov techniques, stabilizing controllers, explicit gains, observers (Andrey)
- Hyperbolic PDEs—wave equations, beams, delay systems (Miroslav and Antranik)
- Navier-Stokes equations (Jennie)
- Magnetohydrodynamics—observer design (Rafael)
- Adaptive control of PDEs (Andrey)
- Motion planning and trajectory tracking (Miroslav)
- Open problems (Miroslav)

Introduction

Miroslav Krstic

Various Approaches to Control of Distributed Parameter Systems

- Controllability

- Optimal control

- Abstract approaches based on semigroups

- Frequency-domain approaches based on robust control (not natural because PDEs come in time domain and conversion to frequency domain is hard; model reduction, implied by the robust control approach, is also hard)

- “Boundary damper” controllers (for a limited class of systems and under a very limited actuation architecture)

Books by

Lions; Komornik; Curtain and Zwart; Lasiecka and Triggiani; Bensoussan, Da Prato, Delfour, and Mitter; Li and Yong; van Keulen; Luo, Guo, and Morgul; Lagnese; Lasiecka; Banks, Smith, and Wang; de Queiroz, Dawson, Nagarkatti, and Zhang; Aamo and Krstic; Gunzburger; Christofides

Where Does Control of PDEs Fit Within the Landscape of Mainstream Control?

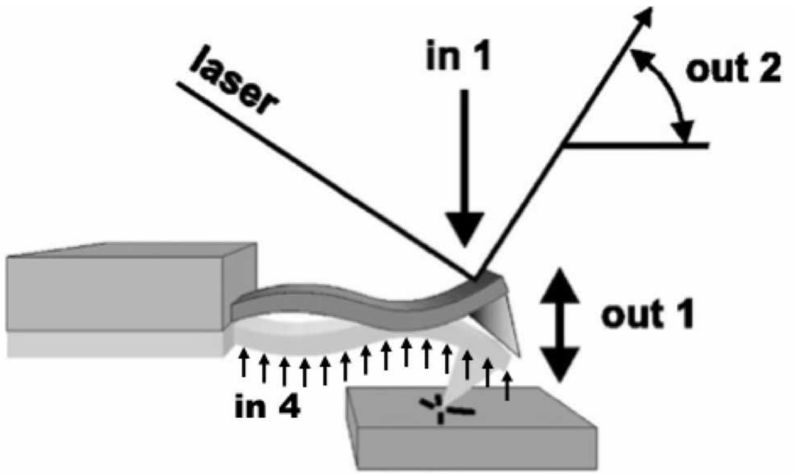
- IEEE TAC, Automatica?

- Seems like a fringe discipline

- Most unfortunate because of all the potential applications

Applications

- flexible structures (aerospace, civil, AFM)



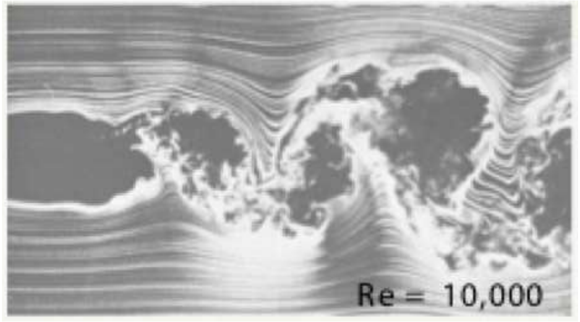
Applications

- flexible structures (aerospace, civil, AFM)

- chemical process control

Applications

- flexible structures (aerospace, civil, AFM)
- chemical process control
- fluids, aerodynamics, turbulence, propulsion, acoustics



Applications

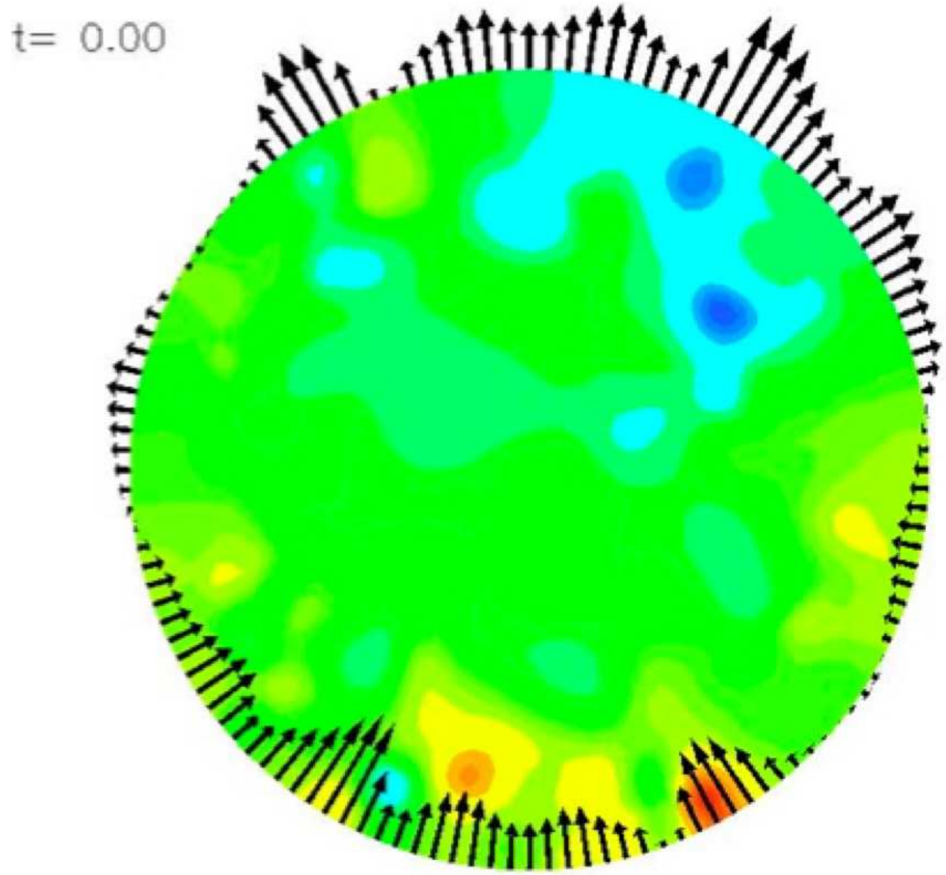
- flexible structures (aerospace, civil, AFM)
- chemical process control
- fluids, aerodynamics, turbulence, propulsion, acoustics
- quantum control

Classes of PDEs

- Parabolic (heat transfer, chemical reactions, etc)
- Hyperbolic (waves—acoustics, strings, etc)
- Other "odd" equations (most physically relevant problems are): Navier-Stokes, Korteweg-de Vries, Kuramoto-Sivashinsky, some beam/plate/shell models, etc

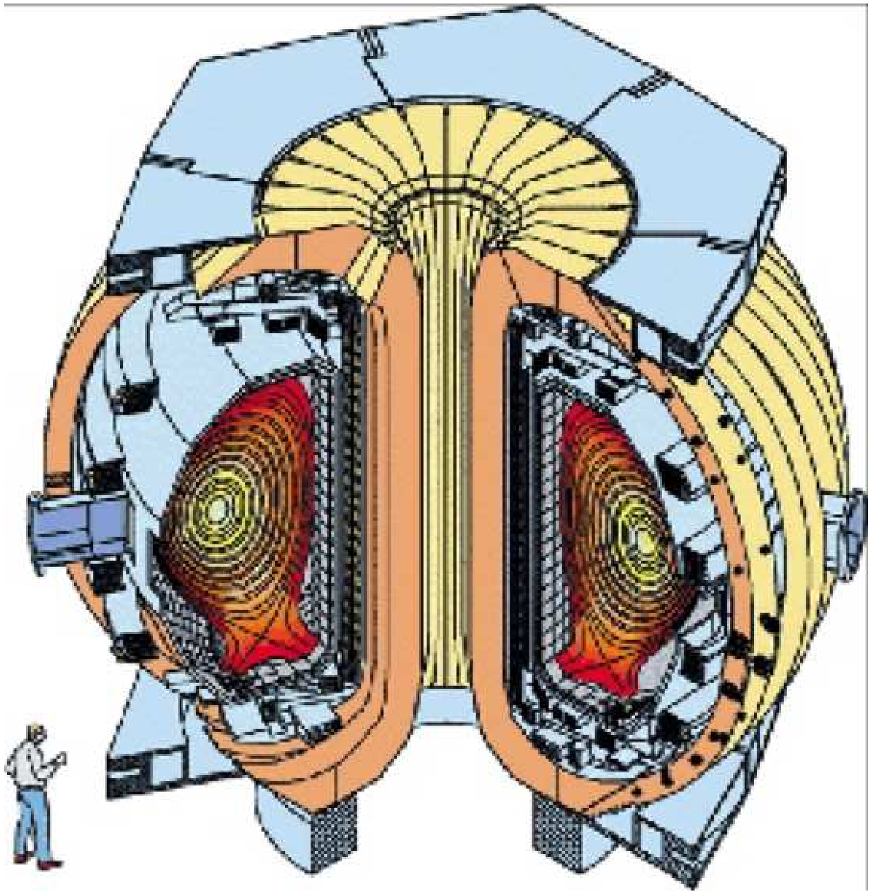
Actuator Location

- Boundary control



Actuator Location

- Boundary control
- In-domain control (a few actuators)

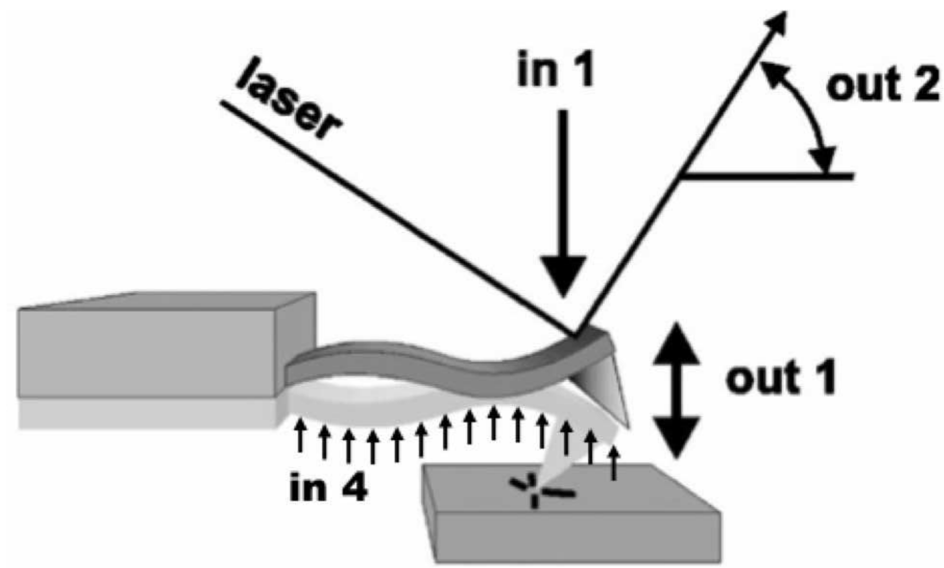


Actuator Location

- Boundary control
- In-domain control (a few actuators)
- Distributed control (lots of actuators)

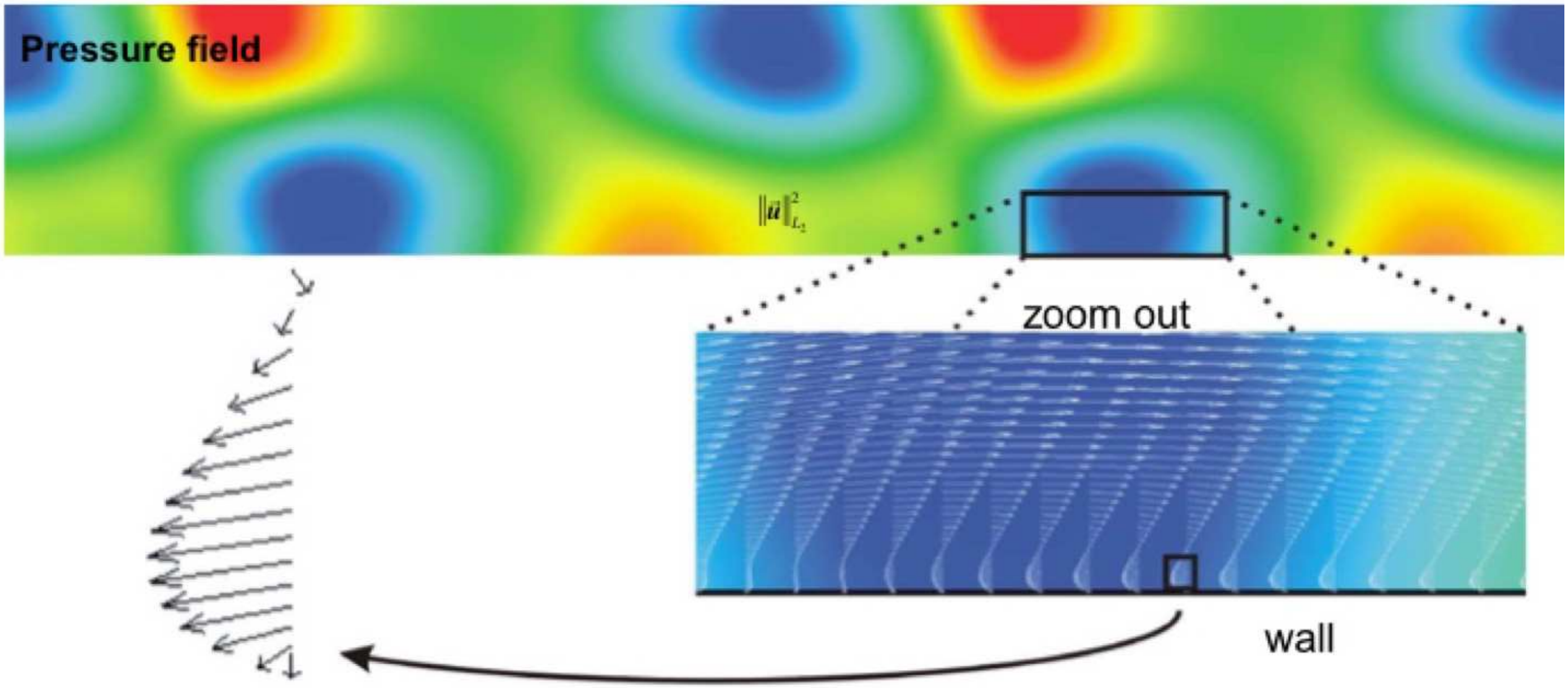
Dimension of Spatial Domain

1D



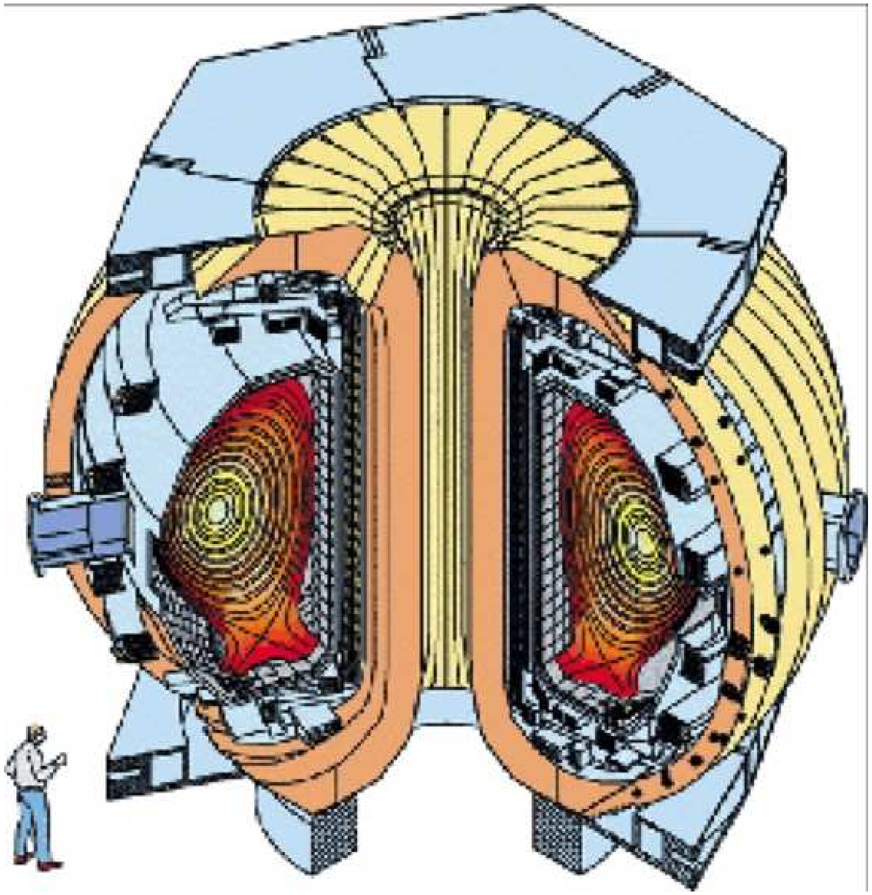
Dimension of Spatial Domain

1D, 2D



Dimension of Spatial Domain

1D, 2D, 3D



Dimension of Spatial Domain

1D, 2D, 3D,...nD?

Control Objectives

- Stabilization and performance addressed in many publications

- Trajectory tracking barely touched

Benchmark PDEs

- heat equation
- reaction-advection-diffusion equations
- wave equation
- beam models (Euler-Bernoulli, Rayleigh, Timoshenko)
- Burgers
- Navier-Stokes
- MHD
- Korteweg-de Vries
- complex-valued PDEs: Ginzburg-Landau, Schrodinger
- Sine-Gordon
- ...

Basic Issues in PDEs

Eigenvalues, eigenfunctions, exact solutions,...

Stability

- no useful “general Lyapunov theory” for infinite dimensional systems
- spatial norms
- Poincare, Agmon, and Sobolev inequalities
- energy boundedness vs. pointwise (in space) boundedness

Choices of Boundary Controls

- Dirichlet (fluids)

- Neumann (thermal)

Static vs. Dynamic Behavior in PDEs

Equilibrium/static problems = PDEs themselves (or, in the 1D case, ODEs).

Nonlinear Issues

- blow up in time (superlinear nonlinearities like in chemical reactions)

- blow up in space (shock waves—Burgers, etc.)

- boundedness despite quadratic nonlinearities (Navier-Stokes)

On with the Course!

- boundary control only

- focus on unstable PDEs

- no Riccati equations

- all basic categories of PDEs, from all major applications areas, will be covered

No theorems.

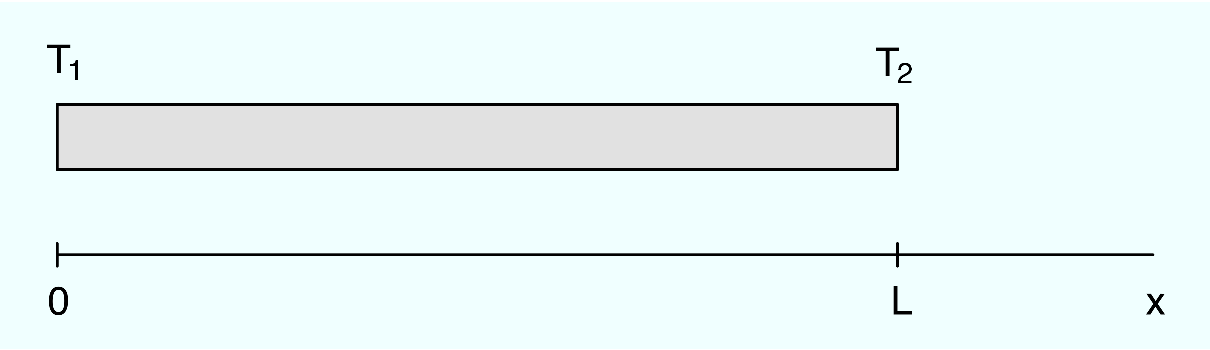
PDE Basics

Andrey Smyshlyaev

mini-course at UCSB, 2006

Introduction

Simplest physical model: heating rod



Heat equation $T_t(x,t) = \varepsilon T_{xx}(x,t)$

Left boundary condition $T(0,t) = T_1$

Right boundary condition $T(L,t) = T_2$

Initial condition $T(x,0) = T_0(x)$

Want to represent this equation in a form suitable for our course.

Procedure

1. Scale x to normalize length: $\xi = \frac{x}{L} \quad \Rightarrow \quad T_t = \frac{\varepsilon}{L^2} T_{\xi\xi}$

2. Scale t to normalize diffusion coefficient: $\tau = \frac{\varepsilon}{L^2} t \quad \Rightarrow \quad T_\tau = T_{\xi\xi}$

3. Find steady-state solution $\bar{T}$:

$$\begin{aligned} \bar{T}''(\xi) &= 0 \\ \bar{T}(0) &= T_1 \quad \Rightarrow \quad \bar{T}(\xi) = T_1 + \xi(T_2 - T_1) \\ \bar{T}(1) &= T_2 \end{aligned}$$

4. Introduce the error variable $w = T - \bar{T}$

$$\begin{aligned} w_\tau(\xi,\tau) &= w_{\xi\xi}(\xi,\tau) \\ w(0,\tau) &= 0 \\ w(1,\tau) &= 0 \\ w(\xi,0) &= w_0(\xi) \end{aligned}$$

Procedure

1. Scale x to normalize length: $\xi = \frac{x}{L} \quad \Rightarrow \quad T_t = \frac{\varepsilon}{L^2} T_{\xi\xi}$

2. Scale t to normalize diffusion coefficient: $\tau = \frac{\varepsilon}{L^2} t \quad \Rightarrow \quad T_\tau = T_{\xi\xi}$

3. Find steady-state solution $\bar{T}$:

$$\begin{aligned} \bar{T}''(\xi) &= 0 \\ \bar{T}(0) &= T_1 \quad \Rightarrow \quad \bar{T}(\xi) = T_1 + \xi(T_2 - T_1) \\ \bar{T}(1) &= T_2 \end{aligned}$$

4. Introduce the error variable $w = T - \bar{T}$

$$\begin{aligned} w_\tau &= w_{\xi\xi} \\ w(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Finally, suppress initial condition and time and space dependence

Basic types of boundary conditions

- Dirichlet: $w(0) = 0$ (temperature)
- Neumann: $w_x(0) = 0$ (heat flux)
- Robin (mixed): $w(0) + qw_x(0) = 0$

Stability of PDEs

Andrey Smyshlyaev

mini-course at UCSB, 2006

Stability of PDEs

Heat equation

$$
\begin{aligned}
w_t &= w_{xx} \\
w(0) &= 0 \\
w(1) &= 0
\end{aligned}
$$

As in finite dimension, there are two ways to analyze stability properties:

— Find the exact solution [usually not possible]

— Use Lyapunov theory to show stability without solving the PDE
[there is no general Lyapunov theory for PDEs]

For this simple plant both methods can be applied

Not so for more complicated systems

Lyapunov Stability

Most common Lyapunov function for PDEs is L_2 spatial norm:

$$V = \frac{1}{2}\int_0^1 w^2(x)\,dx := \|w\|^2$$

Time derivative along the solutions:

$$\begin{aligned}\dot{V} = \frac{dV}{dt} &= \int_0^1 w(x)w_t(x)\,dx \\ &= \int_0^1 w(x)w_{xx}(x)\,dx \\ &= w(x)w_x(x)|_0^1 - \int_0^1 w_x^2(x)\,dx \qquad \text{(integration by parts)} \\ &= -\int_0^1 w_x^2(x)\,dx\end{aligned}$$

So the system is stable, but is it asymptotically stable?

Need to express $\|w_{\color{red}x}\|$ in terms of $\|w\|$

Useful inequalities

Cauchy-Schwartz Inequality

$$\begin{aligned}\int_0^1 uw dx &\leq \|u\|\|w\| \\ &\leq \frac{\gamma}{2}\|u\|^2 + \frac{1}{2\gamma}\|w\|^2\end{aligned}$$

Poincare Inequality

$$\begin{aligned}\int_0^1 w^2 dx &\leq 2w^2(1) + 4\int_0^1 w_x^2 dx \\ \int_0^1 w^2 dx &\leq 2w^2(0) + 4\int_0^1 w_x^2 dx\end{aligned}$$

In particular, if one of the boundary conditions is zero, then

$$\|w\| \leq 2\|w_x\|$$

Proof of Poincare Inequality:

$$\begin{aligned}\int_0^1 w^2 dx &= xw^2|_0^1 - 2\int_0^1 xww_x\, dx \\ &= w^2(1) - 2\int_0^1 xww_x\, dx \\ &\leq w^2(1) + \frac{1}{2}\int_0^1 w^2\, dx + 2\int_0^1 x^2 w_x^2\, dx\end{aligned}$$

We get

$$\begin{aligned}\frac{1}{2}\int_0^1 w^2 dx &\leq w^2(1) + 2\int_0^1 x^2 w_x^2 dx \\ &\leq w^2(1) + 2\int_0^1 w_x^2 dx\end{aligned}$$

Finally

$$\int_0^1 w^2 dx \leq 2w^2(1) + 4\int_0^1 w_x^2\, dx$$

Back to Lyapunov function:

$$
\begin{aligned}
\dot{V} &= -\int_0^1 w_x^2\,dx \\
&\leq -\frac{1}{4}\int_0^1 w^2\,dx \qquad \text{(Poincare inequlaity)} \\
&\leq -\frac{1}{2}V
\end{aligned}
$$

Therefore

$$V(t) \leq V(0)\mathrm{e}^{-t/2} \ \text{ or } \ \|w(t)\| \leq \mathrm{e}^{-t/4}\|w_0\|$$

We showed that $\|w\| \to 0$ as $t \to \infty$

This does not imply that $w(x,t) \to 0$ as $t \to \infty$ for all x (“spikes” in space are possible)

Pointwise Stability

Would like to show that

$$\max_{x\in[0,1]} |w(x,t)| \leq K\mathrm{e}^{-\frac{t}{4}} \max_{x\in[0,1]} |w(x,0)|$$

This result cannot be proved. However, it is possible to show a slightly weaker result

$$\max_{x\in[0,1]} |w(x,t)| \leq K\mathrm{e}^{-\frac{t}{4}} \|w_0\|_{H_1}$$

We define H_1 norm as

$$\|w\|_{H_1} := \int_0^1 w^2\,dx + \int_0^1 w_x^2\,dx$$

Note that by using Poincare inequality it is possible to drop the integral of w^2 for most problems

The following inequality bounds the maximum norm by L_2 and H_1 norms and boundary condition:

Agmon Inequality

$$
\begin{aligned}
\max_{x\in[0,1]} |w(x)|^2 &\leq w^2(0) + 2\|w\|\|w_x\| \\
\max_{x\in[0,1]} |w(x)|^2 &\leq w^2(1) + 2\|w\|\|w_x\|
\end{aligned}
$$

Proof:

$$
\int_0^x w(\xi) w_\xi(\xi)\, d\xi = \frac{1}{2} w^2(\xi)|_0^x = \frac{1}{2} w^2(x) - \frac{1}{2} w^2(0)
$$

Using triangle inequality we get

$$
\begin{aligned}
\frac{1}{2} w^2(x) &\leq \frac{1}{2} w^2(0) + \int_0^x |w(\xi||w_\xi(\xi)|\, d\xi \\
w^2(x) &\leq w^2(0) + 2\int_0^1 |w(\xi)||w_\xi(\xi)|\, dx \\
\max_{x\in[0,1]} |w(x)|^2 &\leq w^2(0) + 2\|w\|\|w_x\|
\end{aligned}
$$

Back to our problem:

$$\begin{aligned} w_t &= w_{xx} \\ w(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Let us use the Lyapunov function

$$V = \frac{1}{2}\int_0^1 w^2(x)\,dx + \frac{1}{2}\int_0^1 w_x^2(x)\,dx$$

$$\begin{aligned} \dot{V} &= \int_0^1 w w_{xx}\,dx + \int_0^1 w_x w_{tx}\,dx \\ &= w(x)w_x(x)|_0^1 - \int_0^1 w_x^2\,dx + w_t(x)w_x(x)_0^1 - \int_0^1 w_{xx}w_t\,dx \\ &= -\int_0^1 w_x^2\,dx - \int_0^1 w_{xx}^2\,dx \\ &\le -\frac{1}{2}\|w_x\|^2 - \frac{1}{2}\|w_x\|^2 \\ &\le -\frac{1}{8}\|w\|^2 - \frac{1}{2}\|w_x\|^2 \\ &\le -\frac{1}{4}V \end{aligned}$$

We have

$$\|w(t)\|^2+\|w_x(t)\|^2 \leq \mathrm{e}^{-t/2}\left(\|w_0\|^2+\|w_{0x}\|^2\right)$$

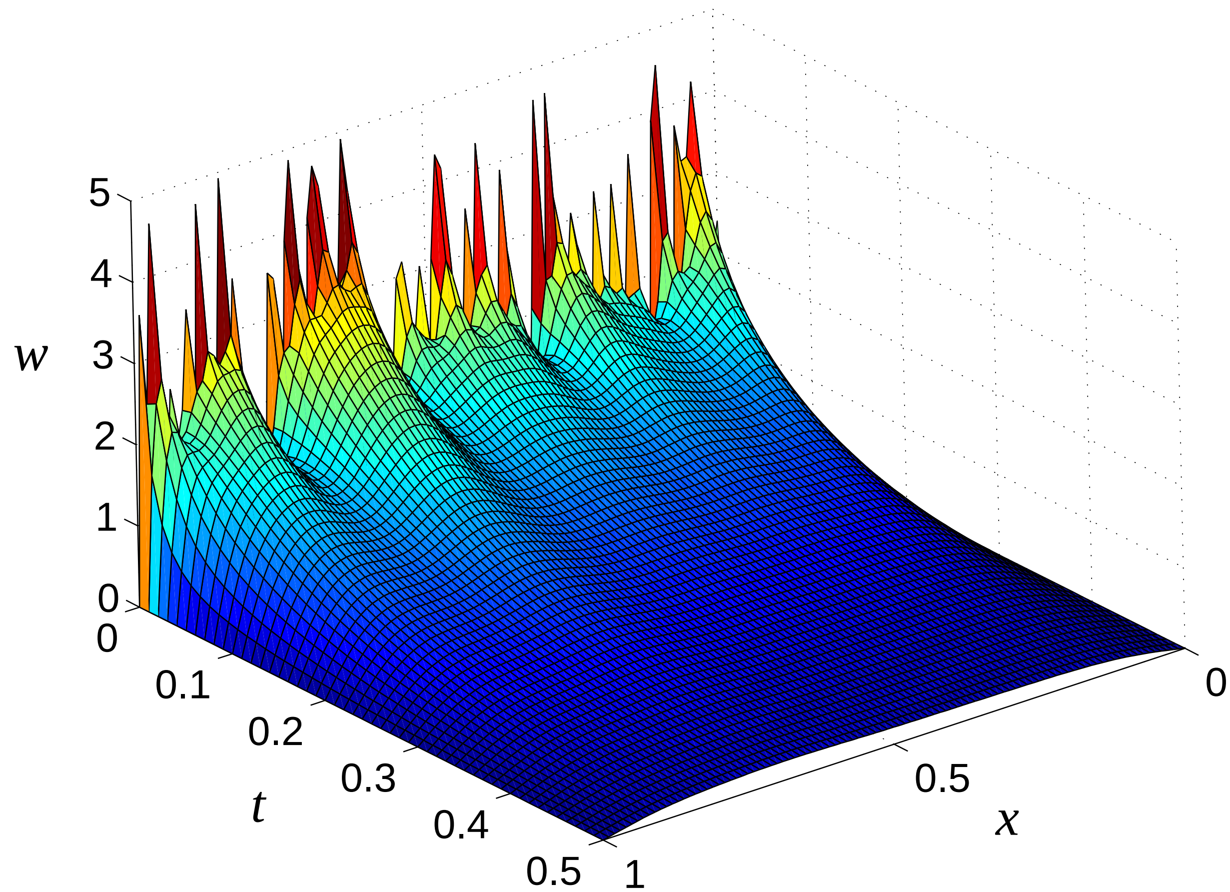
where $w_0 = w(x,0)$ is the initial condition.

Finally,

$$\begin{aligned}\max_{x\in[0,1]} |w(x,t)|^2 &\leq 2\|w(t)\|\|w_x(t)\| \qquad \text{(Agmon inequality)}\\ &\leq \|w(t)\|^2+\|w_x(t)\|^2 \\ &\leq \mathrm{e}^{-t/2}\left(\|w_0\|^2+\|w_{0x}\|^2\right)\end{aligned}$$

We showed that the equilibrium $w \equiv 0$ is asymptotically stable for all $x \in [0,1]$.

Typical response



Eigenvalues, Eigenfunctions, Exact Solutions

Andrey Smyshlyaev

mini-course at UCSB, 2006

Exact Solutions

Exist mostly for the plants with constant parameters.

Two standard methods for finding exact solutions: separation of variables and Laplace transform.

Separation of Variables

Heat equation with reaction:

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

Postulate the solution in the form $u(x,t) = X(x)T(t)$.

Substitute $u(x,t) = X(x)T(t)$ in the equation:

$$X(x)\dot{T}(t) = X''(x)T(t) + \lambda X(x)T(t)$$

Divide by $X(x)T(t)$:

$$\frac{\dot{T}}{T} = \frac{X'' + \lambda X}{X} = \sigma$$

ODE for T:

$$\begin{aligned} \dot{T} &= \sigma T \\ T &= e^{\sigma t} \qquad \text{(without loss of generality)} \end{aligned}$$

ODE for X:

$$\begin{aligned} X'' + (\lambda - \sigma)X &= 0 \\ X(0) &= X(1) = 0 \end{aligned}$$

Solution for $X(x)$:

$$X(x) = {\color{red}A}\sin(\sqrt{\lambda - \sigma}\,x) + {\color{red}B}\cos(\sqrt{\lambda - \sigma}\,x)$$

$$X(x) = A\sin(\sqrt{\lambda - \sigma}\,x) + B\cos(\sqrt{\lambda - \sigma}\,x)$$

$$\begin{aligned}
X(0) = 0 \quad &\Rightarrow \quad B = 0 \\
X(1) = 0 \quad &\Rightarrow \quad A\sin(\sqrt{\lambda - \sigma}) = 0 \\
&\Rightarrow \quad \sqrt{\lambda - \sigma} \; = \; \pi n, \quad \text{where } n = 0, 1, 2, \ldots \\
&\Rightarrow \quad \sigma = \lambda - \pi^2 n^2
\end{aligned}$$

Solution

$$u_n(x,t) = A_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x), \quad n = 0, 1, 2, \ldots$$

Since the PDE is linear, the sum of solutions is also a solution. Therefore the formal general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x)$$

To find A_n we use the knowledge of the initial condition $u(x,0) = u_0(x)$

Set $t = 0 \;\Rightarrow\; u_0(x) = \sum_{n=1}^{\infty} A_n \sin(\pi n x)$

Multiply both sides with $\sin(\pi m x) \;\Rightarrow\; u_0(x)\sin(\pi m x) = \sin(\pi m x)\sum_{n=1}^{\infty} A_n \sin(\pi n x)$

Use the orthogonality property $\int_0^1 \sin(\pi m x)\sin(\pi n x)dx = \left\{ \begin{array}{c} 1/2 \;\; n = m \\ 0 \;\; n \neq m \end{array} \right\}$
to get

$$\int_0^1 u_0(x)\sin(\pi m x)dx = \frac{1}{2}A_m$$

The exact solution is eigenvalues effect of initial conditions

$$u(x,t) = 2\sum_{n=1}^{\infty} e^{\overbrace{(\lambda-\pi^2 n^2)}^{\text{eigenvalues}}t}\underbrace{\sin(\pi n x)}_{\text{eigenfunctions}}\overbrace{\int_0^1 \sin(\pi n x)u_0(x)\,dx}^{\text{effect of initial conditions}}$$

eigenfunctions

The stability condition is $\lambda < \pi^2$. Note that it is much less conservative than the one obtained using Lyapunov method (which gives $\lambda < 1/4$).

Example. Find values of the parameter g for which the system

$$\begin{aligned} u_t &= u_{xx} + g u(0) \\ u_x(0) &= u(1) = 0 \end{aligned}$$

is unstable.

Let $u(x,t) = e^{\sigma t} X(x)$. Substitute this solution in the PDE to get an ODE

$$X''(x) - \sigma X = -g X(0)$$

which has a general solution

$$X(x) = A \sinh(\sqrt{\sigma} x) + B \cosh(\sqrt{\sigma} x) + \frac{g}{\sigma} X(0)$$

To find B, let $x = 0$:

$$X(0) = B + \frac{g}{\sigma} X(0) \quad \Rightarrow \quad B = X(0)\left(1 - \frac{g}{\sigma}\right)$$

Boundary condition at $x = 0$ gives

$$X'(0) = 0 \quad \Rightarrow \quad A = 0$$

We have

$$X(x) = X(0)\left[\frac{g}{\sigma} + \left(1 - \frac{g}{\sigma}\right)\cosh(\sqrt{\sigma}x)\right]$$

Boundary condition at $x = 1$ gives

$$X(1) = 0 \quad \Rightarrow \quad \cosh(\sqrt{\sigma}) \quad = \quad \frac{g}{g - \sigma}$$

This equation cannot be solved in closed form. But we can still find the region of stability.

Solve for g in terms of σ:

$$g \quad = \quad \frac{\sigma\cosh(\sqrt{\sigma})}{\cosh(\sqrt{\sigma}) - 1}$$

Take the limit $\sigma \longrightarrow 0$:

$$g \quad = \quad \lim_{\sigma\to 0}\frac{\sigma\cosh(\sqrt{\sigma})}{\cosh(\sqrt{\sigma}) - 1} = \lim_{\sigma\to 0}\frac{\sigma(1 + \sigma/2)}{1 + \sigma/2 - 1} = 2$$

Therefore, the PDE is unstable for $g > 2$.

Backstepping Design for Parabolic PDEs

Andrey Smyshlyaev

mini-course at UCSB, 2006

Backstepping Control Design

Unstable heat equation

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u(0) &= 0 \\ u(1) &= \text{control} \end{aligned}$$

Backstepping transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy$$

Target system

$$\begin{aligned} w_t &= w_{xx} \\ w(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Controller is obtained by setting $x = 1$ in the transformation

$$u(1) = \int_0^1 k(1,y)u(y)\,dy$$

Useful knowledge from calculus: Leibniz Integral Rule

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} f_z(x,z)dx + f(b(z),z)b'(z) - f(a(z),z)a'(z)$$

Notation:

$$\begin{aligned} k_x(x,x) &= \frac{\partial}{\partial x}k(x,y)|_{y=x} \\ k_y(x,x) &= \frac{\partial}{\partial y}k(x,y)|_{y=x} \\ \frac{d}{dx}k(x,x) &= k_x(x,x) + k_y(x,x) \end{aligned}$$

Kernel PDE Derivation

$$
\begin{aligned}
w(x) &= u(x) - \int_0^x k(x,y)u(y)dy \\
w_x(x) &= u_x(x) - \int_0^x k_x(x,y)u(y)dy - k(x,x)u(x) \\
w_{xx}(x) &= u_{xx}(x) - \int_0^x k_{xx}(x,y)u(y)dy - k_x(x,x)u(x) - \frac{d}{dx}(k(x,x)u(x))
\end{aligned}
$$

Time derivative:

$$
\begin{aligned}
w_t(x) &= u_t(x) - \int_0^x k(x,y)u_t(y)dy \\
&= u_{xx}(x) + \lambda u(x) - \int_0^x k(x,y)[u_{yy}(y) + \lambda u(y)]\,dy \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + \int_0^x k_y(x,y)u_y(y)dy \\
&\quad - \int_0^x \lambda k(x,y)u(y)dy \quad \text{(integration by parts)} \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) - k_y(x,0)u(0) \\
&\quad - \int_0^x k_{yy}(x,y)u(y)dy - \int_0^x \lambda k(x,y)u(y)dy \quad \text{(integration by parts)}
\end{aligned}
$$

$$
\begin{aligned}
w_t - w_{xx} &= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) - k_y(x,0)u(0) \\
&\quad - \int_0^x k_{yy}(x,y)u(y)\,dy - \int_0^x \lambda k(x,y)u(y)\,dy \\
&\quad - \left[u_{xx}(x) - \int_0^x k_{xx}(x,y)u(y)\,dy - k_x(x,x)u(x) - u(x)\frac{d}{dx}k(x,x) - k(x,x)u_x(x))\right] \\
&= u(x)\left[\lambda + 2\frac{d}{dx}k(x,x)\right] + k(x,0)u_x(0) \\
&\quad + \int_0^x u(y)[k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y)]\,dy
\end{aligned}
$$

For right hand side to be zero, 3 conditions should be satisfied:

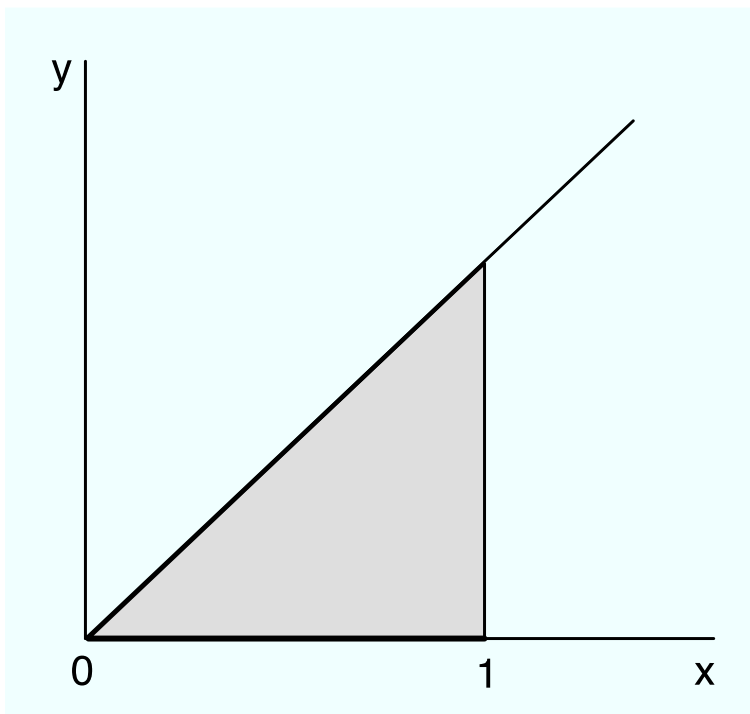
$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) &= \lambda k(x,y) \\
k(x,0) &= 0 \\
\lambda + 2\frac{d}{dx}k(x,x) &= 0
\end{aligned}
$$

Are these 3 conditions compatible? In other words, is this PDE well posed?

Control kernel PDE

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) &= \lambda k(x,y) \\
k(x,0) &= 0 \\
k(x,x) &= -\frac{\lambda x}{2}
\end{aligned}
$$

Domain



Converting Kernel PDE to Integral Equation

Introduce the change of variables

$$
\begin{aligned}
\xi &= x+y \\
\eta &= x-y \\
k(x,y) &= G(\xi,\eta)
\end{aligned}
$$

Then we have

$$
\begin{aligned}
k_x &= G_\xi + G_\eta \\
k_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta} \\
k_y &= G_\xi - G_\eta \\
k_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta}
\end{aligned}
$$

The kernel PDE in new variables is

$$
\begin{aligned}
4G_{\xi\eta}(\xi,\eta) &= \lambda G(\xi,\eta) \\
G(\xi,\xi) &= 0 \\
G(\xi,0) &= -\frac{\lambda\xi}{4}
\end{aligned}
$$

Integrate $4G_{\xi\eta} = \lambda G$ with respect to η from 0 to η:

$$G_{\xi}(\xi,\eta) = G_{\xi}(\xi,0) + \int_0^{\eta} \frac{\lambda}{4} G(\xi,s)\,ds$$

Integrate the result with respect to ξ from η to ξ and use boundary conditions to get

$$G(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_{\eta}^{\xi}\int_0^{\eta} G(\tau,s)\,ds\,d\tau$$

How to solve this integral equation?

Method of Successive Approximations

Very simple idea: start with a guess, compute the right hand side of the equation, use the solution as the next guess and repeat. The result will converge to the solution of the integral equation.

Let us start with initial guess

$$G_0(\xi,\eta) \quad = \quad 0$$

and define

$$G_{n+1}(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} G_n(\tau,s)\,ds\,d\tau$$

Performing the necessary integrations and observing the resulting pattern one can guess the general term of this series and then prove by induction that it is correct:

$$G_{n+1}(\xi,\eta) - G_n(\xi,\eta) = -\frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{\lambda}{4}\right)^{n+1}, \qquad n = 0,1,2,\ldots$$

The solution to the integral equation is

$$\begin{aligned} G(\xi,\eta) &= \lim_{n\to\infty} G_n(\xi,\eta) \\ &= -\sum_{n=0}^{\infty} [G_{n+1}(\xi,\eta) - G_n(\xi,\eta)] \\ &= -\sum_{n=0}^{\infty} \frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!} \left(\frac{\lambda}{4}\right)^{n+1} \end{aligned}$$

This series can be summed up:

$$G(\xi,\eta) = -\frac{\lambda}{2}(\xi-\eta)\frac{I_1\left(\sqrt{\lambda\xi\eta}\right)}{\sqrt{\lambda\xi\eta}}$$

or in the original variables

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}$$

Bessel Functions J_n and I_n

The function $y(x) = J_n(x)$ is a solution to the following ODE

$$x^2 y''_{xx} + x y'_x + (x^2 - n^2) y = 0$$

Series representation

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(m+n)!}$$

Other properties

$$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x))$$

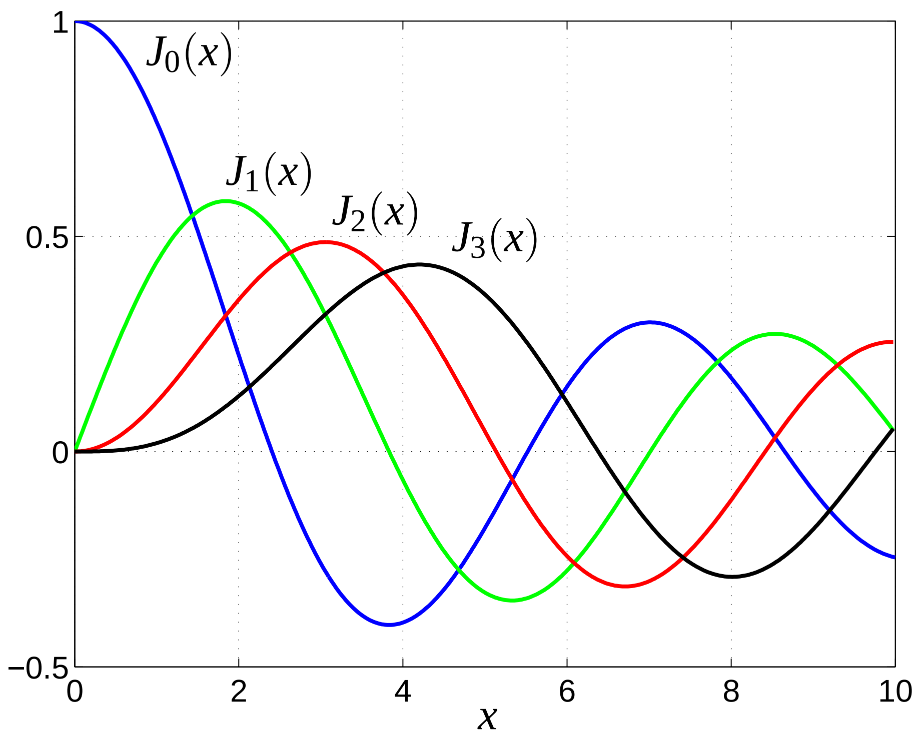
Differentiation

$$\frac{d}{dx} J_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) \qquad \frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}$$

Asymptotic properties

$$J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \longrightarrow 0$$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right), \qquad x \longrightarrow \infty$$



The function $y(x) = I_n(x)$ is a solution to the following ODE

$$x^2 y''_{xx} + x y'_x - (x^2 + n^2) y = 0$$

Series representation

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(m+n)!}$$

Relationship with $J_n(x)$

$$I_n(x) = i^{-n} J_n(ix), \qquad I_n(ix) = i^n J_n(x)$$

Other properties

$$2n I_n(x) = x(I_{n-1}(x) - I_{n+1}(x))$$

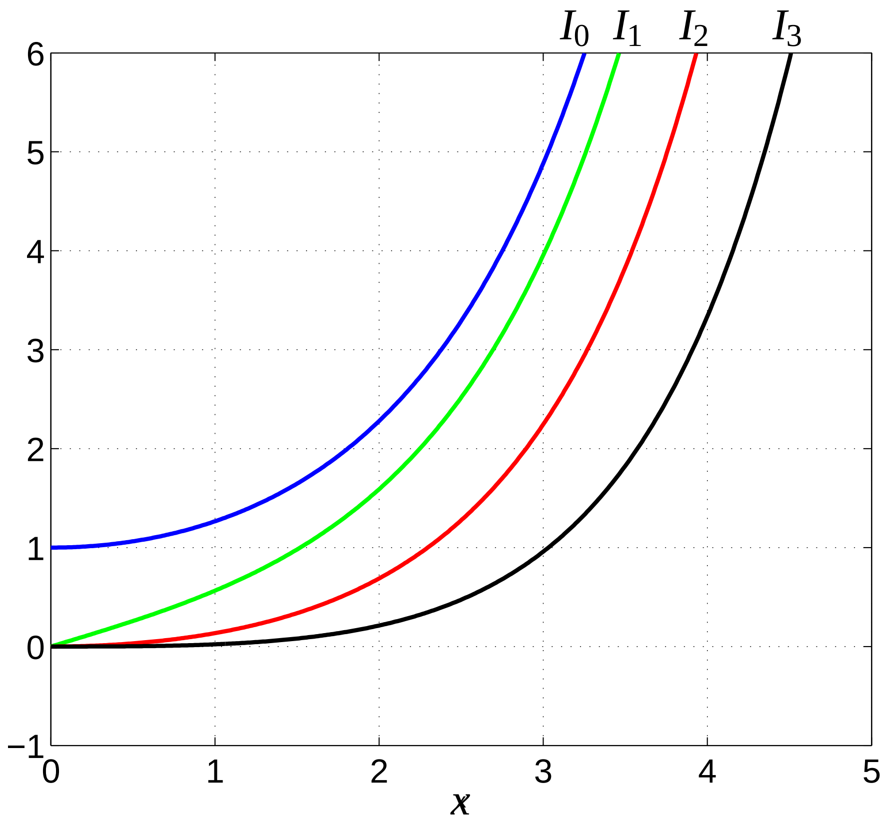
Differentiation

$$\frac{d}{dx} I_n(x) = \frac{1}{2}(I_{n-1}(x) + I_{n+1}(x)) \qquad \frac{d}{dx}(x^n I_n(x)) = x^n I_{n-1}$$

Asymptotic properties

$$I_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \longrightarrow 0$$

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \qquad x \longrightarrow \infty$$



Inverse Transformation

Remember the transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)dy$$

We found $k(x,y)$ and w-system is exp. stable. Does this imply that u is exp. stable?

Depends on the properties of $k(x,y)$. Since our kernel $k(x,y)$ is twice continuously differentiable, it turns out that this is enough for inverse transformation to exist.

Let us find the inverse transformation

$$u(x) = w(x) + \int_0^x l(x,y)w(y)dy$$

It can be shown that $l(x,y)$ satisfies the following PDE

$$\begin{aligned} l_{xx}(x,y) - l_{yy}(x,y) &= -\lambda l(x,y) \\ l(x,0) &= 0 \qquad\qquad \Rightarrow \quad l(\lambda) = -k(-\lambda)! \\ l(x,x) &= -\frac{\lambda x}{2} \end{aligned}$$

We have

$$
\begin{aligned}
l(x,y) &= (-\lambda)y\frac{I_1\left(\sqrt{(-\lambda)(x^2-y^2)}\right)}{\sqrt{-\lambda(x^2-y^2)}} = -\lambda y\frac{I_1\left(j\sqrt{\lambda(x^2-y^2)}\right)}{j\sqrt{\lambda(x^2-y^2)}} \\
&= -\lambda y\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}
\end{aligned}
$$

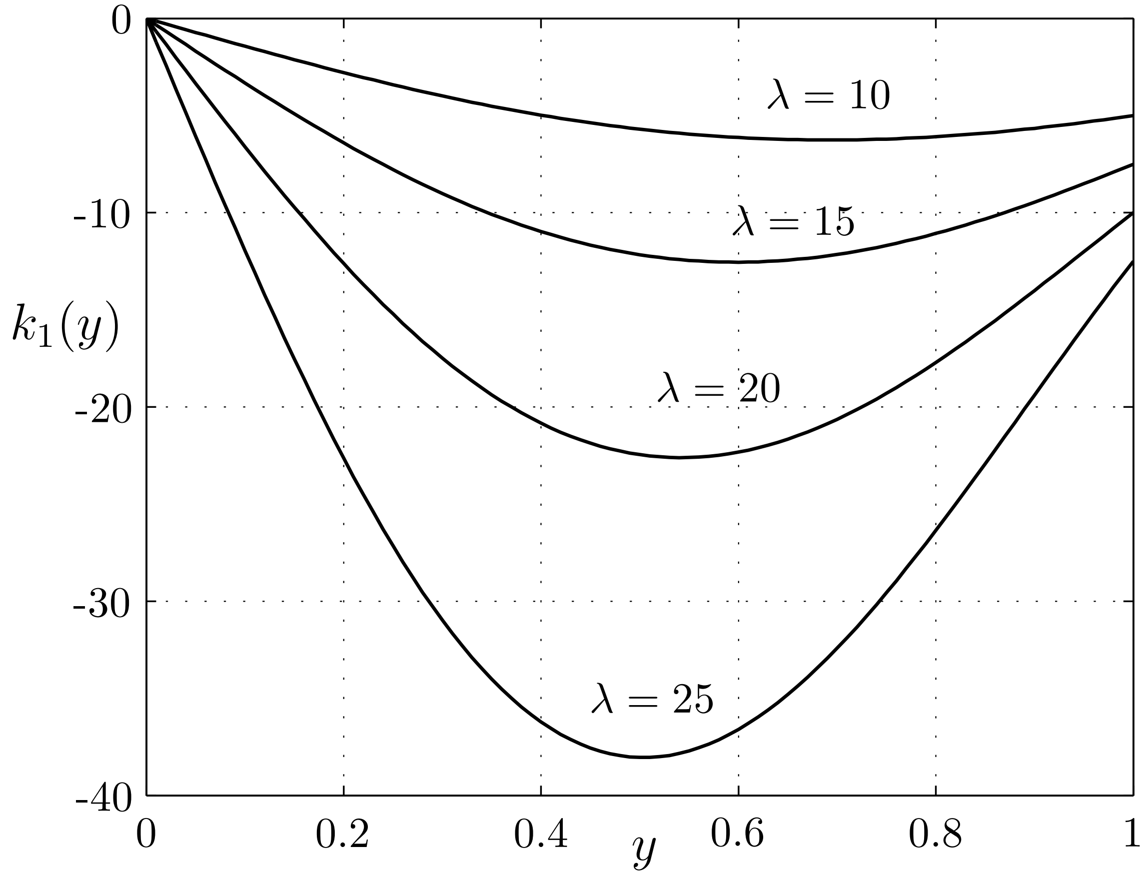
Therefore the inverse transformation is

$$
u(x) = w(x) - \int_0^x \lambda y\frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}w(y)\,dy
$$

Since $w(x,t)\to 0$ as $t\to\infty$, we get $u(x,t)\to 0$ for all $x\in[0,1]$ with a boundary controller

$$
u(1) = -\int_0^1 \lambda y\frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}u(y)\,dy
$$

Control Gain



Neumann Controller

Unstable heat equation with Neumann actuation

$$
\begin{aligned}
u_t &= u_{xx} + \lambda u \\
u(0) &= 0 \\
u_x(1) &= \text{control}
\end{aligned}
$$

Exactly the same transformation as in case of Dirichlet actuation:

$$
w(x) = u(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}} u(y)\, dy
$$

But with a target system modified at $x = 1$ (easy to show that it is stable)

$$
\begin{aligned}
w_t &= w_{xx} \\
w(0) &= 0 \\
w_x(1) &= 0
\end{aligned}
$$

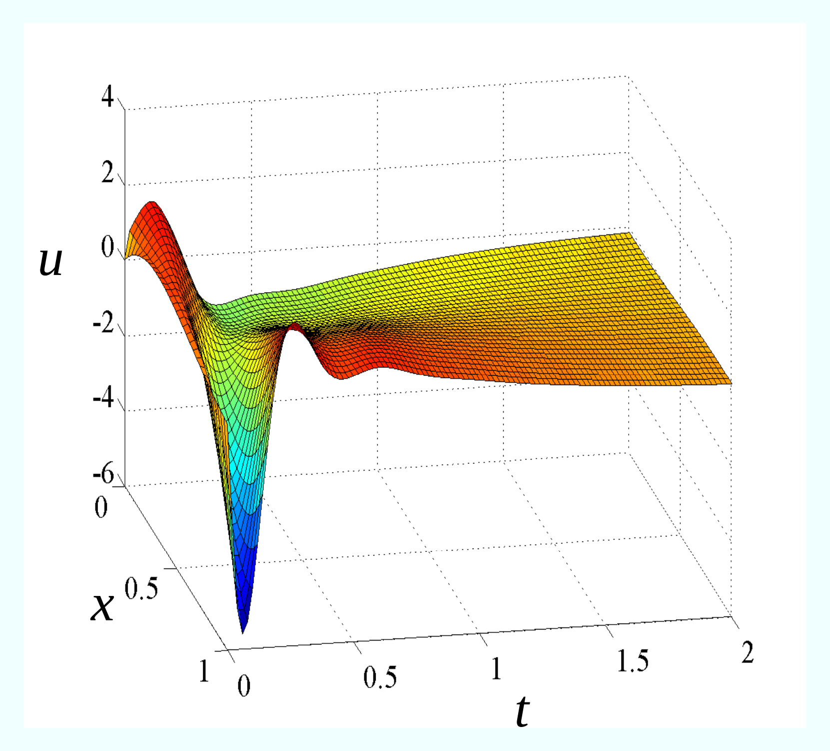
Simply differentiate the transformation with respect to x:

$$
\begin{aligned}
w(x) &= u(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} u(y)\, dy \\
w_x(x) &= u_x(x) + \frac{\lambda x}{2} u(x) + \int_0^x \lambda y x \frac{I_2\left(\sqrt{\lambda(x^2-y^2)}\right)}{x^2-y^2} u(y)\, dy
\end{aligned}
$$

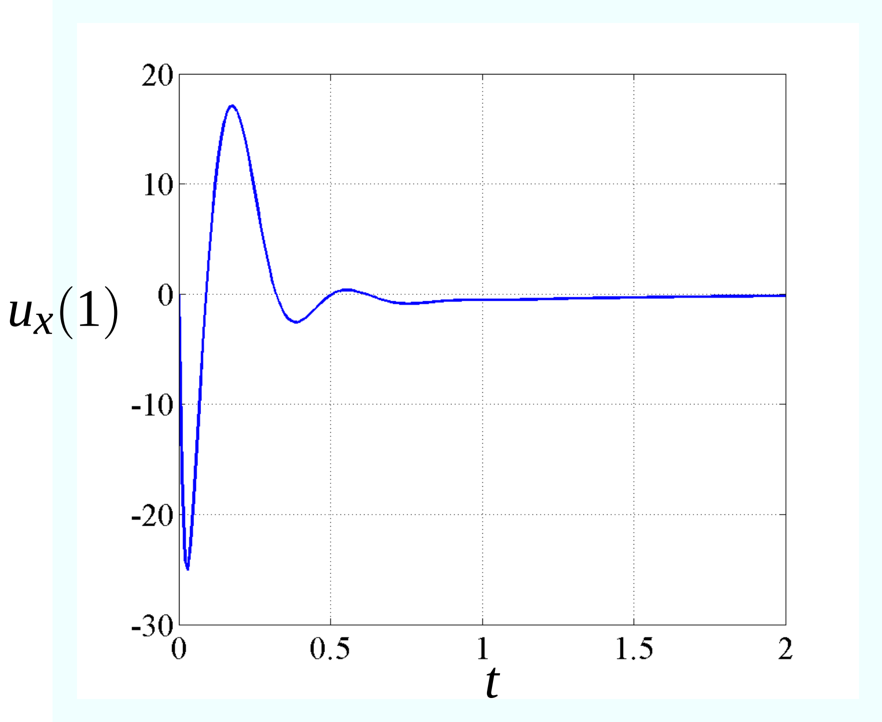
and evaluate at $x = 1$ to get Neumann controller:

$$
u_x(1) = -\frac{\lambda}{2} u(1) - \int_0^1 \lambda y \frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{1-y^2} u(y) dy
$$

Closed Loop Simulation



Closed loop state



Control effort

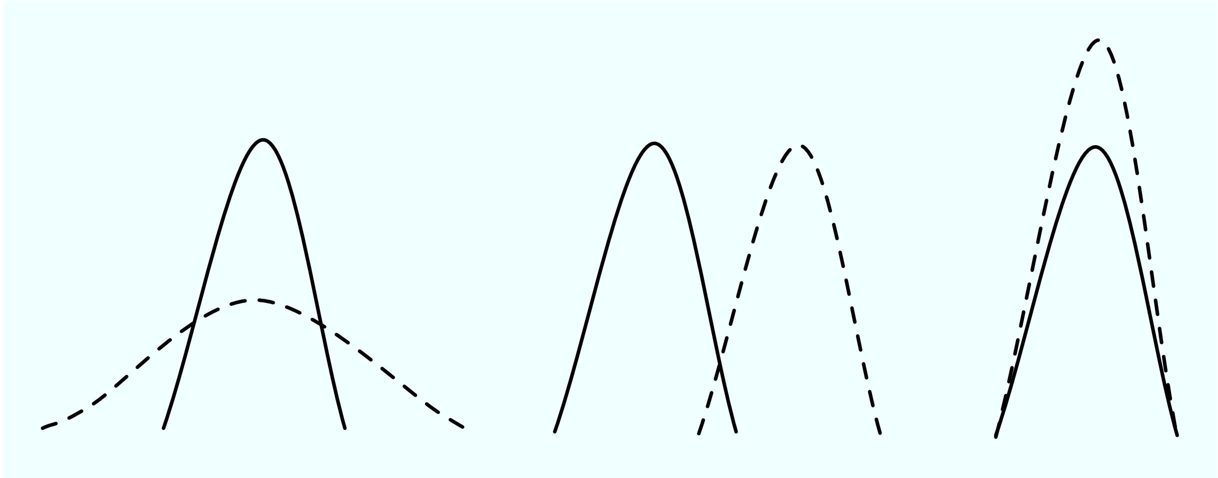
Reaction-Advection-Diffusion Systems

Plant

$$
\begin{aligned}
u_t &= \varepsilon(x)u_{xx} + b(x)u_x + \lambda(x)u \\
u_x(0) &= -qu(0)
\end{aligned}
$$

These equations come from thermal / fluid / chemical problems.

What each term does:



Diffusion Advection Reaction

(smoothing) (transfer) (destabilizing)

Reaction-Advection-Diffusion Systems

Plant

$$\begin{aligned} u_t &= \varepsilon(x) u_{xx} + b(x) u_x + \lambda(x) u \\ u_x(0) &= -q u(0) \end{aligned}$$

Spatially varying coefficients arise for several reasons:

— linearization

— non-homogenous materials

— unusually shaped domains

Using special transformation we can eliminate $b(x)$ and make $\varepsilon(x)$ constant.

Gauge transformation

$$z = \sqrt{\varepsilon_0} \int_0^x \frac{ds}{\sqrt{\varepsilon(s)}}, \quad \text{where } \varepsilon_0 = \left(\int_0^1 \frac{ds}{\sqrt{\varepsilon(s)}} \right)^{-2}$$

$$v(z) = u(x)\varepsilon(x)^{-\frac{1}{4}} \exp\left\{ \int_0^x \frac{b(s)}{2\varepsilon(s)} ds \right\}$$

Transformed plant

$$\begin{aligned} v_t &= \varepsilon_0 v_{xx} + \lambda_0(x) v \\ v_x(0) &= -q_0 v(0) \end{aligned}$$

where

$$\begin{aligned} \lambda_0(x) &= \lambda(x) + \frac{\varepsilon''(x)}{4} - \frac{b'(x)}{2} - \frac{3}{16} \frac{(\varepsilon'(x))^2}{\varepsilon(x)} + \frac{1}{2} \frac{(b(x)\varepsilon'(x))}{\varepsilon(x)} - \frac{1}{4} \frac{b^2(x)}{\varepsilon(x)} \\ q_0 &= q\sqrt{\frac{\varepsilon(0)}{\varepsilon_0}} - \frac{b(0)}{2\sqrt{\varepsilon_0 \varepsilon(0)}} - \frac{\varepsilon'(0)}{4\sqrt{\varepsilon_0 \varepsilon(0)}} \end{aligned}$$

Backstepping transformation

$$w(x) = v(x) - \int_0^x k(x,y)v(y)\,dy$$

Target system

$$\begin{aligned} w_t &= \varepsilon_0 w_{zz} - cw \\ w_z(0) &= 0 \\ w(1) &= 0 \end{aligned}$$

Kernel PDE

$$\begin{aligned} k_{zz}(z,y) - k_{yy}(z,y) &= \frac{\lambda_0(y)+c}{\varepsilon_0} k(z,y) \\ k_y(z,0) &= -q_0 k(z,0) \\ k(z,z) &= -q_0 - \frac{1}{2\varepsilon_0}\int_0^z (\lambda_0(y)+c)dy \end{aligned}$$

Kernel PDE can no longer be solved in closed form, but the solution can be computed numerically (order of magnitude faster in computation time than solving a Ricatti equation).

Other Spatially Causal Plants

Plant

$$
\begin{aligned}
u_t &= u_{xx} + g(x)u(0) + \int_0^x f(x,y)u(y)dy \\
u_x(0) &= 0 \\
u(1) &= \text{control}
\end{aligned}
$$

Control gain PDE

$$
\begin{aligned}
k_{xx} - k_{yy} &= -f(x,y) + \int_y^x k(x,\xi) f(\xi,y) d\xi \\
k_y(x,0) &= g(x) - \int_0^x k(x,y)g(y)dy \\
k(x,x) &= 0
\end{aligned}
$$

Example. Let $f \equiv 0$, then $k_{xx} - k_{yy} = 0$ which has a general solution

$$
k(x,y) = \phi(x-y) + \psi(x+y)
$$

$$k(x,y) = \phi(x-y) + \psi(x+y)$$

Setting $y = x$ we get

$$\phi(0) + \psi(2x) \;=\; 0 \quad \Rightarrow \quad \phi(0) = 0 \quad \text{and} \quad \psi \equiv 0$$

Substituting $k(x,y) = \phi(x-y)$ into the boundary condition of the gain PDE we get

$$\phi'(x) = g(x) - \int_0^x \phi(x-y)g(y)\,dy$$

Applying Laplace tranform with respect to x we get

$$\begin{aligned} -s\phi(s) + \phi(0) &= g(s) - \phi(s)g(s) \\ \phi(s) &= \frac{g(s)}{g(s)-s} \end{aligned}$$

Thus, the controller can be designed in closed form for any $g(x)$!

For example, if $g(x) = g$, then $g(s) = g/s$ and we get

$$\phi(s) = \frac{g}{g - s^2} = -\sqrt{g}\frac{\sqrt{g}}{s^2 - g}$$

This gives

$$\phi(x) = -\sqrt{g}\sinh(\sqrt{g}x) \quad \Rightarrow \quad k(x,y) = -\sqrt{g}\sinh(\sqrt{g}(x-y))$$

Therefore, for the plant

$$\begin{aligned} u_t &= u_{xx} + gu(0) \\ u_x(0) &= 0 \end{aligned}$$

stabilizing controller is

$$u(1) = -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))u(y)dy$$

Boundary Observers and Output Feedback

Andrey Smyshlyaev

mini-course at UCSB, 2006

Observers

Plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u_x(0) &= 0 \end{aligned}$$

Possible input-output architectures:

- Anti-collocated: $u(0)$ measured and $u(1)$ or $u_x(1)$ actuated

- Collocated: $u_x(1)$ measured and $u(1)$ actuated (fluid problems)

  $u(1)$ measured and $u_x(1)$ actuated (thermal problems)

Anti-Collocated Setup

Plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u_x(0) &= 0 \end{aligned}$$

Input: $u(1)$ Output: $u(0)$

Observer

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + \lambda \hat{u} + p_1(x)[u(0) - \hat{u}(0)] \\ \hat{u}_x(0) &= p_{10}[u(0) - \hat{u}(0)] \\ \hat{u}(1) &= u(1) \end{aligned}$$

The function $p_1(x)$ and the constant p_{10} are observer gains.

Compare with finite dimension:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The error $\tilde{u} = u - \hat{u}$ satisfies

$$
\begin{aligned}
\tilde{u}_t &= \tilde{u}_{xx} + \lambda \tilde{u} - p_1(x)\tilde{u}(0) \\
\tilde{u}_x(0) &= -p_{10}\tilde{u}(0) \\
\tilde{u}(1) &= 0
\end{aligned}
$$

We use the transformation

$$
\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)\,dy
$$

to convert the error system into the stable target system:

$$
\begin{aligned}
\tilde{w}_t &= \tilde{w}_{xx} \\
\tilde{w}_x(0) &= 0 \\
\tilde{w}(1) &= 0
\end{aligned}
$$

Taking a derivative with respect to time we get

$$\begin{aligned}
\tilde{u}_t &= \tilde{w}_t - \int_0^x p(x,y)\tilde{w}_{yy}(y)\,dy \\
&= \tilde{w}_{xx} - p(x,x)\tilde{w}_x(x) + p(x,0)\tilde{w}_x(0) + \int_0^x p_y(x,y)\tilde{w}_y(y)\,dy \\
&= \tilde{w}_{xx} - p(x,x)\tilde{w}_x(x) + p_y(x,x)\tilde{w}(x) - p_y(x,0)\tilde{w}(0) - \int_0^x p_{yy}(x,y)\tilde{w}(y)\,dy
\end{aligned}$$

Taking derivatives with respect to x we get

$$\begin{aligned}
\tilde{u}_x &= \tilde{w}_x - p(x,x)\tilde{w}(x) - \int_0^x p_x(x,y)\tilde{w}(y)dy \\
\tilde{u}_{xx} &= \tilde{w}_{xx} - \frac{d}{dx}(p(x,x))\tilde{w}(x) - p(x,x)\tilde{w}_x(x) - p_x(x,x)\tilde{w}(x) - \int_0^x p_{xx}(x,y)\tilde{w}(y)dy
\end{aligned}$$

From the error system we get

$$\begin{aligned}
\tilde{u}_t - \tilde{u}_{xx} - \lambda\tilde{u} + p_1(x)\tilde{u}(0) &= 0 \\
&= [-p_y(x,0) + p_1(x)]\tilde{w}(0) + \left[2\frac{d}{dx}(p(x,x)) - \lambda\right]\tilde{w}(x) \\
&\quad + \int_0^x [p_{xx}(x,y) - p_{yy}(x,y) + \lambda p(x,y)]\tilde{w}(y)dy
\end{aligned}$$

This gives 3 conditions:

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\lambda p(x,y) \\ \frac{d}{dx}p(x,x) &= \frac{\lambda}{2} \\ p_1(x) &= p_y(x,0) \end{aligned}$$

Boundary conditions of the error system give 2 more conditions

$$\begin{aligned} \tilde{u}_x(0) &= -p_{10}\tilde{u}(0) && \Rightarrow && p(0,0) = p_{10} \\ \tilde{u}(1) &= 0 && \Rightarrow && p(1,y) = 0 \end{aligned}$$

Observer kernel PDE

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\lambda p(x,y) \\ p(1,y) &= 0 \\ p(x,x) &= -\frac{\lambda}{2}(1-x) \end{aligned}$$

Observer gains

$$\begin{aligned} p_1(x) &= p_y(x,0) \\ p_{10} &= p(0,0) \end{aligned}$$

Change of variables

$$\bar{x} = 1 - y \qquad \bar{y} = 1 - x \qquad \bar{p}(\bar{x},\bar{y}) = p(x,y)$$

Observer PDE in new variables

$$\begin{aligned}
\bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \lambda \bar{p}(\bar{x},\bar{y}) \\
\bar{p}(\bar{x},0) &= 0 \\
\bar{p}(\bar{x},\bar{x}) &= -\frac{\lambda}{2}\bar{x}
\end{aligned}$$

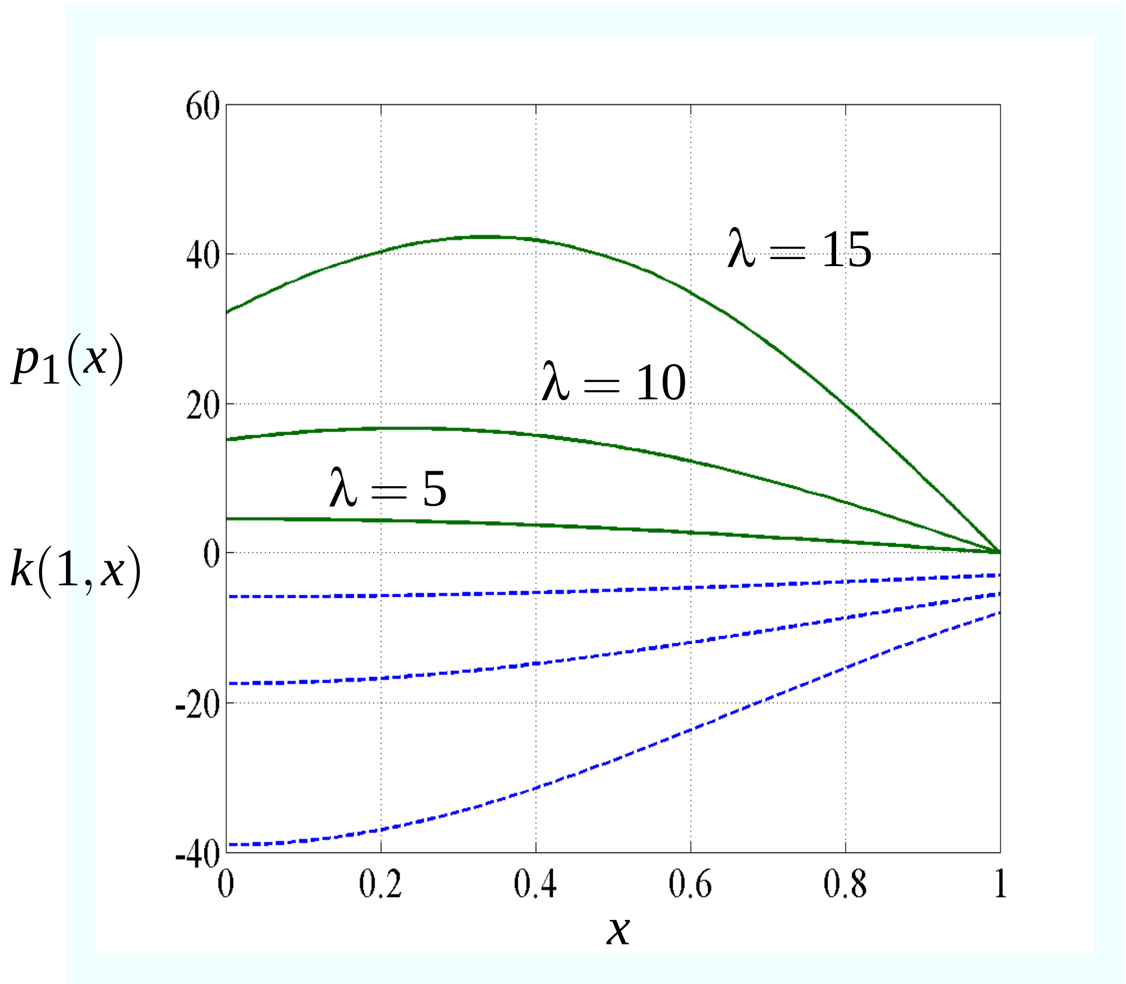
The solution is

$$\bar{p}(\bar{x},\bar{y}) = -\lambda \bar{y} \frac{I_1\left(\sqrt{\lambda(\bar{x}^2 - \bar{y}^2)}\right)}{\sqrt{\lambda(\bar{x}^2 - \bar{y}^2)}} = -\lambda(1-x)\frac{I_1\left(\sqrt{\lambda(x-y)(2-x-y)}\right)}{\sqrt{\lambda(x-y)(2-x-y)}}$$

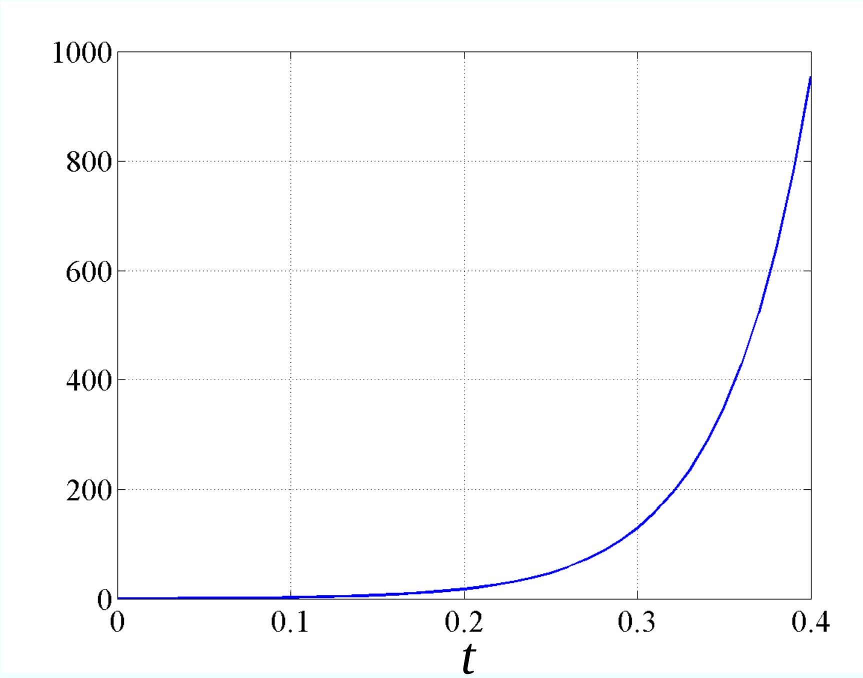
Observer gains

$$\begin{aligned}
p_1(x) &= p_y(x,0) = -\frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right) \\
p_{10} &= p(0,0) = -\frac{\lambda}{2}
\end{aligned}$$

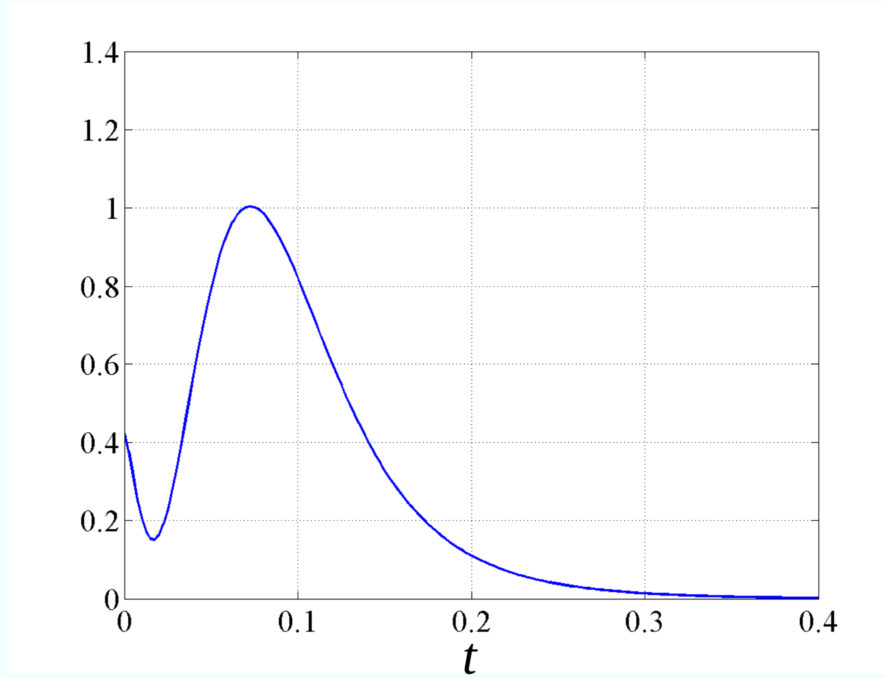
Observer Gains



Observer Simulation



L_2 norm of the open loop state



L_2 norm of the observer error

Collocated Setup

Plant

$$\begin{aligned} u_t &= u_{xx} + \lambda u \\ u_x(0) &= 0 \end{aligned}$$

Input: $u(1)$ Output: $u_x(1)$

Observer

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + \lambda \hat{u} + p_1(x)[u_x(1) - \hat{u}_x(1)] \\ \hat{u}_x(0) &= 0 \\ \hat{u}(1) &= u(1) + p_{10}[u_x(1) - \hat{u}_x(1)] \end{aligned}$$

The error $\tilde{u} = u - \hat{u}$ satisfies

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + \lambda \tilde{u} - p_1(x)\tilde{u}_x(1) \\ \tilde{u}_x(0) &= 0 \\ \tilde{u}(1) &= -p_{10}\tilde{u}_x(1) \end{aligned}$$

Using the transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_x^1 p(x,y)\tilde{w}(y)\,dy$$

we transform the error system into

$$\begin{aligned} \tilde{w}_t &= \tilde{w}_{xx} \\ \tilde{w}_x(0) &= 0 \\ \tilde{w}(1) &= 0 \end{aligned}$$

Observer gain PDE

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\lambda p(x,y) \\ p(x,x) &= -\frac{\lambda}{2}x \\ p_x(0,y) &= 0 \end{aligned}$$

The observer gains are

$$\begin{aligned} p_1(x) &= p(x,1) \\ p_{10} &= 0 \end{aligned}$$

Using the change of variables

$$\bar{x} = y \qquad \bar{y} = x$$

we get the same PDE as for the control kernel

$$\begin{aligned} \bar{p}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{p}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \lambda p(\bar{x},\bar{y}) \\ \bar{p}_{\bar{y}}(\bar{x},0) &= 0 \\ \bar{p}(\bar{x},\bar{x}) &= -\frac{\lambda}{2}\bar{x} \end{aligned}$$

The solution is

$$p(\bar{x},\bar{y}) = -\lambda \bar{x} \frac{I_1\left(\sqrt{\lambda(\bar{x}^2 - \bar{y}^2)}\right)}{\sqrt{\lambda(\bar{x}^2 - \bar{y}^2)}} = -\lambda y \frac{I_1\left(\sqrt{\lambda(y^2 - x^2)}\right)}{\sqrt{\lambda(y^2 - x^2)}}$$

Observer gain

$$p_1(x) = -\lambda \frac{I_1\left(\sqrt{\lambda(1 - x^2)}\right)}{\sqrt{\lambda(1 - x^2)}} = k(1,x) \quad \text{(duality)}$$

Output Feedback

Plant

$$
\begin{aligned}
u_t &= u_{xx} + \lambda u \\
u(0) &= 0
\end{aligned}
$$

Input: $u_x(1)$ Output: $u(1)$

Observer

$$
\begin{aligned}
\hat{u}_t &= \hat{u}_{xx} + \lambda \hat{u} + \frac{\lambda x}{1-x^2} I_2\left(\sqrt{\lambda(1-x^2)}\right)[u(1) - \hat{u}(1)] \\
\hat{u}(0) &= 0 \\
\hat{u}_x(1) &= u_x(1) - \frac{\lambda}{2}[u(1) - \hat{u}(1)]
\end{aligned}
$$

Controller

$$
u_x(1) = -\frac{\lambda}{2} u(1) - \int_0^1 \frac{\lambda y}{1-y^2} I_2\left(\sqrt{\lambda(1-y^2)}\right) \hat{u}(y)\, dy
$$

Frequency Domain Representation

Plant

$$\begin{aligned} u_t &= u_{xx} + g u(0) \\ u_x(0) &= 0 \end{aligned}$$

Output: $u(0)$ Input: $u(1)$

To derive the transfer function from $u(1)$ to $u(0)$, take the Laplace transform of the plant:

$$\begin{aligned} s u(x,s) &= u''(x,s) + g u(0,s) \\ u'(0,s) &= 0 \end{aligned}$$

General solution for this ODE:

$$u(x,s) = A \sinh(\sqrt{s} x) + B \cosh(\sqrt{s} x) + \frac{g}{s} u(0,s)$$

and boundary condition gives

$$u'(0,s) = A\sqrt{s} = 0 \Rightarrow A = 0$$

We have

$$u(x,s) = B\cosh(\sqrt{s}x) + \frac{g}{s}u(0,s)$$

Setting $x = 0$ we get

$$B = u(0,s)\left(1 - \frac{g}{s}\right)$$

so that

$$u(x,s) = u(0,s)\left[\frac{g}{s}\left(1 - \frac{g}{s}\right)\cosh(\sqrt{s}x)\right]$$

Setting $x = 1$ we get the open-loop transfer function

$$u(0,s) = \frac{s}{g + (s-g)\cosh(\sqrt{s})}u(1,s)$$

There are infinitely many poles and no zeros (infinite relative degree).

Let us now design compensator. The observer is

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + g u(0) \\ \hat{u}_x(0) &= 0 \\ \hat{u}(1) &= -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))\hat{u}(y)dy \end{aligned}$$

Applying Laplace transform we get

$$\begin{aligned} s\hat{u}(x,s) &= \hat{u}''(x,s) + g u(0,s) \\ \hat{u}'(0,s) &= 0 \\ \hat{u}(1,s) &= -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))\hat{u}(y,s)dy \end{aligned}$$

The solution of this ODE is

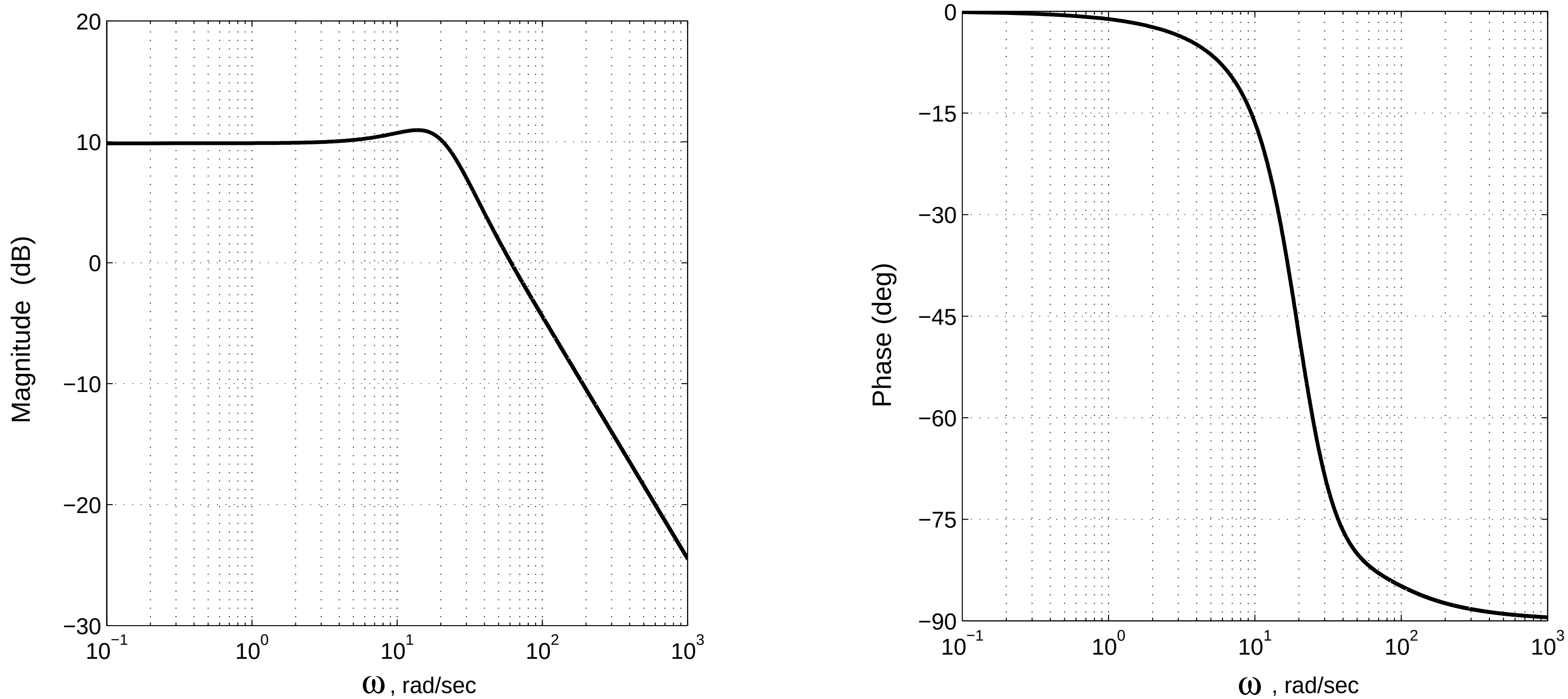
$$\hat{u}(x,s) = \hat{u}(0,s)\cosh(\sqrt{s}x) + \frac{g}{s}\left(1-\cosh(\sqrt{s}x)\right)u(0,s)$$

Setting $x = 1$ and using the boundary condition we express $\hat{u}(0,s)$ as a function of $u(0,s)$:

$$\hat{u}(0,s) = \frac{\cosh(\sqrt{s}) - \cosh(\sqrt{g})}{s\cosh(\sqrt{s}) - g\cosh(\sqrt{g})} g u(0,s)$$

Finally, the compensator is

$$u(1,s) = \frac{g}{s}\left(-1 + \frac{(s-g)\cosh(\sqrt{s}\cosh(\sqrt{g}))}{s\cosh(\sqrt{s}) - g\cosh(\sqrt{g})}\right) u(0,s)$$



Approximation:

$$u(1,s) \approx 60\frac{s+17}{s^2+25s+320}u(0,s)$$

Hyperbolic PDEs—Wave Equations

Miroslav Krstic, Andrey Smyshlyaev, and Antranik Siranosian

mini-course at UCSB, 2006

Wave Equation with “Free End” Damping

String/Cable of unit length:

$$\begin{aligned} u_{tt} &= u_{xx} && \text{(wave equation)} \\ u_x(0) &= 0 && \text{(“free” end)} \\ u(1) &= 0 && \text{(“pinned” end)} \end{aligned}$$

Energy/Lyapunov function

$$E = \frac{1}{2}\left(\|u_x\|^2 + \|u_t\|^2\right)$$

u_x	=	“shear”	potential energy
u_t	=	velocity	kinetic energy

$$
\begin{aligned}
\dot{E} &= \int_0^1 u_x u_{xt} dx + \int_0^1 u_t u_{tt} dx \qquad \text{(chain rule)} \\
&= \int_0^1 u_x u_{xt} dx + \int_0^1 u_t u_{xx} dx \\
&= \int_0^1 u_x u_{xt} dx + u_t(x) u_x(x)|_0^1 - \int_0^1 u_{tx} u_x dx \qquad \text{(integration by parts)} \\
&= u_t(x) u_x(x)|_0^1 \\
&= 0 \qquad \text{(using BCs)}
\end{aligned}
$$

Conservation of energy: $E(t) \equiv E(0)$. The system is marginally/neutrally stable. Inifinitely many eigenvalues on the imaginary axis.

A classical method of asymptotically stabilizing the system is to add "boundary damping:"

$$u_x(0) = c_0 u_t(0).$$

"Passive" control in both senses of the word (no active actuator and exploits the PRness and 'zero-state-observability' of the plant).

Asymptotic stability proof by Lyapunov possible but tricky. Eigenvalue calculation easier.

First, the solution postulated as

$$u(x,t) = \mathrm{e}^{\sigma t}\phi(x).$$

Substituting this into the PDE gives

$$\sigma^2 \mathrm{e}^{\sigma t}\phi(x) = \mathrm{e}^{\sigma t}\phi''(x),$$

and using the two BCs gives

$$\begin{aligned} \mathrm{e}^{\sigma t}\phi(1) &= 0 \\ \mathrm{e}^{\sigma t}\phi'(0) &= c_0 \sigma \mathrm{e}^{\sigma t}\phi(0). \end{aligned}$$

Sturm-Louiville problem for wave eqn with boundary damping:

$$\begin{aligned} \phi'' - \sigma^2\phi &= 0 \\ \phi'(0) &= c_0\sigma\phi(0) \\ \phi(1) &= 0. \end{aligned}$$

The solution given by

$$\phi(x) = e^{\sigma x} + Be^{-\sigma x}$$

From the BC at $x = 1$ we get

$$B = -e^{-2\sigma}.$$

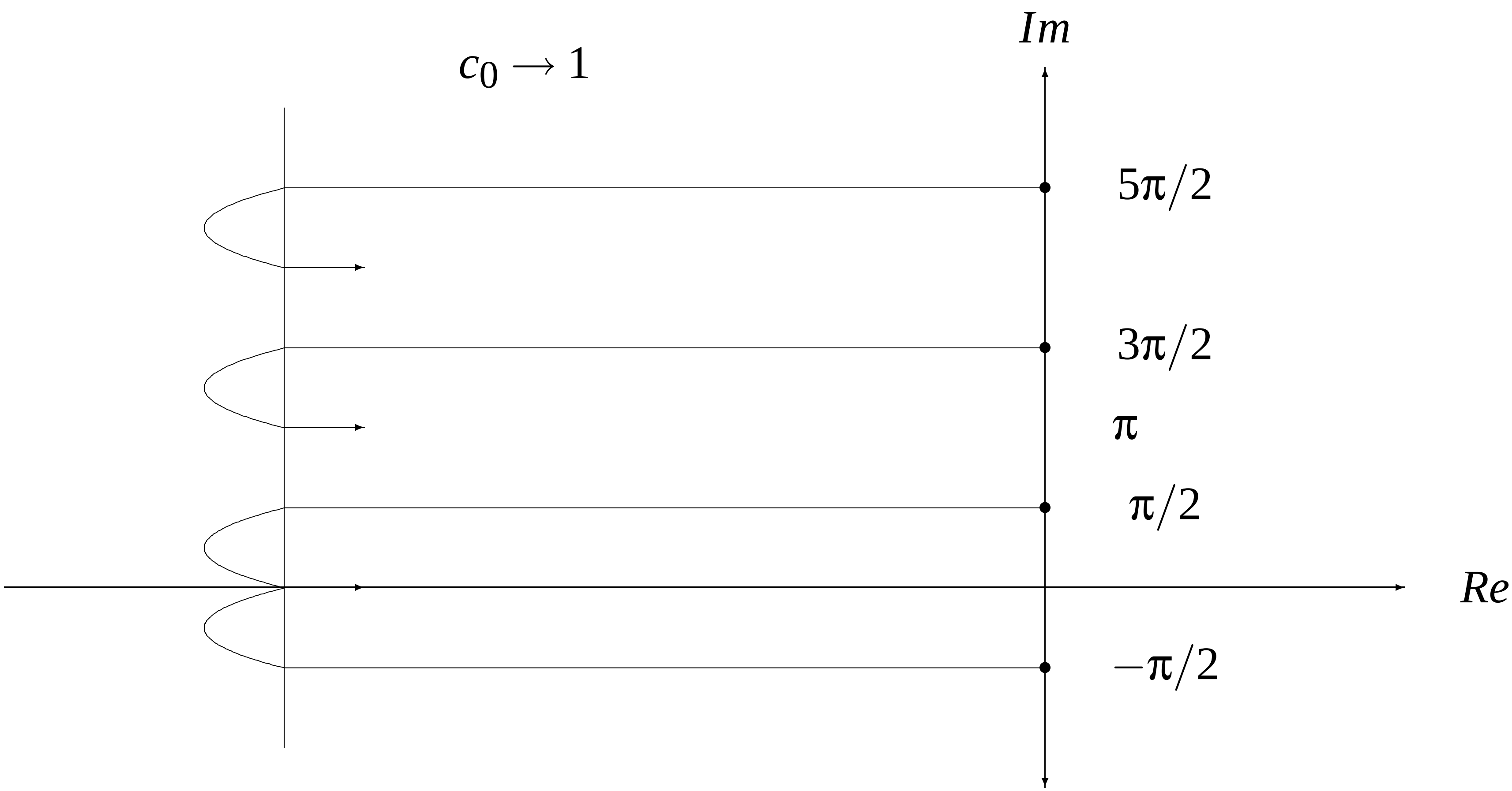
From the BC at $x = 0$ we get

$$\begin{aligned} \phi'(0) - c_0\sigma\phi(0) &= 0 \\ \sigma(1 + e^{2\sigma}) - c_0\sigma(1 - e^{2\sigma}) &= 0 \\ e^{2\sigma} &= -\frac{1 - c_0}{1 + c_0}. \end{aligned}$$

Solving for σ gives

$$\sigma = -\frac{1}{2}\ln\left|\frac{1 + c_0}{1 - c_0}\right| + j\pi \begin{cases} n + \frac{1}{2} & 0 \leq c_0 < 1 \\ n & c_0 > 1 \end{cases}$$

Eigenvalues at $-\infty$ for $c_0 = 1$ (solution $\rightarrow 0$ in finite time).



In real systems with (even the slightest) damping, the ideal c_0 is not unity. The dependence on c_0 is extremely sensitive around $c_0 = 1$.

The "boundary damper" feedback is very effective in adding damping to eigenvalues but it requires actuation on the free end $x = 0$, which is seldom feasible.

Backstepping: Actuation at the "Base"

Suppose we apply the "boundary damper" feedback using an active actuator at the base, while keeping the other end of the string/cable free:

$$\begin{aligned} u_{tt} &= u_{xx} \\ u_x(0) &= 0 \\ u_x(1) &= -c_1 u_t(1)\,, \quad \text{where } c_1 > 0. \end{aligned}$$

(The sign of the gain must change to accommodate the switch from one boundary to the other, which is equivalent to the reversal of the direction of the x axis.)

Due to the Neumann BC the system has one eigenvalue at the origin $\sigma = 0$.

As a result the system has any arbitrary constant $u(x) =$ const. as an equilibrium profile.

To deal with this multitude of arbitrary equilibriums a more sophisticated (backstepping) controller is needed at $x = 1$ if the boundary condition at $x = 0$ is to remain free.

Target system for the backstepping controller:

$$\begin{aligned} w_{tt} &= w_{xx} \\ w_x(0) &= c_0 w(0) \\ w_x(1) &= -c_1 w_t(1)\,, \quad \text{where } c_1 > 0. \end{aligned}$$

The BC $w_x(0) = c_0 w(0)$ doesn't use ∂_t, i.e., it is not of “damping” type but of “Robin” type.

The idea with the BC $w_x(0) = c_0 w(0)$ is to use large c_0 to make it behave like $w(0) \approx 0$.

Lyapunov function for target system:

$$V = \frac{1}{2}\left(\|w_x\|^2 + \|w_t\|^2 + c_0 w^2(0)\right) + \delta \int_0^1 (1+x) w_x(x) w_t(x) dx$$

Positive definiteness: With Poincare's and Young's inequalities, one can show that for sufficiently small δ, $\exists\ m_1, m_2 > 0$ s.t.

$$m_1 U \leqslant V \leqslant m_2 U\,, \quad \text{where} \quad U = \|w_x\|^2 + \|w_t\|^2 + w^2(0)$$

$$
\begin{aligned}
\dot{V} &= \int_0^1 w_x w_{tx} dx + \int_0^1 w_t w_{tt} dx + c_0 w(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{tt}) dx
\end{aligned}
$$

substituting target system

$$
\begin{aligned}
&= \int_0^1 w_x w_{tx} dx + \int_0^1 w_t w_{xx} dx + w_x(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{xx}) dx
\end{aligned}
$$

integration by parts

$$
\begin{aligned}
&= \int_0^1 w_x w_{tx} dx + w_t w_x|_0^1 - \int_0^1 w_t w_{xt} dx + w_x(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{xx}) dx
\end{aligned}
$$

canceling terms

$$
\begin{aligned}
&= \delta \left(\int_0^1 w_{xt} w_t dx + \int_0^1 w_x w_{xx} dx + \int_0^1 x w_{xt} w_t dx + \int_0^1 x w_x w_{xx} dx \right) \\
&\quad + w_t(1) w_x(1)
\end{aligned}
$$

Notice that $w_{xt}w_t dx = \frac{d}{dx}\frac{w_t^2}{2}$ and $w_x w_{xx} dx = \frac{d}{dx}\frac{w_x^2}{2}$ and use integration by parts on the latter two integrals involving an extra x term:

$$
\begin{aligned}
\dot{V} &= w_t(1)w_x(1) + \frac{\delta}{2}\left[(1+x)(w_x^2+w_t^2)\right]\Big|_0^1 - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right] \\
&= -c_1 w_t^2 + \delta(w_t^2(1) + w_x^2(1)) - \frac{\delta}{2}\left[w_x^2(0) + w_t^2(0)\right] - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right]
\end{aligned}
$$

$$
\dot{V} = -\left(c_1 - \delta(1+c_1^2)\right)w_t^2(1) - \frac{\delta}{2}\left(w_t^2(0) + c_0^2 w^2(0)\right) - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right]
$$

which is negative definite for $\delta < \frac{c_1}{1+c_1^2}$. One can further show that

$$
U(t) \leq M\mathrm{e}^{-t/M}U(0)
$$

for some possibly large M.

This exponential stability result legitimizes our “target system.”

Backstepping Design. The transformation

$$w(x) = u(x) + c_0 \int_0^x u(y)dy$$

and the boundary controller

$$u_x(1) = -c_1 u_t(1) - c_0 u(1) - c_1 c_0 \int_0^1 u_t(y)dy$$

transform $u_{tt} = u_{xx},\ u_x(0) = 0$ into $w_{tt} = w_{xx},\ w_x(0) = c_0 w(0),\ w_x(1) = -c_1 w_t(1)$.

<u>Homework:</u> Prove this result.

So, $k(x,y) = c_0$!

Gain selection guideline: c_0 large and c_1 around 1.

Term-by-term discussion: $-c_1 u_t(1) - c_0 u(1)$ is PD control; $-c_1 c_0 \int_0^1 u_t(y)dy$ is a spatially averaged velocity and is a backstepping "damping" term.

Dirichlet implementation:

$$u(1) = -\frac{1}{c_1 s + c_0}[u_x(1)] - \frac{s}{c_1 s + c_0}\left[\int_0^1 u(y)dy\right]$$

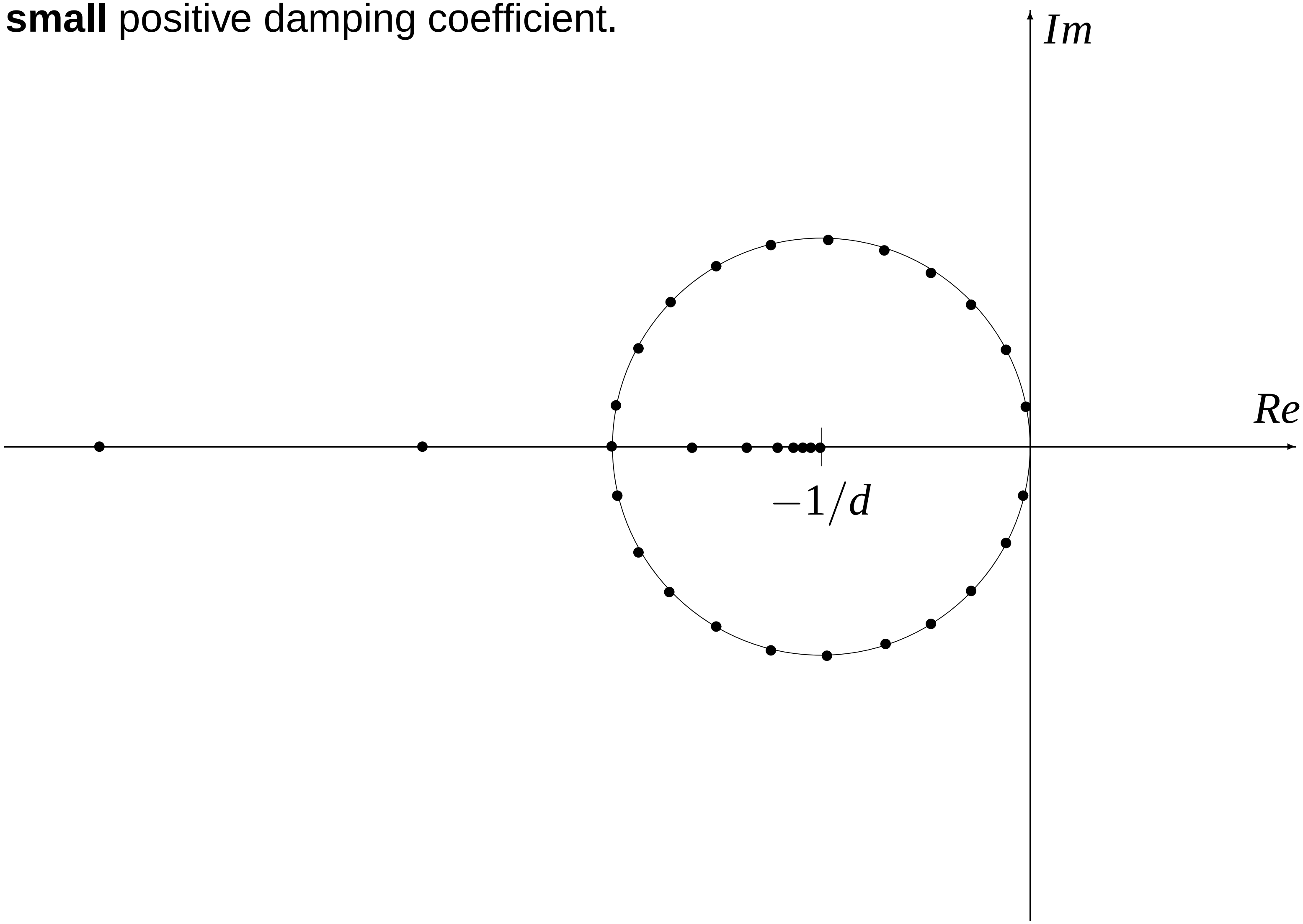
The corresponding *observer-based* feedback is

$$
\begin{aligned}
\hat{u}_{tt} &= \hat{u}_{xx} \\
\hat{u}_x(0) &= \tilde{c}_0(\hat{u}_t(0) - u_t(0)) \\
\hat{u}(1) &= u(1) \\
u_x(1) &= -c_0 u(1) - c_1\left(u_t(1) + c_0 \int_0^1 \hat{u}_t(y)\,dy\right).
\end{aligned}
$$

Wave Equation with Kelvin-Voigt Damping

$$\begin{aligned} u_{tt} &= u_{xx} + d u_{xxt} \\ u_x(0) &= 0 \qquad\qquad \text{(free end)} \end{aligned}$$

where d is a **small** positive damping coefficient.



Target system:

$$
\begin{aligned}
w_{tt} &= (1+d\partial_t)(w_{xx}-cw) \\
w_x(0) &= 0 \\
w(1) &= 0
\end{aligned}
$$

Damping in the PDE, no need for damping in the BCs.

Increasing c moves the eigenvalues along the circle in the negative real direction.

This adds damping (real part becomes more negative) but also increases the natural frequency (imaginary part grows higher). The c term increases stiffness.

Trade-off between settling time and overshoot.

Transformation and boundary controller

$$\begin{aligned} w(x) &= u(x) - \int_0^x k(x,y)u(y)dy \\ u(1) &= \int_0^1 k(1,y)u(y)dy \end{aligned}$$

with kernel

$$k(x,y) = -cx\frac{I_1\left(\sqrt{c(x^2-y^2)}\right)}{\sqrt{c(x^2-y^2)}}.$$

Same as for the reaction-diffusion equation $u_t = u_{xx} + \lambda u$ with BC $u_x(0) = 0$!

Remarks. The controller does not depend on d. For a fixed c, as $d \longrightarrow 0$, the controller's effect reduces to moving the eigenvalues up the $j\omega$-axis, which results in "jittery" response and uses a lot of control energy. However in systems with moderate d, this technique works great for adding more damping.

Flexible Beam PDEs

Miroslav Krstic, Antranik Siranosian, and Andrey Smyshlyaev

mini-course at UCSB, 2006

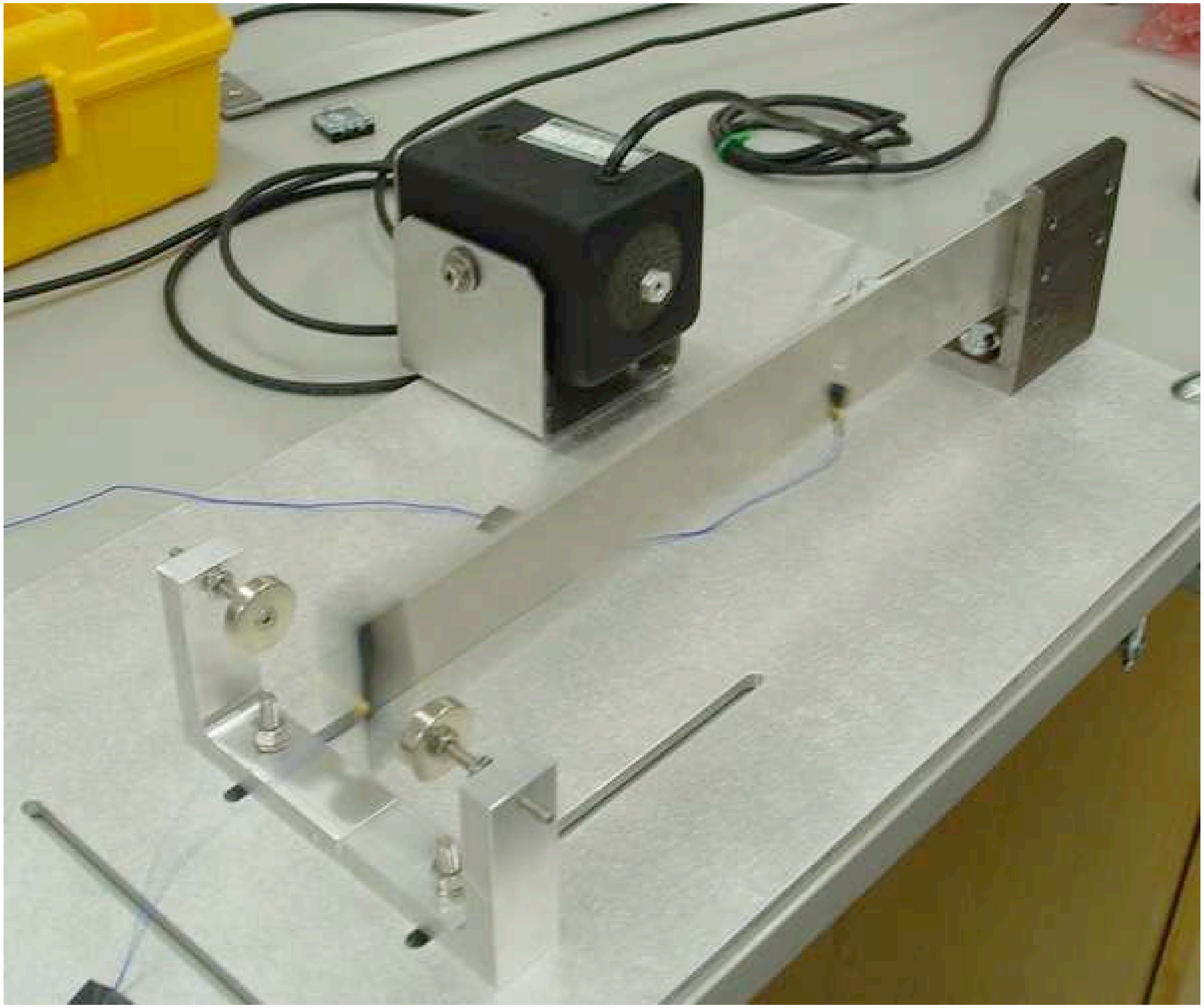
Euler-Bernoulli beam model:

$$\begin{aligned} u_{tt} + u_{xxxx} &= 0 \\ u_{xx}(0) &= u_{xxx}(0) = 0 \qquad \text{(free end condition)} \\ u(0) &= u_x(0) = 0 \qquad \text{(clamped end condition)} \end{aligned}$$

“Boundary damper” feedback at the tip with the base clamped:

$$u_{xx}(0) = c_0 u_{xt}(0), \qquad c_0 > 0.$$

Eigenvalues the same as the wave equation but with different vertical spacing.



Photograph of cantilevered beam testbed. (Shaker disconnected.)

Shear Beam Model—Backstepping Control from the Base

The shear beam model:

$$u_{tt} - \varepsilon u_{xxtt} + u_{xxxx} = 0,$$

where ε is inversely proportional to the "shear modulus."

(u_{xxtt} not a damping term. It reduces stiffness.)

Equivalent representation:

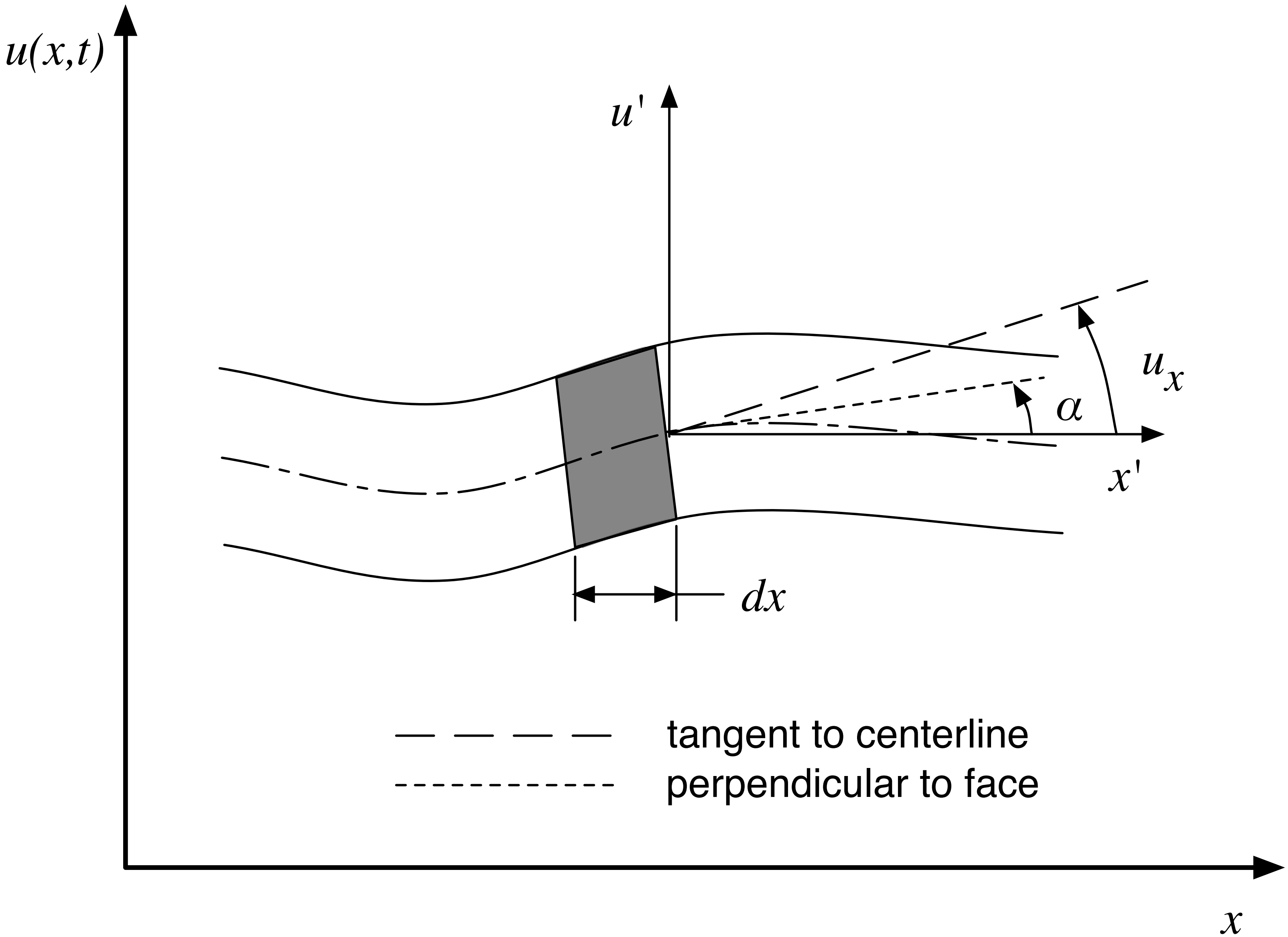
$$\begin{aligned} \varepsilon u_{tt} &= u_{xx} - \alpha_x \\ 0 &= \varepsilon \alpha_{xx} - \alpha + u_x, \end{aligned}$$

where $\alpha(x,t)$ is the deflection angle due to the bending.

The free end BC:

$$\begin{aligned} u_x(0) &= \alpha(0) \\ \alpha_x(0) &= 0 \end{aligned}$$

Differential element of a beam:



To get the shear beam model

$$u_{tt} - \varepsilon u_{xxtt} + u_{xxxx} = 0 \tag{1}$$

from

$$\begin{aligned} \varepsilon u_{tt} &= u_{xx} - \alpha_x & (2) \\ 0 &= \varepsilon \alpha_{xx} - \alpha + u_x & (3) \end{aligned}$$

the following steps are made:

a) (2)$_x +$ (3) $= (\star)$.

b) $(\star)_x = (\star\star)$.

c) $(\star\star) - \frac{1}{\varepsilon}$(2) $=$(1).

Remark. In the Timoshenko beam model the equation (3) also contains α_{tt}. The shear beam model is a singular perturbation of the Timoshenko model.

The spatial ODE for α is a TPBV problem. It can be solved via Laplace transform (w.r.t. x):

$$\alpha(x) = \cosh(bx)\alpha(0) + b\sinh(bx)u(0) - b^2 \int_0^x \cosh(b(x-y))u(y)dy$$

where $b = \frac{1}{\varepsilon}$.

The constant $\alpha(0)$ can be expressed in terms of $\alpha(1)$, which is a control input, by setting $x = 1$ in $\alpha(x)$:

$$\alpha(0) \quad = \quad \frac{1}{\cosh(b)} \left[\alpha(1) - b\sinh(b)u(0) + b^2 \int_0^1 \cosh(b(1-y))u(y)dy \right]$$

The term $\int_0^1 \cosh(b(1-y))u(y)dy$ is non-strict-feedback. To eliminate it, we choose

$$\alpha(1) = b\sinh(b)u(0) - b^2 \int_0^1 \cosh(b(1-y))u(y)dy$$

and make $\alpha(0) = 0$.

Then

$$\alpha(x) = b\sinh(bx)u(0) - b^2 \int_0^x \cosh(b(x-y))u(y)dy$$

is strict-feedback in $u(x)$.

Taking ∂_x of $\alpha(x)$ and substituing into the u-PDE yields

$$\varepsilon u_{tt} = u_{xx} + b^2 u - b^2 \cosh(bx) u(0) + b^3 \int_0^x \sinh(b(x-y)) u(y) dy,$$

which is a strict-feedback problem.

Target system:

$$\begin{aligned} \varepsilon w_{tt} &= w_{xx} \\ w_x(0) &= c_0 w(0) \\ w_x(1) &= -c_1 w_t(1) \end{aligned}$$

Same as for the wave/string equation.

Boundary control:

$$u_x(1) = k(1,1)u(1) + \int_0^1 k_x(1,y)u(y)dy - c_1 u_t(1) + c_1 \int_0^1 k(1,y)u_t(y)dy$$

Kernel PDE:

$$\begin{aligned} k_{xx} &= k_{yy} + b^2 k - b^3 \sinh(b(x-y)) + b^3 \int_x^y k(x,\xi)\sinh(b(\xi - y))d\xi \\ k(x,x) &= -\frac{b^2}{2}x - c_0 \\ k_y(x,0) &= b^2 \left(\int_0^x k(x,y)\cosh(by)dy - \cosh(bx) \right) \end{aligned}$$

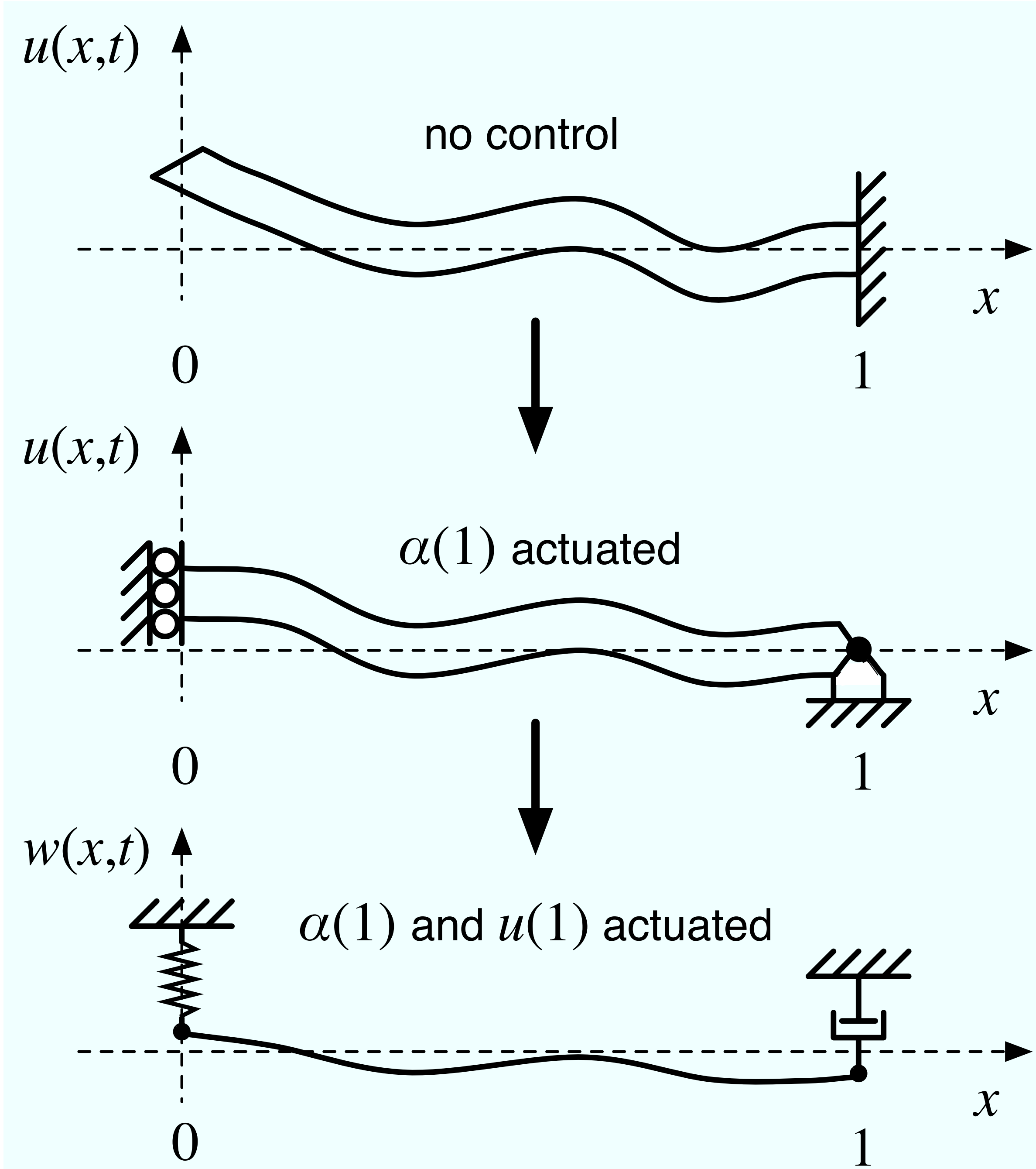
Can be implemented with Dirichlet actuation:

$$u(1) = -\frac{1}{c_1 s + c_0 + \frac{b^2}{2}}[\cdots]$$

Remark.

The controller makes a beam behave dynamically like a string. Is this a good idea?

It can be shown that for beams with high shear modulus the “target” string is *very short* and has a “damper” BC. So, it seems like a very good idea.



$u(x,t)$
no control
x
0
1
$u(x,t)$
$\alpha(1)$ actuated
x
0
1
$w(x,t)$
$\alpha(1)$ and $u(1)$ actuated
x
0
1

Schrodinger Equation

$$u_t = -ju_{xx}$$

Special case of Ginzburg-Landau equation.

Note: $u(x,t)$ is complex valued.

Its nonlinear version of interest in quantum control.

Its linear version equivalent to the Euler-Bernoulli beam model.

Think of $-j$ like the diffusion coefficient in parabolic PDEs.

If the uncontrolled BC is $u_x(0) = 0$, the boundary controller is

$$u(1) = \int_0^1 k(1,y)u(y)dy$$

Gain kernel:

$$\begin{aligned} k(1,y) &= \sqrt{\frac{c}{2\left(1-y^2\right)}}\left[\mathrm{bei}_1\left(\sqrt{c\left(1-y^2\right)}\right) - \mathrm{ber}_1\left(\sqrt{c\left(1-y^2\right)}\right)\right. \\ &\quad \left. + j\left(\mathrm{ber}_1\left(\sqrt{c\left(1-y^2\right)}\right) + \mathrm{bei}_1\left(\sqrt{c\left(1-y^2\right)}\right)\right)\right] \end{aligned}$$

where $\mathrm{ber}_1(\cdot)$ and $\mathrm{bei}_1(\cdot)$ are Kelvin functions.

Target system: $w_t = -jw_{xx} - cw$ (well damped).

Delay Systems and First Order Hyperbolic PDEs

Miroslav Krstic

mini-course at UCSB, 2006

First Order Hyperbolic PDEs

Traffic flow, chemical reactors, heat exchangers, delays.

The general first order hyperbolic PDE tractable by backstepping:

$$\begin{aligned} u_t &= u_x + g(x)u(0) + \int_0^x f(x,y)u(y)dy \\ u(1) &= \text{control}\,. \end{aligned}$$

Only one spatial derivative $\longrightarrow$ only one boundary condition.

For g or f positive and large $\longrightarrow$ open-loop unstable.

Transformation and boundary controller

$$\begin{aligned} w(x) &= u(x) - \int_0^x k(x,y)u(y)dy \\ u(1) &= \int_0^1 k(1,y)u(y)\,dy\,. \end{aligned}$$

Target system

$$\begin{aligned} w_t &= w_x \\ w(1) &= 0. \end{aligned}$$

Solution

$$w(x,t) = \begin{cases} w_0(t+x) & 0 \le t < 1 \\ 0 & t \ge 1, \end{cases}$$

where $w_0(x)$ is the initial condition. Pure delay—converges to zero in finite time.

Kernel PDE (well posed):

$$\begin{aligned} k_x + k_y &= \int_y^x k(x,\xi) f(\xi,y) d\xi - f(x,y) \\ k(x,0) &= \int_0^x k(x,y) g(y) dy - g(x). \end{aligned}$$

Example 1.

$$u_t = u_x + g\mathrm{e}^{bx}u(0)$$

Transformation/controller kernel

$$k(x,y) = -g\mathrm{e}^{(b+g)(x-y)}$$

Example 2.

$$u_t = u_x + \int_0^x f\mathrm{e}^{b(x-y)}u(y)\,dy$$

Transformation/controller kernel

$$k(x,y) = -f\mathrm{e}^{b(x-y)}y\frac{I_1\left(2\sqrt{fx(x-y)}\right)}{\sqrt{fx(x-y)}}$$

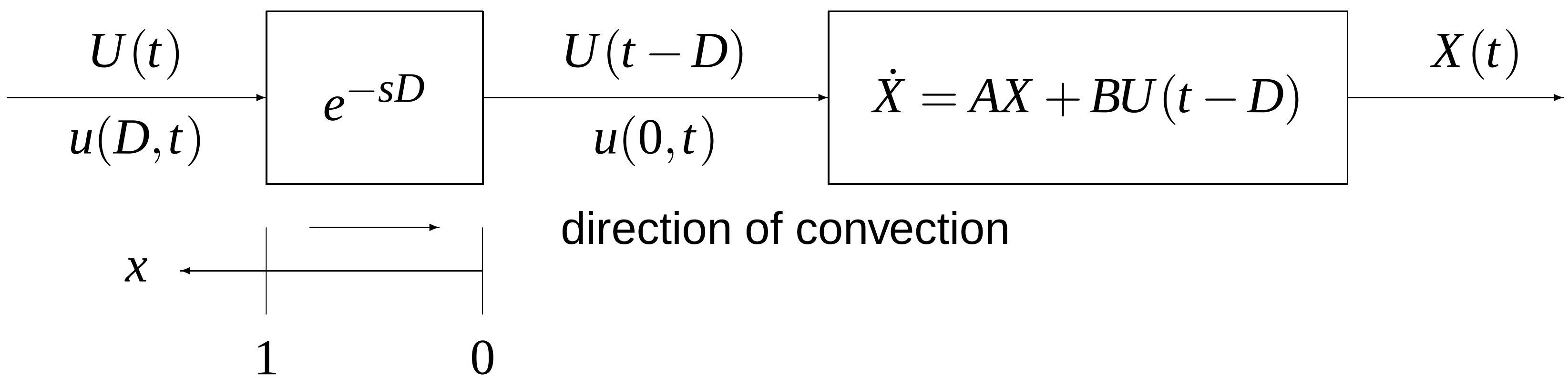
Systems with Delay

$$\dot{X} = AX + BU(t-D)$$

Assume: (A,B) controllable and matrix K found such that $A+BK$ is Hurwitz.

A hyperbolic PDE representation:

$$\begin{aligned} \dot{X} &= AX + Bu(0,t) \\ u_t &= u_x \\ u(D,t) &= U(t) \end{aligned}$$



Note that $u(x,t) = U(t+x-D)$.

Consider the backstepping transformation

$$w(x) = u(x) - \int_0^x q(x,y)u(y)dy - \gamma(x)^T X$$

and the target system

$$\begin{aligned}
\dot{X} &= (A+BK)X + Bw(0) \\
w_t &= w_x \\
w(D) &= 0\,.
\end{aligned}$$

Since w becomes zero in finite time, the w-system is exponentially stable.

As usual, let us calculate the time and spatial derivatives of the transformation:

$$\begin{aligned}
w_x &= u_x - q(x,x)u(x) - \int_0^x q_x(x,y)u(y)dy - \gamma'(x)^T X \\
w_t &= u_t - \int_0^x q(x,y)u_t(y)dy - \gamma(x)^T \left[AX + Bu(0)\right]
\end{aligned}$$

We get three conditions:

$$
\begin{aligned}
q_x + q_y &= 0 \\
q(x,0) &= \gamma(x)^T B \\
\gamma' &= A^T \gamma
\end{aligned}
$$

The first two conditions form a familiar first order hyperbolic PDE and the third one is a simple ODE.

To find the initial condition for the ODE, we set $x = 0$ in $w(x)$, which gives $w(0) = u(0) - \gamma(0)^T X$, and hence

$$
\dot{X} = AX + Bu(0) + B\left(K - \gamma(0)^T\right) X \, .
$$

We thus get $\gamma(0) = K^T$.

Therefore the ODE is

$$\begin{aligned} \gamma' &= A^T \gamma \\ \gamma(0) &= K^T \end{aligned}$$

The solution is

$$\gamma(x)^T = K e^{Ax}$$

The q-PDE is

$$\begin{aligned} q_x + q_y &= 0 \\ q(x,0) &= \gamma(x)^T B \end{aligned}$$

The solution is given explicitly:

$$q(x,y) = K e^{A(x-y)} B$$

This gives the control law:

$$u(D) = \int_0^D K e^{A(D-y)} B u(y) dy + K e^{AD} X$$

or

$$U(t) = K \left[e^{AD} X(t) + \int_{t-D}^{t} e^{A(t-\theta)} B U(\theta) d\theta \right]$$

Same controller as in Artstein (1982) but a better proof (complete Lyapunov function).

The equivalent of Smith Predictor for unstable systems.

Boundary Control of Channel Flow

Stabilization and Decoupling of the Orr-Sommerfeld/Squire equations

Jennie Cochran, Rafael Vazquez, Miroslav Krstic

mini-course at UCSB, 2006

Past Efforts

- LQR based on discretized Navier-Stokes — Bewley; Speyer-Kim
- Model reduction based interior control — Christofides, et al.
- Controllability of Navier-Stokes — Coron; Fursikov-Imanuvilov; Barbu
- Transient energy growth — Bamieh, Dahleh, Jovanovic

Past Efforts

Stabilization of channel flow in 2-D — Vazquez, Krstic

- for Navier-Stokes without discretization or model reduction
- for arbitrary Reynolds numbers
- closed loop linearized Navier-Stokes solved explicitly

Linearized Navier-Stokes Model

(around parabolic streamwise equilibrium flow $U^e = 4y(1-y)$)

$$
\begin{aligned}
u_t &= \frac{1}{Re}(u_{xx} + u_{zz} + u_{yy}) - U^e u_x - U^e_y V - p_x \\
W_t &= \frac{1}{Re}(W_{xx} + W_{zz} + W_{yy}) - U^e W_x - p_z \\
V_t &= \frac{1}{Re}(V_{xx} + V_{zz} + V_{yy}) - U^e V_x - p_y
\end{aligned}
$$

$$u_x + W_z + V_y = 0$$

uncontrolled wall $\quad u\big|_{y=0} = V\big|_{y=0} = W\big|_{y=0} = 0$

controlled wall $\quad u\big|_{y=1} = U_c \qquad W\big|_{y=1} = W_c \qquad V\big|_{y=1} = V_c$

Fourier Transform

System in wave space:

$$\alpha^2 = 4\pi^2(\mathbf{k_x}^2 + \mathbf{k_z}^2) \qquad \beta = 16\pi\mathbf{k_x}i$$

$$\begin{aligned}
u_t &= \frac{1}{Re}(-\alpha^2 u + u_{yy}) + \frac{\beta}{2}y(y-1)u + 4(2y-1)V - 2\pi i\mathbf{k_x}p \\
W_t &= \frac{1}{Re}(-\alpha^2 W + W_{yy}) + \frac{\beta}{2}y(y-1)W - 2\pi i\mathbf{k_z}p \\
V_t &= \frac{1}{Re}(-\alpha^2 V + V_{yy}) + \frac{\beta}{2}y(y-1)V - p_y
\end{aligned}$$

$$2\pi i\mathbf{k_x}u + 2\pi i\mathbf{k_z}W + V_y = 0$$

uncontrolled wall $\quad u\big|_{y=0} = V\big|_{y=0} = W\big|_{y=0} = 0$

controlled wall $\quad u\big|_{y=1} = U_c \qquad W\big|_{y=1} = W_c \qquad V\big|_{y=1} = V_c$

Velocity/Vorticity Transformation

$$2\pi i(k_x u + k_z W) + V_y = 0$$

$$\begin{aligned} Y &= V_y = -2\pi i(k_x u + k_z W) \\ \omega &= -2\pi i(k_z u - k_x W) \quad \text{vorticity} \end{aligned}$$

$$\begin{aligned} u &= \frac{i}{2\pi}\frac{k_x Y + k_z \omega}{k_x^2 + k_z^2} \\ W &= \frac{i}{2\pi}\frac{k_z Y - k_x \omega}{k_x^2 + k_z^2} \\ V &= \int_0^y Y(\eta)\,d\eta\,. \end{aligned}$$

Plant Model

$$
\begin{aligned}
Y_t &= \frac{1}{Re}(-\alpha^2 Y + Y_{yy}) - 8\pi i k_x y(1-y)Y - 8\pi i k_x(1-2y)V - \alpha^2 p \\
\omega_t &= \frac{1}{Re}(-\alpha^2 \omega + \omega_{yy}) - 8\pi i k_x y(1-y)\omega - 8\pi i k_z(1-2y)V
\end{aligned}
$$

Pressure

$$
\begin{aligned}
-\alpha^2 p + p_{yy} &= \beta(2y-1)V \\
p_y(0) &= \frac{1}{Re} V_{yy}(0) \\
p_y(1) &= \frac{1}{Re}(-\alpha^2 V_c + V_{yy}(1)) - (V_c)_t
\end{aligned}
$$

$$
\begin{aligned}
p = {} & \beta \int_0^y V(t,\eta)(2\eta-1)\sinh(\alpha(y-\eta))\,d\eta \\
& + \frac{\cosh(\alpha y)}{\sinh(\alpha)} \left\{ -\beta \int_0^1 V(t,\eta)(2\eta-1)\cosh(\alpha(1-\eta))\,d\eta \right. \\
& \left. + \frac{1}{Re}(-\alpha^2 V_c + V_{yy}(1)) - (V_c)_t \right\} \\
& - \frac{\cosh(\alpha(1-y))}{\sinh(\alpha)} \frac{1}{Re} V_{yy}(0)
\end{aligned}
$$

Pressure

$$
\begin{aligned}
p \;=\;& \beta \int_0^y V(t,\eta)(2\eta-1)\sinh(\alpha(y-\eta))\,d\eta \\
&+ \frac{\cosh(\alpha y)}{\sinh(\alpha)} \left\{ -\beta \int_0^1 V(t,\eta)(2\eta-1)\cosh(\alpha(1-\eta))\,d\eta \right. \\
&\left. + \frac{1}{Re}\left(-\alpha^2 V_c + V_{yy}(1)\right) - (V_c)_t \right\} \\
&- \frac{\cosh(\alpha(1-y))}{\sinh(\alpha)} \frac{1}{Re} V_{yy}(0)
\end{aligned}
$$

- The backstepping method can handle

$$\int_0^y f(y,\eta)u(\eta)\,d\eta$$

- The backstepping method cannot handle

$$\int_0^1 f(\eta)u(\eta)\,d\eta)$$

Pressure

$$
\begin{aligned}
p &= \beta \int_0^y V(t,\eta)(2\eta - 1)\sinh(\alpha(y-\eta))\,d\eta \\
&\quad + \frac{\cosh(\alpha y)}{\sinh(\alpha)} \left\{ -\beta \int_0^1 V(t,\eta)(2\eta-1)\cosh(\alpha(1-\eta))\,d\eta \right. \\
&\quad \left. + \frac{1}{Re}(-\alpha^2 V_c + V_{yy}(1)) - (V_c)_t \right\} \\
&\quad - \frac{\cosh(\alpha(1-y))}{\sinh(\alpha)} \frac{1}{Re} V_{yy}(0)
\end{aligned}
$$

Set V_c as follows

$$
\begin{aligned}
(V_c)_t &= \frac{1}{Re}\left(V_{yy}(1) - V_{yy}(0) - \alpha^2 V_c \right) \\
&\quad - \beta \int_0^1 V(t,\eta)(2\eta-1)\cosh(\alpha(1-\eta))\,d\eta
\end{aligned}
$$

- Full state feedback plus low pass filter (LPF)

Pressure

$$
\begin{aligned}
p &= \beta \int_0^y V(t,\eta)(2\eta-1)\sinh(\alpha(y-\eta))\,d\eta \\
&\quad + \frac{\cosh(\alpha y)}{\sinh(\alpha)} \frac{1}{Re} V_{yy}(0) \\
&\quad - \frac{\cosh(\alpha(1-y))}{\sinh(\alpha)} \frac{1}{Re} V_{yy}(0)
\end{aligned}
$$

Set V_c as follows

$$
\begin{aligned}
V_c &= \frac{1}{s+\frac{\alpha^2}{Re}} \left[\frac{V_{yy}(1)-V_{yy}(0)}{Re} \right. \\
&\quad \left. \times -\beta \int_0^1 V(s,\eta)(2\eta-1)\cosh(\alpha(1-\eta))\,d\eta \right]
\end{aligned}
$$

- Full state feedback plus low pass filter (LPF)

Plant Model

$$
\begin{aligned}
Y_t &= \frac{1}{Re}(-\alpha^2 Y + Y_{yy}) - 8\pi i k_x y(1-y)Y - 8\pi i k_x(1-2y)V - \alpha^2 p \\
\omega_t &= \frac{1}{Re}(-\alpha^2 \omega + \omega_{yy}) - 8\pi i k_x y(1-y)\omega - 8\pi i k_z(1-2y)V
\end{aligned}
$$

$$
\begin{aligned}
p &= \beta \int_0^y V(t,\eta)(2\eta - 1)\sinh(\alpha(y-\eta))d\eta \\
&\quad + \frac{1}{Re} V_{yy}(0) \frac{\cosh(\alpha y) - \cosh(\alpha(1-y))}{\sinh(\alpha)}
\end{aligned}
$$

$$
V_y = Y \qquad V = \int_0^y Y(\eta) d\eta
$$

Plant Model

$$
\begin{aligned}
Y_t \;=\;& \frac{1}{Re}(-\alpha^2 Y + Y_{yy}) - 8\pi i k_x y(1-y)Y - 8\pi i k_x(1-2y)\left(\int_0^y Y(\eta)d\eta\right) \\
& -\alpha^2\left\{\beta\int_0^y\left(\int_0^\eta Y(\sigma)d\sigma\right)(2\eta-1)\sinh(\alpha(y-\eta))d\eta\right. \\
& \left. + \frac{1}{Re}\left(Y_y(0)\right)\frac{\cosh(\alpha y)-\cosh(\alpha(1-y))}{\sinh(\alpha)}\right\}
\end{aligned}
$$

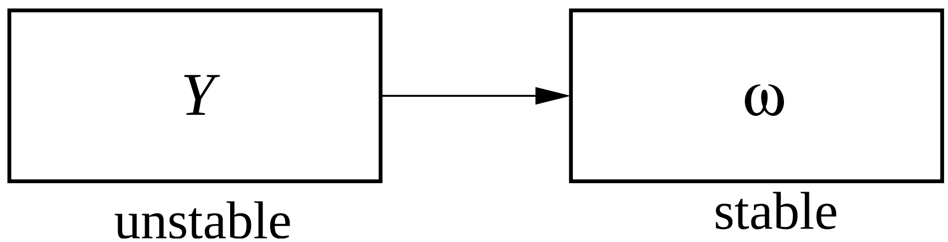
$$
\omega_t \;=\; \frac{1}{Re}(-\alpha^2\omega + \omega_{yy}) - 8\pi i k_x y(1-y)\omega - 8\pi i k_z(1-2y)\left(\int_0^y Y(\eta)d\eta\right)
$$

Plant Model

$$
\begin{aligned}
\varepsilon &= \frac{1}{Re} \\
\phi(y) &= \frac{\beta}{2} y(y-1) = 8\pi i k_x y(y-1) \\
f(y,\eta) &= 8i\Big\{\pi k_x(2y-1) - 4\pi\frac{k_x}{\alpha}\sinh(\alpha(y-\eta)) \\
&\quad -2\pi k_x(2\eta-1)\cosh(\alpha(y-\eta))\Big\} \\
g(y) &= \varepsilon\alpha^2 \frac{\cosh(\alpha(1-y)) - \cosh(\alpha y)}{\sinh(\alpha)} \\
h(y) &= -8\pi k_z i(2y-1)
\end{aligned}
$$

Plant Model

$$
\begin{aligned}
Y_t &= \varepsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y + g(y)Y_y(t,0) + \int_0^y f(y,\eta)Y(t,\eta)\,d\eta \\
\omega_t &= \varepsilon(-\alpha^2 \omega + \omega_{yy}) + \phi(y)\omega + h(y)\int_0^y Y(\eta)\,d\eta
\end{aligned}
$$



Two causes of turbulence:

- Linear: Y subsystem unstable for high Re

- Nonlinear: Y feeds ω, large overshoot, nonlinearities kick in, solution pulled into a strange attractor (the transient growth scenario—Farrell-Ioannou; Trefethen; Bamieh-Dahleh).

Backstepping

Plant:

$$
\begin{aligned}
Y_t &= \varepsilon(-\alpha^2 Y + Y_{yy}) + \phi(y)Y + g(y)Y_y(t,0) + \int_0^{y} f(y,\eta)Y(t,\eta)d\eta \\
\omega_t &= \varepsilon(-\alpha^2\omega + \omega_{yy}) + \phi(y)\omega + h(y)\int_0^{y} Y(\eta)d\eta
\end{aligned}
$$

$$
Y(t,0) = \omega(t,0) = 0 \qquad Y(t,1) = Y_c \qquad \omega(t,1) = \omega_c
$$

Transformation:

$$
\begin{aligned}
\Psi &= Y - \int_0^y K(k_x,k_z,y,\eta)Y(t,k_x,k_z,\eta)d\eta \\
\Omega &= \omega - \int_0^y \Gamma(k_x,k_z,y,\eta)Y(t,k_x,k_z,\eta)d\eta
\end{aligned}
$$

Target:

$$
\begin{aligned}
\Psi_t &= \varepsilon(-\alpha^2\Psi + \Psi_{yy}) + \phi(y)\Psi \\
\Omega_t &= \varepsilon(-\alpha^2\Omega + \Omega_{yy}) + \phi(y)\Omega
\end{aligned}
$$

$$
\begin{aligned}
\Psi(t,0) = \Psi(t,1) = 0 \\
\Omega(t,0) = \Omega(t,1) = 0
\end{aligned}
$$

Kernel PDEs

$$
\begin{aligned}
\varepsilon K_{yy} &= \varepsilon K_{\eta\eta} - f(y,\eta) + (\phi(\eta) - \phi(y))K + \int_{\eta}^{y} K(y,\xi) f(\xi,\eta)\, d\xi \\
\varepsilon K(y,0) &= \int_0^y K(y,\eta) g(\eta)\, d\eta - g(y) \\
\varepsilon K(y,y) &= -g(0)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon \Gamma_{yy} &= \varepsilon \Gamma_{\eta\eta} - h(y) + (\phi(\eta) - \phi(y))\Gamma + \int_{\eta}^{y} \Gamma(y,\sigma) f(\sigma,\eta)\, d\sigma \\
\varepsilon \Gamma(y,0) &= \int_0^y \Gamma(y,\eta) g(\eta)\, d\eta \\
\Gamma(y,y) &= 0
\end{aligned}
$$

Controllers

$$Y(1) = \int_0^1 K(1,\eta)Y(\eta)d\eta$$
$$\omega(1) = \int_0^1 \Gamma(1,\eta)Y(\eta)d\eta$$

$$U_c = -\frac{2\pi i}{\alpha^2}\Big(k_x Y(t,1) + k_z \omega(t,1)\Big)$$
$$W_c = -\frac{2\pi i}{\alpha^2}\Big(k_z Y(y,1) - k_x \omega(t,1)\Big)$$

$$U_c = \int_0^1 \frac{4\pi^2}{\alpha^2}\left(k_x K(1,\eta) + k_z \Gamma(1,\eta)\right)\left(k_x u(t,\eta) + k_z W(t,\eta)\right)d\eta$$
$$W_c = \int_0^1 \frac{4\pi^2}{\alpha^2}\left(k_z K(1,\eta) - k_x \Gamma(1,\eta)\right)\left(k_x u(t,\eta) + k_z W(t,\eta)\right)d\eta$$

Result in Wavespace

The controllers

$$U_c = \int_0^1 \frac{4\pi^2}{\alpha^2}\left(k_x K(1,\eta) + k_z \Gamma(1,\eta)\right)\left(k_x u(t,\eta) + k_z W(t,\eta)\right) d\eta$$

$$W_c = \int_0^1 \frac{4\pi^2}{\alpha^2}\left(k_z K(1,\eta) - k_x \Gamma(1,\eta)\right)\left(k_x u(t,\eta) + k_z W(t,\eta)\right) d\eta$$

$$(V_c)_t = \frac{1}{Re}\left(V_{yy}(1) - V_{yy}(0) - \alpha^2 V_c\right) - \beta \int_0^1 V(t,\eta)(2\eta - 1)\cosh(\alpha(1-\eta))d\eta$$

stabilize

$$\begin{aligned} u_t &= \frac{1}{Re}(-\alpha^2 u + u_{yy}) + \frac{\beta}{2} y(y-1)u + 4(2y-1)V - 2\pi i k_x p \\ W_t &= \frac{1}{Re}(-\alpha^2 W + W_{yy}) + \frac{\beta}{2} y(y-1)W - 2\pi i k_z p \\ V_t &= \frac{1}{Re}(-\alpha^2 V + V_{yy}) + \frac{\beta}{2} y(y-1)V - p_y \end{aligned}$$

$$2\pi i(k_x u + k_z W) + V_y = 0$$

at zero.

Result in Physical Space

- Use Parseval's theorem - energy in wavespace equals energy in physical space
- Actuate limited set of wavenumbers
- For high wavenumbers: system is stable
- For small wavenumbers: explicitly solve for a Taylor series expansion of the kernel pdes
- For the case $k_x = 0$: derive analytical solutions to the kernel pdes

Special Case $k_x = 0$

- averaged streamwise velocity
- transient growth is the largest — Jovanovic, Bamieh; Bewley; Schmid, Henningson
- analytical solutions to the kernel pdes for K and Γ

$$
\begin{aligned}
\kappa &= \alpha\Big|_{k_x=0} = 2\pi k_z \\
K(y,\eta) &= \left(\frac{\kappa^2}{\bar{g}(0)} - \bar{g}(0)\right)\mathrm{e}^{\bar{g}(0)(y-\eta)} - \frac{\kappa^2}{\bar{g}(0)} \\
\Gamma(y,\eta) &= \frac{\kappa i}{\varepsilon}\Big\{\eta(y-\eta)(3y-\eta-2) \\
&\quad + A_0 + A_1(y-\eta) + A_2(y-\eta)^2 + A_3(y-\eta)^3 \\
&\quad + B_0(y-\eta)\cosh(\kappa(y-\eta)) \\
&\quad + C_0\cosh(\kappa(y-\eta)) + C_1\sinh(\kappa(y-\eta)) \\
&\quad + E_0\mathrm{e}^{\bar{g}(0)(y-\eta)}\Big\}
\end{aligned}
$$

Explicit K and Γ

$$
\begin{aligned}
\bar{g}(y) &= \kappa \tanh(\tfrac{\kappa}{2}) \cosh(\kappa\, y) - \kappa \sinh(\kappa\, y) \\
\bar{h}(y) &= -\frac{4\kappa}{\varepsilon} i(2y-1)
\end{aligned}
$$

$$
\begin{aligned}
K_{yy} &= K_{\eta\eta} \\
K(y,y) &= -\bar{g}(0) \\
K(y,0) &= -\left(\bar{g}(y) - \int_0^y K(y,\eta)\bar{g}(\eta)\,d\eta\right)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{yy} &= \Gamma_{\eta\eta} - \bar{h}(y) \\
\Gamma(y,y) &= 0 \\
\Gamma(y,0) &= \int_0^y \Gamma(y,\eta)\bar{g}(\eta)\,d\eta
\end{aligned}
$$

Explicit Solution to K

$$
\begin{aligned}
F(s) &= -\bar{g}(s) + \int_0^s \bar{g}(s-\sigma)F(\sigma)d\sigma \\
\bar{g}'' &= \kappa^2 \bar{g}
\end{aligned}
$$

$$
\begin{aligned}
F'' - \bar{g}(0)F' &= 0 \\
F(0) &= -\bar{g}(0) \\
F'(0) &= \kappa^2 - \bar{g}^2(0)
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= A_1 e^{\bar{g}(0)s} + A_2 \\
A_1 + A_2 &= -\bar{g}(0) \\
\bar{g}(0)A_1 &= \kappa^2 - \bar{g}^2(0)
\end{aligned}
$$

$$
K(y,\eta) = \left(\frac{\kappa^2}{\bar{g}(0)} - \bar{g}(0)\right) e^{\bar{g}(0)(y-\eta)} - \frac{\kappa^2}{\bar{g}(0)}
$$

Explicit Solution to Γ

$$\begin{aligned}
\Gamma_{yy} &= \Gamma_{\eta\eta} - \bar{h}(y) \\
\Gamma(y,y) &= 0 \\
\Gamma(y,0) &= \int_0^y \Gamma(y,\eta)\bar{g}(\eta)d\eta
\end{aligned}$$

$$\begin{aligned}
\xi &= y+\eta \\
\zeta &= y-\eta \\
\Gamma(y,\eta) &= \Gamma\left(\frac{\xi+\zeta}{2},\frac{\xi-\zeta}{2}\right) = \Sigma(\xi,\zeta)
\end{aligned}$$

$$\begin{aligned}
\Sigma_{\xi\zeta} &= -\frac{1}{4}\bar{h}\left(\frac{\xi+\zeta}{2}\right) \\
\Sigma(\xi,0) &= 0 \\
\Sigma(\xi,\xi) &= \int_0^\xi \Sigma(\xi+\tau,\xi-\tau)\bar{g}(\tau)d\tau
\end{aligned}$$

Explicit Solution to Γ

$$
\begin{aligned}
\Sigma_{\xi\zeta} &= -\frac{1}{4}\bar{h}\left(\frac{\xi+\zeta}{2}\right) \\
\Sigma(\xi,0) &= 0 \\
\Sigma(\xi,\xi) &= \int_0^{\xi} \Sigma(\xi+\tau,\xi-\tau)\bar{g}(\tau)d\tau \\
\Sigma(\xi,\zeta) &= \frac{\kappa i}{2\varepsilon}(\xi-\zeta)\zeta(2\zeta+\xi-2)+\int_0^{\zeta}\Sigma(\zeta+\tau,\zeta-\tau)\bar{g}(\tau)d\tau \\
\Sigma(\xi,\zeta) &= \frac{\kappa i}{2\varepsilon}(\xi-\zeta)\zeta(2\zeta+\xi-2)+\frac{1}{\varepsilon}\Delta(\zeta) \\
\Delta(\zeta) &= \Upsilon(\zeta)-\int_0^{\zeta}\Delta(\sigma)\bar{g}(\zeta-\sigma)d\sigma \\
\Upsilon(\zeta) &= \kappa i\int_0^{\zeta}\sigma(\zeta-\sigma)(3\zeta-\sigma-2)\bar{g}(\sigma)d\sigma
\end{aligned}
$$

Explicit Solution to Γ

$$
\begin{aligned}
\Delta &= \Upsilon(\zeta) - \int_0^{\zeta} \Delta(\sigma)\bar{g}(\zeta-\sigma)\,d\sigma \\
\bar{g}'' &= \kappa^2 \bar{g} \\
\Delta'' - \bar{g}(0)\Delta' &= \Upsilon'' - \kappa^2 \Upsilon \\
\Delta(0) &= 0 \\
\Delta'(0) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Delta = {} & A_0 + A_1\zeta + A_2\zeta^2 + A_3\zeta^3 \\
& + B_0\zeta\cosh(\kappa\zeta) + B_1\zeta\sinh(\kappa\zeta) \\
& + C_0\cosh(\kappa\zeta) + C_1\sinh(\kappa\zeta) \\
& + E_0 e^{\bar{g}(0)\zeta}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma(y,\eta) = {} & \frac{\kappa i}{\varepsilon}\Big\{\eta(y-\eta)(3y-\eta-2) \\
& + \frac{4\,\bar{g}(0)^2 - 2\,\bar{g}(0)^3 + \bar{g}(0)\,\alpha^2 - \alpha^2}{\bar{g}(0)^3\alpha^2} + 2\,\frac{\bar{g}(0)-1}{\bar{g}(0)^2}(y-\eta) - \frac{1+\bar{g}(0)}{\bar{g}(0)}(y-\eta)^2 + (y-\eta)^3 \\
& - 8\frac{1}{\alpha^2}(y-\eta)\cosh(\kappa(y-\eta)) \\
& + 4\,\frac{\sinh(\alpha)+\alpha}{\alpha^3}\cosh(\kappa(y-\eta)) + 4\,\frac{\cosh(\alpha)+3}{\alpha^3}\sinh(\kappa(y-\eta)) \\
& - 2\,\frac{5\,\bar{g}(0)^2 - \bar{g}(0)^3 + \bar{g}(0)\,\alpha^2 - \alpha^2}{\bar{g}(0)^3\left(\alpha^2 - \bar{g}(0)^2\right)} e^{\bar{g}(0)(y-\eta)}\Big\}
\end{aligned}
$$

Observer Design for Magnetohydrodynamic and Navier-Stokes Flows

Rafael Vazquez and Miroslav Krstic

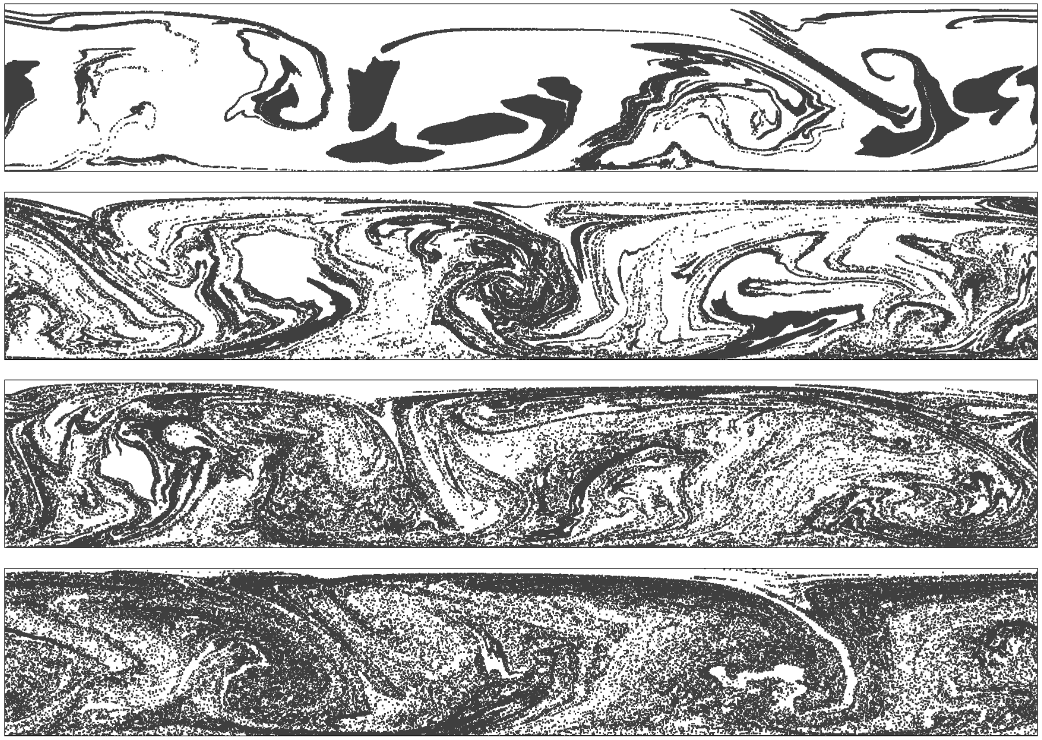
mini-course at UCSB, 2006

Magnetohydrodynamic (MHD) Flows

- Fluids: Navier-Stokes equations

- Conducting fluids: Navier-Stokes equations + Maxwells equations = MHD equations

Give rise to **very complex phenomena**: Chaos , Turbulence

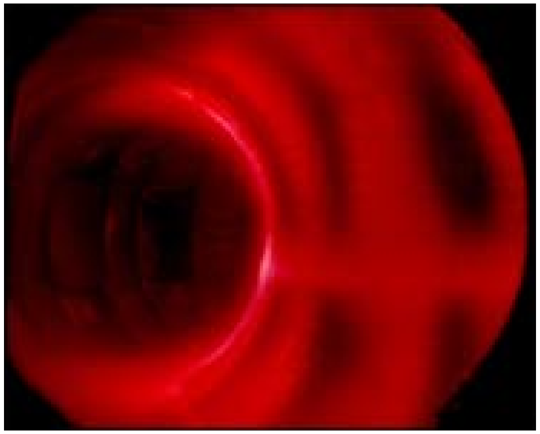


MHD: Observer Design

Problem statement: Design an observer to estimate the value of the velocity/pressure/electromagnetic field at any point.

- Measurements: Pressure, skin friction, current in the boundaries.
- The observer must work for arbitrary Reynolds number.
- The design must be done for the continuum MHD equations. No discretization is allowed.
- The nonlinearities should be considered, to allow estimation in turbulent regimes of the flow.

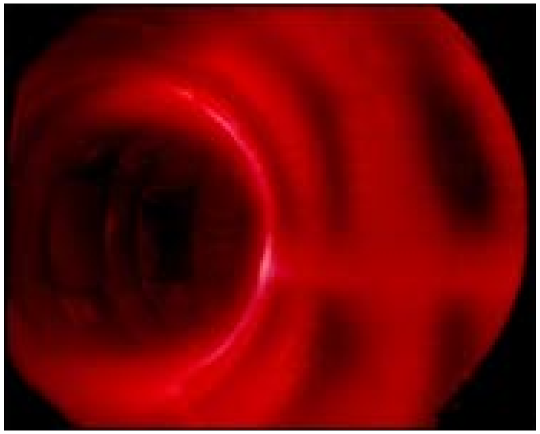
Example Applications



Electrically-Conducting-Fluid Cooling Systems

Use of liquid metals or electrically conducting liquid salts for cooling of computing devices, plasmas in fusion reactors.

Example Applications



Electrically-Conducting-Fluid Cooling Systems

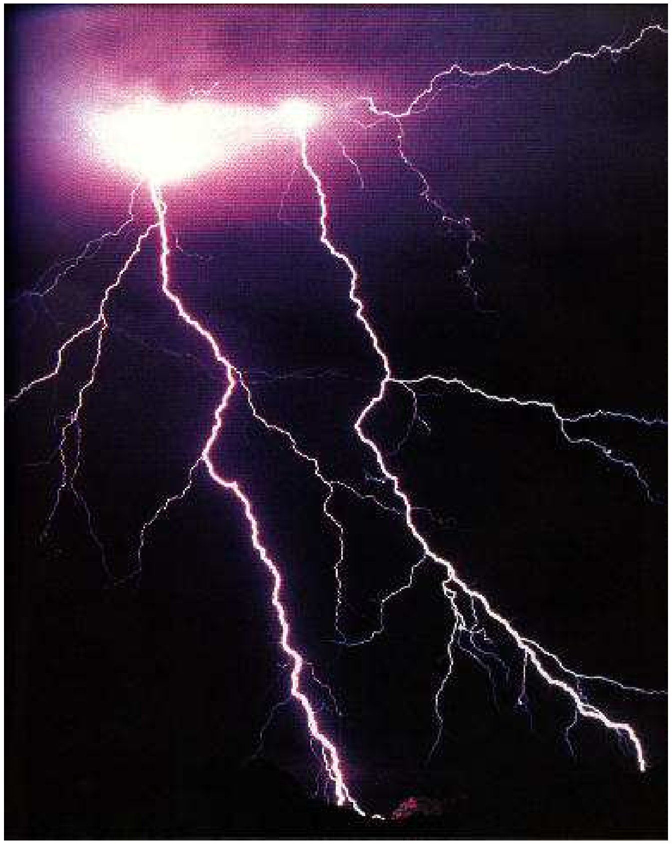
Use of liquid metals or electrically conducting liquid salts for cooling of computing devices, plasmas in fusion reactors.

Feedback control can

- Enhance mixing and heat transfer
- Reduce pumping power

Control designs usually full-state. A state estimator is needed.

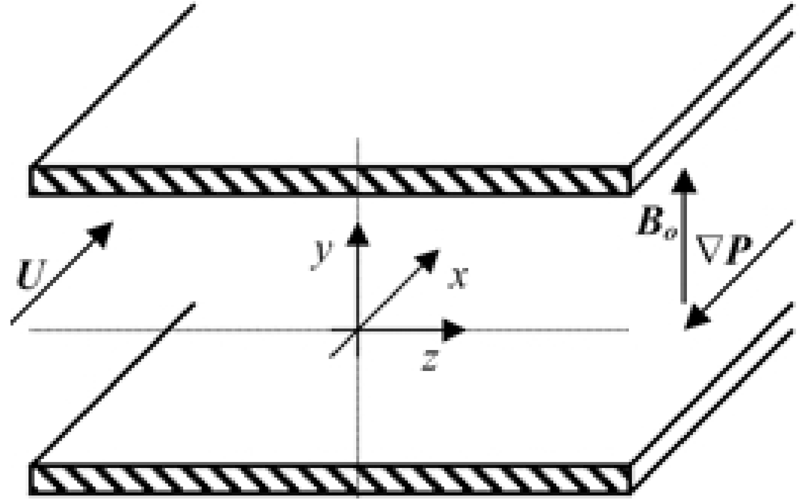
Example Applications



Model-based weather forecasting

The absence of effective state estimators modeling turbulent flows is one of the key obstacles in model-based weather forecasting.

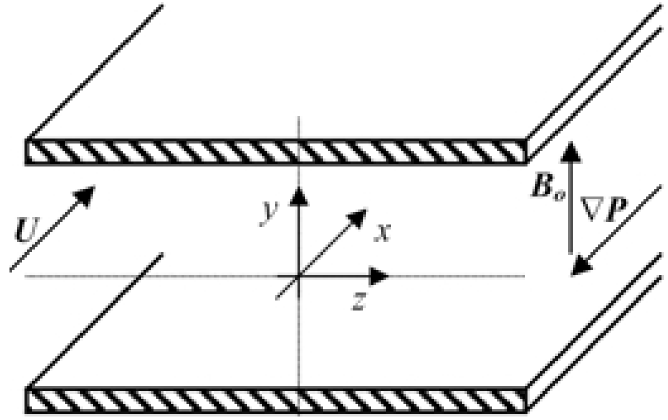
The Hartmann Flow



- Conducting fluid moving between paralell plates
- Imposed pressure gradient in streamwise direction
- Imposed magnetic field normal to the walls
- Benchmark model for turbulent MHD flow

Remark: If fluid is nonconducting (or zero magnetic field) — Navier-Stokes channel flow.

The Hartmann Flow — Inductionless Approximation



Parameters (mechanical and electrical):

L: Distance between plates B_0: Imposed magnetic field ν: fluid viscosity

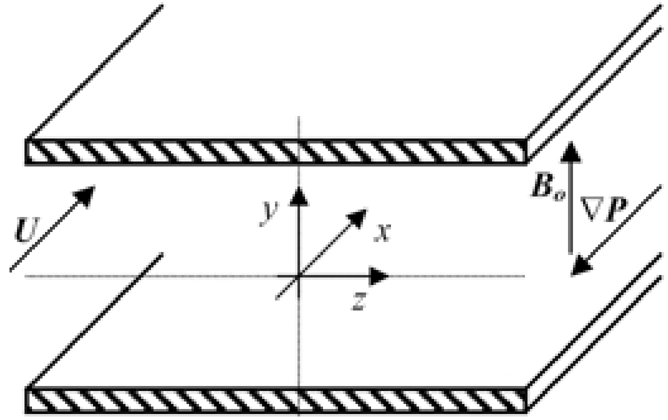
ρ: fluid density σ: fluid conductivity U_0: reference velocity (maximum velocity)

Nondimensional numbers:

Reynolds number: $Re = \frac{U_0 L}{\nu}$ Magnetic Reynolds number: $Re_M = \nu\rho\sigma U_0 L$

Stuart number: $N = \frac{\sigma L B_0^2}{\rho U_0}$ Hartmann number: $H = \sqrt{ReN} = B_0 L \sqrt{\frac{\sigma}{\rho\nu}}$.

The Hartmann Flow — Inductionless Approximation



Parameters:

L: Distance between plates B_0: Imposed magnetic field ν: fluid viscosity

ρ: fluid density σ: fluid conductivity U_0: reference velocity (maximum velocity)

Nondimensional numbers:

Reynolds number: $Re = \frac{U_0 L}{\nu}$ Magnetic Reynolds number: $Re_M = \nu\rho\sigma U_0 L$

Stuart number: $N = \frac{\sigma L B_0^2}{\rho U_0}$ Hartmann number: $H = \sqrt{ReN} = B_0 L \sqrt{\frac{\sigma}{\rho\nu}}$.

Inductionless approximation: Assume $Re_M \ll 1$. Then electromagnetic dynamics can be neglected (more precisely: apply Singular Perturbation theory to magnetic induction equation, which is obtained from Maxwell's equations).

The Hartmann Flow — Nondimensional Equations

Velocity field (U,V,W) equations (Navier-Stokes)

$$\begin{aligned} U_t &= \frac{\triangle U}{Re} - UU_x - VU_y - WU_z - P_x + N\phi_z - NU \\ V_t &= \frac{\triangle V}{Re} - UV_x - VV_y - WV_z - P_y \\ W_t &= \frac{\triangle W}{Re} - UW_x - VW_y - WW_z - P_z - N\phi_x - NW \end{aligned}$$

Electric potential ϕ equation

$$\triangle\phi = U_z - W_x$$

Continuity equation (incompressibility condition)

$$U_x + V_y + W_z = 0$$

Velocity boundary conditions (no slip, no penetration)

$$U(t,x,0,z) = U(t,x,1,z) = W(t,x,0,z) = W(t,x,1,z) = 0, \quad V(t,x,0,z) = V(t,x,1,z) = 0$$

Electric potential boundary conditions (perfectly conducting wall)

$$\phi(t,x,0,z) = \phi(t,x,1,z) = 0$$

The Hartmann Flow — Nondimensional Equations

Nondimensional current (vector field):

$$\begin{aligned} j^x(t,x,y,z) &= -\phi_x - W \\ j^y(t,x,y,z) &= -\phi_y \\ j^z(t,x,y,z) &= -\phi_z + U \end{aligned}$$

Remark: if $N = 0$ (nonconducting fluid or zero magnetic field), recover the Navier-Stokes equations (3D Channel Flow)

$$\begin{aligned} U_t &= \frac{\triangle U}{Re} - UU_x - VU_y - WU_z - P_x \\ V_t &= \frac{\triangle V}{Re} - UV_x - VV_y - WV_z - P_y \\ W_t &= \frac{\triangle W}{Re} - UW_x - VW_y - WW_z - P_z \end{aligned}$$

$$U_x + V_y + W_z = 0$$

$$U(t,x,0,z) = U(t,x,1,z) = V(t,x,0,z) = V(t,x,1,z) = W(t,x,0,z) = W(t,x,1,z) = 0$$

The Hartmann Flow — Equilibrium Profile

Look for steady-state solutions: *assume* only one nonzero nondimensional velocity component, $U^e(y)$.

$$0 = \frac{U^e_{yy}(y)}{Re} - P^e_x - NU^e(y)$$

Solution:

$$\begin{aligned}
U^e(y) &= \frac{\sinh(H(1-y)) - \sinh H + \sinh(Hy)}{2\sinh H/2 - \sinh H} \\
V^e &= W^e = \phi^e = 0 \\
P^e &= \frac{N\sinh H}{2\sinh H/2 - \sinh H}x \\
j^{xe} &= j^{ye} = 0 \\
j^{ze} &= U^e(y)
\end{aligned}$$

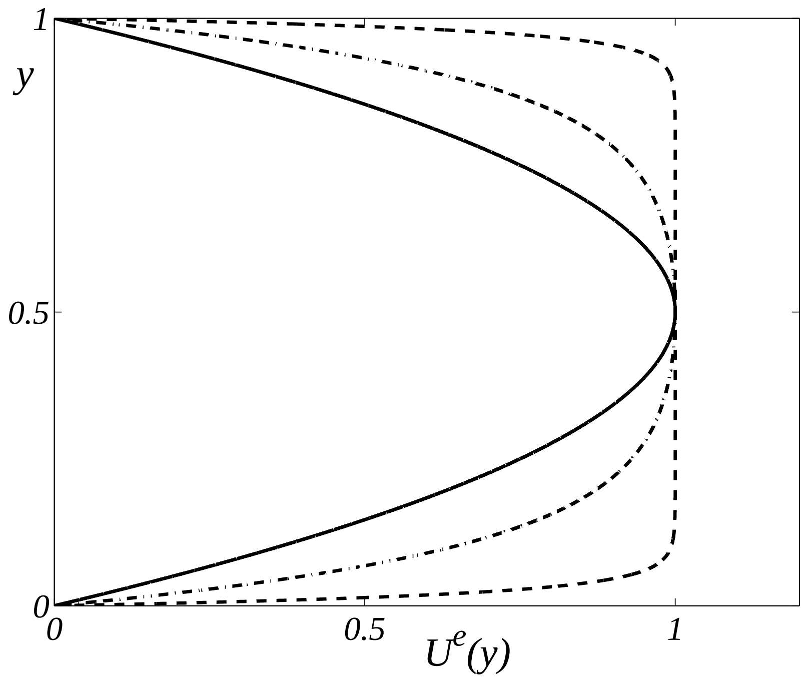
Remark: If fluid is nonconducting (or zero magnetic field) then $N = H = 0$. Taking limit

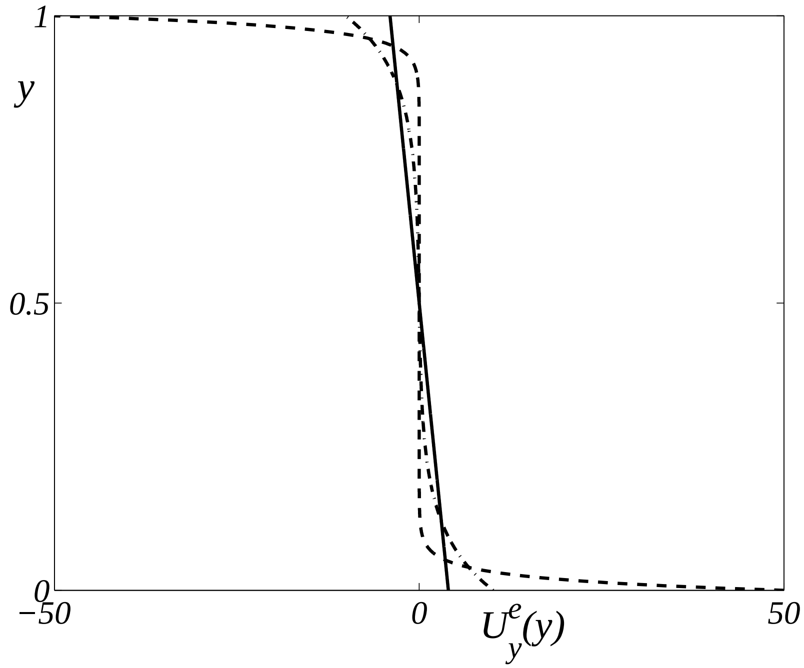
$$\begin{aligned}
U^e(y) &= 4y(1-y) \\
V^e &= W^e = \phi^e = j^{xe} = j^{ye} = j^{ze} = 0 \\
P^e &= -\frac{8}{Re}x
\end{aligned}$$

(Poiseuille profile)

The Hartmann Flow — Equilibrium Profile

Nondimensional equilibrium velocity and velocity gradient:





- Large magnetic fields “flatten” the equilibrium velocity profile

- Equilibrium velocity gradient (skin friction) at the walls approximately proportional to H (magnetic field intensity)

Observer

Copy of the plant + Output Injection (Keeping nonlinearities!)

$$
\begin{aligned}
\hat{U}_t &= \frac{\triangle \hat{U}}{Re} - \hat{U}\hat{U}_x - \hat{V}\hat{U}_y - \hat{W}\hat{U}_z - \hat{P}_x + N\hat{\phi}_z - N\hat{U} - Q^U \\
\hat{V}_t &= \frac{\triangle \hat{V}}{Re} - \hat{U}\hat{V}_x - \hat{V}\hat{V}_y - \hat{W}\hat{V}_z - \hat{P}_y - Q^V \\
\hat{W}_t &= \frac{\triangle \hat{W}}{Re} - \hat{U}\hat{W}_x - \hat{V}\hat{W}_y - \hat{W}\hat{W}_z - \hat{P}_z - N\hat{\phi}_x - N\hat{W} - Q^W
\end{aligned}
$$

$$\triangle \hat{\phi} = \hat{U}_z - \hat{W}_x$$

$$\hat{U}_x + \hat{V}_y + \hat{W}_z = 0$$

Boundary conditions:

$$
\begin{aligned}
\hat{U}(t,x,0,z) &= \hat{W}(t,x,0,z) = \hat{V}(t,x,0,z) = \hat{U}(t,x,1,z) = \hat{W}(t,x,1,z) = \hat{V}(t,x,1,z) = 0 \\
\hat{\phi}(t,x,0,z) &= \hat{\phi}(t,x,1,z) = 0
\end{aligned}
$$

Output Injection

Output injection terms: Gain times measurement

$$\begin{pmatrix} Q^U \\ Q^V \\ Q^W \end{pmatrix} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{L}(x-\xi, y, z-\zeta) \begin{pmatrix} P(\xi,0,\zeta) - \hat{P}(\xi,0,\zeta) \\ U_y(\xi,0,\zeta) - \hat{V}_y(\xi,0,\zeta) \\ W_y(\xi,0,\zeta) - \hat{W}_y(\xi,0,\zeta) \\ \phi_y(\xi,0,\zeta) - \hat{\phi}_y(\xi,0,\zeta) \end{pmatrix} d\xi d\zeta$$

where $\mathbf{L}$ is a matrix of (physical space) output injection gains defined

$$\mathbf{L}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi(k_x,y,k_z)\mathbf{R}(k_x,y,k_z)\mathrm{e}^{2\pi i(k_x x + k_z z)} dk_z dk_x$$

Truncating function:

$$\chi(k_x,k_z) = \begin{cases} 1, & k_x^2 + k_z^2 \leq M^2 \\ 0, & \text{otherwise.} \end{cases}$$

where $M = \frac{1}{2\pi}\sqrt{\frac{(H+4)Re}{2}}$ is a design parameter.
$\mathbf{R}$ is a matrix of (Fourier space) output injection gains:

$$\mathbf{R} = \begin{pmatrix} R^{UP} & R^{UU} & R^{UW} & R^{U\phi} \\ R^{VP} & R^{VU} & R^{VW} & R^{V\phi} \\ R^{WP} & R^{WU} & R^{WW} & R^{W\phi} \end{pmatrix}$$

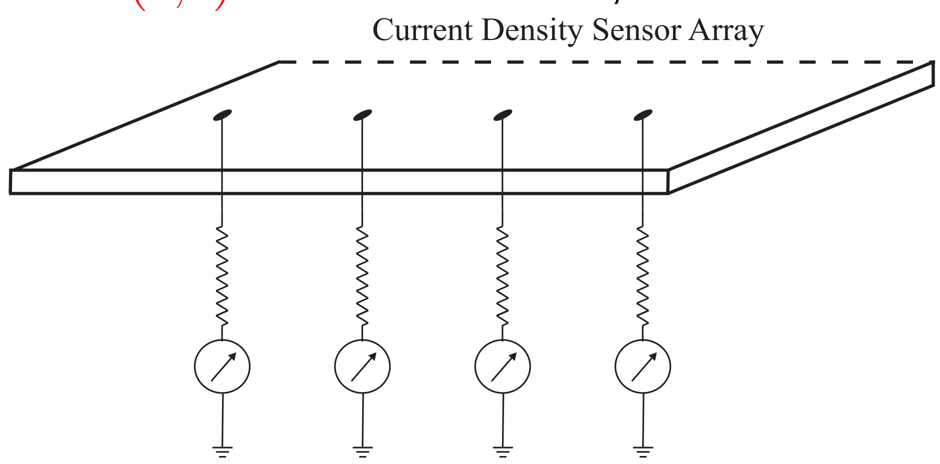
Measurements

- Measurements done on lower wall only

- Mechanical measurements: need pressure $P(x,0,z)$ and $U_y(x,0,z)$, $W_y(x,0,z)$ (proportional to skin friction)

- Electrical measurement: need $\phi_y(x,0,z)$, proportional to current $j^y(x,0,z)$ traversing normal to the wall

Remark: Need to measure (x,z)-valued functions , so distributed sensors must be used



Output Injection Gains

Entries of matrix **R**

$$
\begin{aligned}
R^{UP} &= 2\pi k_x i \cosh(\alpha y) \\
R^{VP} &= \alpha \sinh(\alpha y) \\
R^{WP} &= 2\pi k_z i \cosh(\alpha y) \\
R^{UU} &= \frac{4\pi^2 k_x^2}{\alpha Re} \sinh(\alpha y) + \Pi_1(k_x, y, k_z) \\
R^{VU} &= 2\pi i (k_x + k_z) \frac{1 - \cosh(\alpha y)}{Re} - 2\pi i \int_0^y (k_x \Pi_1(k_x, \eta, k_z) + k_z \Pi_2(k_x, \eta, k_z))\, d\eta \\
R^{WU} &= \frac{4\pi^2 k_x k_z}{\alpha Re} \sinh(\alpha y) + \Pi_2(k_x, y, k_z) \\
R^{UW} &= \frac{4\pi^2 k_x k_z}{\alpha Re} \sinh(\alpha y) + \Pi_3(k_x, y, k_z) \\
R^{VW} &= 2\pi i (k_x + k_z) \frac{1 - \cosh(\alpha y)}{Re} - 2\pi i \int_0^y (k_x \Pi_3(k_x, \eta, k_z) + k_z \Pi_4(k_x, \eta, k_z))\, d\eta \\
R^{WW} &= \frac{4\pi^2 k_z^2}{\alpha Re} \sinh(\alpha y) + \Pi_4(k_x, y, k_z)
\end{aligned}
$$

Output Injection Gains

Entries of matrix $\mathbf{R}$ (Continued):

$$\begin{aligned} R^{U\phi} &= -N\frac{2\pi k_z i}{\alpha}\sinh(\alpha y) \\ R^{V\phi} &= 0 \\ R^{W\phi} &= N\frac{2\pi k_x i}{\alpha}\sinh(\alpha y) \end{aligned}$$

where $\alpha^2 = 4\pi^2(k_x^2 + k_z^2)$, and

$$\begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} l(k_x, y, 0, k_z) \\ 0 \\ \theta_1(k_x, y, 0, k_z) \\ \theta_2(k_x, y, 0, k_z) \end{pmatrix}.$$

The matrix $\mathbf{A}$ is

$$\mathbf{A} = -\frac{4\pi^2}{\alpha^2} \begin{pmatrix} k_x^2 & k_x k_z & k_x k_z & k_z^2 \\ k_x k_z & k_z^2 & -k_x^2 & -k_x k_z \\ k_x k_z & -k_x^2 & k_z^2 & -k_x k_z \\ k_z^2 & -k_x k_z & -k_x k_z & k_x^2 \end{pmatrix},$$

Remark: $\det(\mathbf{A}) = -1$, so its inverse is well-defined .

Output Injection Gains

The functions $l(k_x, y, \eta, k_z)$, $\theta_1(k_x, y, \eta, k_z)$, and $\theta_2(k_x, y, \eta, k_z)$ are the solution of

$$\begin{aligned}
\varepsilon l_{\eta\eta} &= \varepsilon l_{yy} - (\beta(y) - \beta(\eta))\, l - f + \int_\eta^y f(y,\xi) l(\xi,\eta) d\xi \\
\varepsilon \theta_{1\eta\eta} &= \varepsilon \theta_{1yy} - (\beta(y) - \beta(\eta))\, \theta_1(y,\eta) - h_1 + h_1 \int_\eta^y l(\xi,\eta) d\xi + \int_\eta^y h_2(y,\xi) \theta_1(\xi,\eta) d\xi \\
\varepsilon \theta_{2\eta\eta} &= \varepsilon \theta_{2yy} - (\beta(y) - \beta(\eta))\, \theta_2 - h_2 + \int_\eta^y h_2(y,\xi) \theta_2(\xi,\eta) d\xi
\end{aligned}$$

These are hyperbolic partial integro-differential equations in the domain $\mathcal{T} = \{(y,\eta) : 0 \leq y \leq 1, 0 \leq \eta \leq y\}$, with bounday conditions

$$\begin{aligned}
l(k_x, y, y, k_z) &= \theta_1(k_x, y, y, k_z) = \theta_2(k_x, y, y, k_z) = 0, \\
l(k_x, 1, \eta, k_z) &= \theta_1(k_x, 1, \eta, k_z) = \theta_2(k_x, 1, \eta, k_z) = 0
\end{aligned}$$

Backstepping theory (previous talks) guarantees existence, uniqueness and regularity of solutions for above equations!

Output Injection Gains

Functions appearing in previous slides

$$
\begin{aligned}
\varepsilon &= \frac{1}{Re} \\
\beta &= 2\pi i k_x U^e \\
f &= 4\pi i k_x \left\{ \frac{U_y^e}{2} + \int_\eta^y U_y^e(\sigma) \frac{\sinh(\alpha(y-\sigma))}{\alpha} d\sigma \right\} + N\alpha \sinh(\alpha(y-\sigma)) \\
h_1 &= 2\pi i k_z U_y^e \\
h_2 &= -N\alpha \sinh(\alpha(y-\eta))
\end{aligned}
$$

Main Result: Observer Convergence

Define error variables $\tilde{U} = U - \hat{U}, \tilde{V} = V - \hat{V}, \tilde{W} = W - \hat{W}, \tilde{P} = P - \hat{P}, \tilde{\phi} = \phi - \hat{\phi}$ and fluctuation variable $u = U - U^e$.

Consider the observer error variables. There exists positive constants C_1 and C_2 such that, if the L^2 norms of the initial conditions for $\tilde{U}$, $\tilde{V}$, and $\tilde{W}$ are less than C_1, i.e.,

$$\int_{-\infty}^{\infty}\int_0^1\int_{-\infty}^{\infty}\left(\tilde{U}^2+\tilde{V}^2+\tilde{W}^2\right)(0,x,y,z)dxdydz < C_1,$$

and if the turbulent kinetic energy of u, V and W (defined as the L^2 norm of the fluctuation with respect to the equilibrium profile) is less than C_2 for all time, i.e., $\forall t \geq 0$,

$$\int_{-\infty}^{\infty}\int_0^1\int_{-\infty}^{\infty}\left(u^2+V^2+W^2\right)(t,x,y,z)dxdydz < C_2,$$

then the L^2 norms of $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ converge to zero :

$$\lim_{t\to\infty}\int_{-\infty}^{\infty}\int_0^1\int_{-\infty}^{\infty}\left(\tilde{U}^2+\tilde{V}^2+\tilde{W}^2\right)(t,x,y,z)dxdydz = 0.$$

Summary of Observer Features

- Observer structure: Copy of plant + Output Injection
- Nonlinear terms are kept
- Only measurements in the lower wall needed
- Computations of output injection gains requires solving of linear PIDEs. Effective symbolic or numerical schemes available. No Ricatti equations.
- All other gains explicit.
- Observer theoretical convergence guaranteed when initial estimates close enough to real estimates and state near equilibrium

Observer Design and Convergence Proof

Main tools:

- Fourier transform (wave number space)
- Separate analysis for large and small wave numbers
- Solution of TPBVP for pressure and electric potential (in terms of strict-feedback integrals and measurements of states)
- Transformation of variables (U, V, W) to (Y, ω)
- Backstepping observer design for unstable (small) wave numbers
- Lyapunov analysis for stable (large) wave numbers

Observer Design and Convergence Proof

Observer error equations:

$$
\begin{aligned}
\tilde{U}_t &= \frac{\triangle \tilde{U}}{Re} - U^e(y)\tilde{U}_x + \mathcal{N}^U(\tilde{U},\tilde{V},\tilde{W},u,V,W) - U^e_y(y)\tilde{V} - \tilde{P}_x + N\tilde{\phi}_z - N\tilde{U} + Q^U \\
\tilde{V}_t &= \frac{\triangle \tilde{V}}{Re} - U^e(y)\tilde{V}_x + \mathcal{N}^V(\tilde{U},\tilde{V},\tilde{W},u,V,W) - \tilde{P}_y + Q^V \\
\tilde{W}_t &= \frac{\triangle \tilde{W}}{Re} - U^e(y)\tilde{W}_x + \mathcal{N}^W(\tilde{U},\tilde{V},\tilde{W},u,V,W) - \tilde{P}_z - N\tilde{\phi}_x - N\tilde{W} + Q^W
\end{aligned}
$$

where we have introduced

$$
\begin{aligned}
\mathcal{N}^U &= \tilde{U}\tilde{U}_x - u\tilde{U}_x - \tilde{U}u_x + \tilde{V}\tilde{U}_y - V\tilde{U}_y - \tilde{V}u_y + \tilde{W}\tilde{U}_z - W\tilde{U}_z - \tilde{W}u_z, \\
\mathcal{N}^V &= \tilde{U}\tilde{V}_x - u\tilde{V}_x - \tilde{U}V_x + \tilde{V}\tilde{V}_y - V\tilde{V}_y - \tilde{V}V_y + \tilde{W}\tilde{V}_z - W\tilde{V}_z - \tilde{W}V_z, \\
\mathcal{N}^W &= \tilde{U}\tilde{W}_x - u\tilde{W}_x - \tilde{U}W_x + \tilde{V}\tilde{W}_y - V\tilde{W}_y - \tilde{V}W_y + \tilde{W}\tilde{W}_z - W\tilde{W}_z - \tilde{W}W_z,
\end{aligned}
$$

Assuming:

- $(\hat{U},\hat{V},\hat{W})$ close to actual state (U,V,W) (i.e., observer error state close to zero)
- The fluctuation (u,V,W) around the equilibrium state close to zero

Then, $\mathcal{N}_U(\tilde{U},\tilde{V},\tilde{W},u,V,W)$, $\mathcal{N}_V(\tilde{U},\tilde{V},\tilde{W},u,V,W)$ and $\mathcal{N}_W(\tilde{U},\tilde{V},\tilde{W},u,V,W)$ are small and dominated by linear terms in the equations. They can be neglected.

Observer Design and Convergence Proof

Observer error linearized equations

$$
\begin{aligned}
\tilde{U}_t &= \frac{\triangle \tilde{U}}{Re} - U^e(y)\tilde{U}_x - U^e_y(y)\tilde{V} - \tilde{P}_x + N\tilde{\phi}_z - N\tilde{U} + Q^U \\
\tilde{V}_t &= \frac{\triangle \tilde{V}}{Re} - U^e(y)\tilde{V}_x - \tilde{P}_y + Q^V \\
\tilde{W}_t &= \frac{\triangle \tilde{W}}{Re} - U^e(y)\tilde{W}_x - \tilde{P}_z - N\tilde{\phi}_x - N\tilde{W} + Q^W .
\end{aligned}
$$

Since the plant is linear and spatially invariant, we use a Fourier transform in the x and z coordinates (the spatially invariant directions) defined as

$$
\begin{aligned}
f(k_x, y, k_z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z) \mathrm{e}^{-2\pi i(k_x x + k_z z)} dz dx, \\
f(x, y, z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x, y, k_z) \mathrm{e}^{2\pi i(k_x x + k_z z)} dk_z dk_x.
\end{aligned}
$$

Watch out! We use the same symbol f for both Fourier and physical space functions.

k_x and k_z are known as the “wave numbers” in hydrodynamics.

Observer Design and Convergence Proof

Observer error equations in Fourier space

$$
\begin{aligned}
\tilde{U}_t &= \frac{-\alpha^2\tilde{U}+\tilde{U}_{yy}}{Re}-\beta\tilde{U}-U^e_y\tilde{V}-2\pi k_x i\tilde{P}+2\pi k_z iN\tilde{\phi}-N\tilde{U}\\
&\quad+\chi(k_x,k_z)\left\{R^{UP}P_0+R^{UU}U_{y0}+R^{UW}W_{y0}+R^{U\phi}\phi_{y0}\right\}\\
\tilde{V}_t &= \frac{-\alpha^2\tilde{V}+\tilde{V}_{yy}}{Re}-\beta\tilde{V}-\tilde{P}_y\\
&\quad+\chi(k_x,k_z)\left\{R^{VP}P_0+R^{VU}U_{y0}+R^{VW}W_{y0}+R^{V\phi}\phi_{y0}\right\}\\
\tilde{W}_t &= \frac{-\alpha^2\tilde{W}+W_{yy}}{Re}-\beta\tilde{W}-2\pi k_z i\tilde{P}-2\pi k_x iN\tilde{\phi}-N\tilde{W}\\
&\quad+\chi(k_x,k_z)\left\{R^{WP}P_0+R^{WU}U_{y0}+R^{WW}W_{y0}+R^{W\phi}\phi_{y0}\right\}
\end{aligned}
$$

$$2\pi i k_x\tilde{U}+\tilde{V}_y+2\pi k_z\tilde{W}=0$$

$$-\alpha^2\tilde{\phi}+\hat{\phi}_{yy}=2\pi i\left(k_z\tilde{U}-k_x\tilde{W}\right)$$

Measurements: $P_0=\tilde{P}(k_x,0,k_z)$, $U_{y0}=\tilde{U}_y(k_x,0,k_z)$, $W_{y0}=\tilde{W}_y(k_x,0,k_z)$, $\phi_{y0}=\tilde{\phi}_y(k_x,0,k_z)$.

Observer Design and Convergence Proof

The system is uncoupled for each wave number. Therefore different wave numbers can be studied independently.

- Small wave numbers: $k_x^2 + k_z^2 \leq M^2$ (*observed* wave number range). Since $\chi = 1$ there is output injection.

- Large wave numbers: $k_x^2 + k_z^2 > M^2$ (*unobserved* wave number range). Since $\chi = 0$ there is no output injection.

Remark: If stability for all wave numbers is established, then stability in physical space follows. The number M is a design parameter that ensures stability for the unobserved wave number range.

Observed Wave Numbers

Consider $k_x^2 + k_z^2 \leq M^2$. Then $\chi = 1$, and there is output injection.

Eliminating the pressure and electric potential: From plant equations and continuity equations, an equation for pressure error can be derived

$$-\alpha^2 \tilde{P} + \tilde{P}_{yy} = -4\pi k_x i U_y^e(y) \tilde{V} + N V_y$$

It is a Poisson equation.

Solving in terms of pressure in the lower wall:

$$\begin{aligned}\tilde{P} \;&=\; -\frac{4\pi k_x i}{\alpha} \int_0^y U_y^e(\eta) \sinh\left(\alpha(y-\eta)\right) \tilde{V}(k_x,\eta,k_z) d\eta + N \int_0^y \frac{\sinh\left(\alpha(y-\eta)\right)}{\alpha} \tilde{V}_y(k_x,\eta,k_z) d\eta \\ &\quad + \cosh(\alpha y) P_0 + \frac{\sinh(\alpha y)}{\alpha} \tilde{P}_y(k_x,0,k_z)\end{aligned}$$

Since $R^{VP}(k_x,0,k_z) = R^{VU}(k_x,0,k_z) = R^{VW}(k_x,0,k_z) = R^{V\phi}(k_x,0,k_z) = 0$, therefore evaluating the V equation at $y=0$ one finds that

$$\tilde{P}_y(k_x,0,k_z) = \frac{\tilde{V}_{yy}(k_x,0,k_z)}{Re} = -2\pi i \frac{k_x U_{y0} + k_z \tilde{W}_{y0}}{Re}$$

Observed Wave Numbers

Then the pressure can be expressed in terms of a (strict-feedback) integral of the state $\tilde{V}$ and measurements

$$\tilde{P} = -\frac{4\pi k_x i}{\alpha}\int_0^y U_y^e(\eta)\sinh(\alpha(y-\eta))\,\tilde{V}(k_x,\eta,k_z)d\eta + N\int_0^y \frac{\sinh(\alpha(y-\eta))}{\alpha}\tilde{V}_y(k_x,\eta,k_z)d\eta$$
$$+\cosh(\alpha y)P_0 - 2\pi i\frac{\sinh(\alpha y)}{Re\alpha}\left(k_x U_{y0} + k_z W_{y0}\right)$$

Equation for the potential

$$-\alpha^2\tilde{\phi} + \tilde{\phi}_{yy} = 2\pi i\left(k_z\tilde{U} - k_x\tilde{W}\right)$$

Solution in terms of strict-feedback integral and measurement

$$\tilde{\phi} = \frac{2\pi i}{\alpha}\int_0^y \sinh(\alpha(y-\eta))\left(k_z\tilde{U}(k_x,\eta,k_z) - k_x\tilde{W}(k_x,\eta,k_z)\right)d\eta + \frac{\sinh(\alpha y)}{\alpha}\phi_{y0}$$

Observed Wave Numbers

Substituting P and ϕ, most output injection terms cancel away, leaving

$$
\begin{aligned}
\tilde{U}_t &= \frac{-\alpha^2\tilde{U}+\tilde{U}_{yy}}{Re} - \beta\tilde{U} - U_y^e(y)\tilde{V} - N\tilde{U} - \frac{8\pi k_x^2}{\alpha}\int_0^y U_y^e(\eta)\sinh(\alpha(y-\eta))\,\tilde{V}(k_x,\eta,k_z)d\eta \\
&\quad -2\pi i k_x N\int_0^y \frac{\sinh(\alpha(y-\eta))}{\alpha}\tilde{V}_y(k_x,\eta,k_z)d\eta \\
&\quad +\Pi_1 U_{y0} + \Pi_3 W_{y0} - \frac{4\pi^2 k_z N}{\alpha}\int_0^y \sinh(\alpha(y-\eta))\left(k_z\tilde{U}(k_x,\eta,k_z) - k_x\tilde{W}(k_x,\eta,k_z)\right)d\eta \\
\tilde{W}_t &= \frac{-\alpha^2\tilde{W}+W_{yy}}{Re} - \beta\tilde{W} - N\tilde{W} + \Pi_2 U_{y0} + \Pi_4 W_{y0} \\
&\quad -\frac{8\pi k_x k_z}{\alpha}\int_0^y U_y^e(\eta)\sinh(\alpha(y-\eta))\,\tilde{V}(k_x,\eta,k_z)d\eta \\
&\quad -2\pi i k_z N\int_0^y \frac{\sinh(\alpha(y-\eta))}{\alpha}\tilde{V}_y(k_x,\eta,k_z)d\eta \\
&\quad +\frac{4\pi^2 k_x N}{\alpha}\int_0^y \sinh(\alpha(y-\eta))\left(k_z\tilde{U}(k_x,\eta,k_z) - k_x\tilde{W}(k_x,\eta,k_z)\right)d\eta
\end{aligned}
$$

Remark: There is no equation for $\tilde{V}$, since from continuity equation and $\tilde{V}(k_x,0,k_z)=0$:

$$
\tilde{V} = -2\pi i\int_0^y \left(k_x\tilde{U}(k_x,\eta,k_z) + k_z\tilde{W}(k_x,\eta,k_z)\right)d\eta
$$

Observed Wave Numbers

Apply the following change of variables

$$
\begin{aligned}
Y &= 2\pi i\left(k_x\tilde{U}+k_z\tilde{W}\right)\\
\omega &= 2\pi i\left(k_z\tilde{U}-k_x\tilde{W}\right)
\end{aligned}
$$

and inverse

$$
\begin{aligned}
U &= \frac{2\pi i}{\alpha^2}(k_xY+k_z\omega)\\
W &= \frac{2\pi i}{\alpha^2}(k_zY-k_x\omega)
\end{aligned}
$$

The plant in (Y,ω) variables is

$$
\begin{aligned}
Y_t &= \frac{-\alpha^2Y+Y_{yy}}{Re}-\beta Y-NY+l(k_x,y,0,k_z)Y_{y0}+\int_0^y f(k_x,y,\eta,k_z)Y(k_x,\eta,k_z)d\eta\\
\omega_t &= \frac{-\alpha^2\omega+\omega_{yy}}{Re}-\beta\omega-N\omega+\theta_1(k_x,y,0,k_z)Y_{y0}+\theta_2(k_x,y,0,k_z)\omega_{y0}\\
&\quad+h_1\int_0^y Y(k_x,\eta,k_z)d\eta+\int_0^y h_2(y,\eta)\omega(k_x,\eta,k_z)d\eta
\end{aligned}
$$

where $Y_{y0}=Y(k_x,0,k_z)$and $\omega_{y0}=\omega(k_x,0,k_z)$.

Observed Wave Numbers

Use the backstepping transformation

$$
\begin{aligned}
Y &= \Psi - \int_0^y l(k_x, y, \eta, k_z)\Psi(k_x, \eta, k_z)d\eta \\
\omega &= \Omega - \int_0^y \theta_1(k_x, y, \eta, k_z)\Psi(k_x, \eta, k_z)d\eta - \int_0^y \theta_2(k_x, y, \eta, k_z)\Omega(k_x, \eta, k_z)d\eta
\end{aligned}
$$

to map (for each k_x, k_z in the observed range) the (Y, ω) plant into the *target* system

$$
\begin{aligned}
\Psi_t &= \frac{-\alpha^2\Psi + \Psi_{yy}}{Re} - \beta\Psi - N\psi \\
\Omega_t &= \frac{-\alpha^2\Omega + \Omega_{yy}}{Re} - \beta\Omega - N\Omega
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
\Psi(k_x, 0, k_z) &= \Psi(k_x, 1, k_z) = 0 \\
\Omega(k_x, 0, k_z) &= \Omega(k_x, 1, k_z) = 0
\end{aligned}
$$

Kernel equations deduced from transformation and target system. Output injection gains computed from kernels.

Stability of observed wave number range guaranteed using backstepping theory!

Unobserved Wave Numbers

When $k_x^2 + k_z^2 > M^2$, $\chi = 0$: No output injection.

Plant in (Y, ω) variables:

$$
\begin{aligned}
Y_t &= \frac{-\alpha^2 Y + Y_{yy}}{Re} - \beta Y - 2\pi k_x i U_y^e \tilde{V} + \alpha^2 \tilde{P} - NY \\
\omega_t &= \frac{-\alpha^2 \omega + \omega_{yy}}{Re} - \beta \omega - 2\pi k_z i U_y^e \tilde{V} - \alpha^2 N \tilde{\phi} - N\omega
\end{aligned}
$$

and a Poisson equation for the potential in terms of ω

$$
-\alpha^2 \tilde{\phi} + \phi_{yy} = \omega
$$

Want to do Lyapunov analysis. If stability in (Y, ω) variables is proven for large wave numbers, stability in $(\tilde{U}, \tilde{V}, \tilde{W})$ variables follows.

Unobserved Wave Numbers

Consider the Lyapunov function

$$\Lambda = \int_0^1 \frac{|Y|^2 + |\omega|^2 + \alpha^2 |\tilde{V}|^2}{2} dy,$$

Notation: $\int_0^1 f = \int_0^1 f(k_x, y, k_z) dy$, f^* complex conjugate of f.

Estimation of $\dot{\Lambda}$:

$$\begin{aligned} \dot{\Lambda} \;=\; & -\frac{2\alpha^2}{Re}\Lambda - \frac{1}{Re}\int_0^1 \left(|Y_y|^2 + |\omega_y|^2 + \alpha^2|\tilde{V}_y|^2\right) - N\int_0^1 \left(|Y|^2 + |\omega|^2\right) \\ & -\alpha^2 N \int_0^1 \frac{\tilde{\phi}^*\omega + \tilde{\phi}\omega^*}{2} + \int_0^1 2\pi i U_y^e(y) \frac{\tilde{V}^*(k_x Y + k_z\omega) - \tilde{V}(k_x Y^* + k_z \omega^*)}{2} \\ & +\alpha^2 \int_0^1 \frac{P^* Y + P Y^* - P_y^* \tilde{V} - P_y \tilde{V}^*}{2} \end{aligned}$$

Unobserved Wave Numbers

Lemma 1

$$-\alpha^2 \int_0^1 \frac{\tilde{\phi}^* \omega + \tilde{\phi} \omega^*}{2} \le \int_0^1 |\omega|^2.$$

Lemma 2

$$|U_y^e(y)| \le 4 + H.$$

Integrating by parts and applying Lemma 1

$$\begin{aligned} \dot{\Lambda} \ \le \ & -\frac{2\alpha^2}{Re}\Lambda - \frac{1}{Re}\int_0^1 \left(|Y_y|^2 + |\omega_y|^2 + \alpha^2 |\tilde{V}_y|^2 \right) \\ & + \int_0^1 2\pi i U_y^e(y) \frac{\tilde{V}^*(k_x Y + k_z \omega) - \tilde{V}(k_x Y^* + k_z \omega^*)}{2} - N \int_0^1 |Y|^2 \end{aligned}$$

Using Lemma 2 and applying Young's and Poincare's inequalities

$$\begin{aligned} \dot{\Lambda} \ \le \ & -2\frac{1+\alpha^2}{Re}\Lambda - N\int_0^1 |Y|^2 dy + 2\pi(4+H)\int_0^1 \left(|\tilde{V}| (|k_x||Y| + |k_z||\omega|) \right) dy \\ \le \ & \left(4 + H - 2\frac{1+\alpha^2}{Re} \right) \Lambda \end{aligned}$$

Unobserved Wave Numbers

Since

$$\dot{\Lambda} \quad \leq \quad \left(4 + H - 2\frac{1+\alpha^2}{Re}\right)\Lambda$$

if $\alpha^2 \geq \frac{Re(4+H)}{2}$, i.e., $k_x^2 + k_z^2 \geq M^2$, then

$$\dot{\Lambda} \leq -\frac{2}{Re}\Lambda,$$

from where stability in the unobserved wave number range follows.

Adaptive Control of PDEs

Andrey Smyshlyaev

mini-course at UCSB, 2006

Literature on Adaptive Control of PDEs:

- High-gain adaptive feedback (non-identifier based) under a relative degree one assumption—Logemann and coauthors (Martensson, Ryan, Townley).

- MRAC (with identifiability proofs but with actuation throughout the PDE domain)—Bentsman, Orlov, Hong; Demetriou, Rosen, and coworkers; Solo and Bamieh.

Approaches to identifier design

- Lyapunov
- Estimation based/Certainty equivalence
  - with passive identifier (often called “observer-based” method)
  - with swapping identifier (often called the “gradient” method)

Backstepping adaptive controllers for PDEs can be designed using all three major approaches.

We only present here **output feedback design with swapping identifier.**

PDE with unknown functional parameter

$$\begin{aligned} u_t &= u_{xx} + \lambda(x) u \\ u_x(0) &= 0 \\ u(1) &= \text{control} \end{aligned}$$

PDE with unknown functional parameter

$$
\begin{aligned}
u_t &= \varepsilon(x)u_{xx} + b(x)u_x + \lambda(x)u + g(x)u(0) + \int_0^x f(x,y)u(y)\,dy \\
u_x(0) &= -qu(0) \\
u(1) &= \text{control}
\end{aligned}
$$

PDE with unknown functional parameter

$$\begin{aligned} u_t &= u_{xx} + \lambda(x) u \\ u_x(0) &= 0 \\ u(1) &= \text{control} \end{aligned}$$

Measurement: $u(0)$

Infinite relative degree plant with arbitrarily many (finite) number of unstable poles

Scalar input and output, infinite–dimensional state, infinitely many unknown parameters

Idea: using backstepping transformation change the form of the plant into one with unknown parameters multiplying the output:

$$\begin{aligned} v_t &= v_{xx} + \theta(x) v(0) \\ v_x(0) &= \theta_1 v(0) \\ v(1) &= u(1) \end{aligned}$$

Observer canonical form

$$\begin{aligned} v_t &= v_{xx} + \theta(x)v(0) \\ v_x(0) &= \theta_1 v(0) \\ v(1) &= u(1) \end{aligned}$$

To put the plant into this form we use the transformation

$$v(x) = u(x) - \int_0^x p(x,y)u(y)\,dy$$

$$\begin{aligned} p_{xx}(x,y) &= p_{yy}(x,y) + \lambda(y)p(x,y) \\ p(1,y) &= 0 \\ p(x,x) &= \frac{1}{2}\int_x^1 \lambda(s)\,ds \end{aligned}$$

$\theta(x)$ and θ_1 are the new unknown parameters:

$$\theta(x) = -p_y(x,0) \qquad \theta_1 = -p(0,0)$$

Observer canonical form

$$
\begin{aligned}
v_t &= v_{xx} + \theta(x) v(0) \\
v_x(0) &= \theta_1 v(0) \\
v(1) &= u(1)
\end{aligned}
$$

We are going to directly estimate $\theta(x)$ and θ_1 without identifying the original plant parameter $\lambda(x)$.

Note:

— $v(0) = u(0)$ and therefore $v(0)$ is measured.

— $v(1) = u(1)$ so that the controller for v-system gives the controller for the original system

Input filter

$$
\begin{aligned}
\psi_t &= \psi_{xx} \\
\psi_x(0) &= 0 \\
\psi(1) &= u(1)
\end{aligned}
$$

Only one filter (the plant has no zeros)

Output filters

$$\begin{aligned} F_t &= F_{xx} + \delta(x-\xi)u(0) \qquad \xi \in [0,1] \\ F_x(0) &= 0 \\ F(1) &= 0 \end{aligned}$$

$$\begin{aligned} \phi_t &= \phi_{xx} \\ \phi_x(0) &= u(0) \\ \phi(1) &= 0 \end{aligned}$$

Algebraic way to represent $F(x,\xi)$ through $\phi(x)$:

$$\begin{aligned} F_{xx}(x,\xi) &= F_{\xi\xi}(x,\xi) \\ F(0,\xi) &= -\phi(\xi) \\ F_x(0,\xi) &= 0 \\ F_\xi(x,0) &= F(x,1) = 0 \end{aligned}$$

Three forms of the output filter $F(x,\xi,t)$

Dynamic form

$$\begin{aligned} F_t &= F_{xx} + \delta(x-\xi)u(0) \\ F_x(0) &= 0 \\ F(1) &= 0 \end{aligned}$$

Algebraic form

$$\begin{aligned} F_{xx}(x,\xi) &= F_{\xi\xi}(x,\xi) \\ F(0,\xi) &= -\phi(\xi) \\ F_x(0,\xi) &= 0 \\ F_\xi(x,0) &= F(x,1) = 0 \end{aligned}$$

Explicit algebraic form

$$F(x,\xi) = -\sum_{n=0}^{\infty} \cos\frac{\pi(2n+1)x}{2} \cos\frac{\pi(2n+1)\xi}{2} \int_0^1 \cos\frac{\pi(2n+1)s}{2} \phi(s)\, ds$$

Observer error

$$e(x) = v(x) - \left[\psi(x) + \theta_1 \phi(x) + \int_0^1 \theta(\xi) F(x,\xi)\, d\xi\right]$$

is exponentially stable:

$$\begin{aligned} e_t &= e_{xx} \\ e_x(0) &= 0 \\ e(1) &= 0 \end{aligned}$$

Static parametric model

$$e(0) = v(0) - \left[\psi(0) + \theta_1 \phi(0) - \int_0^1 \theta(\xi) \phi(\xi)\, d\xi\right]$$

Update laws (least squares)

$$\hat{\theta}_t(x,t) = \frac{\int_0^1 \gamma(x,y,t)\phi(y)\,dy + \gamma_0(x,t)\phi(0)}{1+\|\phi\|^2+\phi^2(0)}\left(v(0)-\psi(0)-\hat{\theta}_1\phi(0)+\int_0^1 \hat{\theta}(\xi)\phi(\xi)\,d\xi\right)$$

$$\dot{\hat{\theta}}_1 = \frac{\int_0^1 \gamma_0(y,t)\phi(y)\,dy + \gamma_1(t)\phi(0)}{1+\|\phi\|^2+\phi^2(0)}\left(v(0)-\psi(0)-\hat{\theta}_1\phi(0)+\int_0^1 \hat{\theta}(\xi)\phi(\xi)\,d\xi\right)$$

Riccati adaptation gains

$$\begin{aligned}
\gamma_t(x,y,t) &= -\frac{\int_0^1 \gamma(x,s)\phi(s)\,ds \int_0^1 \gamma(y,s)\phi(s)\,ds + \gamma_0(x)\gamma_0(y)\phi^2(0)}{1+\|\phi\|^2+\phi^2(0)} \\
&\quad -\frac{\phi(0)\gamma_0(y)\int_0^1 \gamma(x,s)\phi(s)\,ds + \phi(0)\gamma_0(x)\int_0^1 \gamma(y,s)\phi(s)\,ds}{1+\|\phi\|^2+\phi^2(0)}
\end{aligned}$$

$$\dot{\gamma}_0(x) = -\frac{\left(\int_0^1 \gamma(x,s)\phi(s)\,ds + \gamma_0(x)\phi(0)\right)\left(\int_0^1 \gamma_0(s)\phi(s)\,ds + \gamma_1\phi(0)\right)}{1+\|\phi\|^2+\phi^2(0)}$$

$$\dot{\gamma}_1 = -\frac{\left(\int_0^1 \gamma_0(s)\phi(s)\,ds + \gamma_1\phi(0)\right)^2}{1+\|\phi\|^2+\phi^2(0)}$$

Controller

$$u(1) = \int_0^1 \hat{k}(1,y)\left(\psi(y) + \hat{\theta}_1\phi(y) + \int_0^1 \hat{\theta}(\xi)F(y,\xi)\,d\xi\right)dy$$

with $\hat{k}(x,y)$ given by the PDE

$$\begin{aligned}
\hat{k}_{xx} - \hat{k}_{yy} &= 0 \\
\hat{k}_y(x,0) &= \hat{\theta}_1\hat{k}(x,0) + \hat{\theta}(x) - \int_0^x \hat{k}(x,y)\hat{\theta}(y)\,dy \\
\hat{k}(x,x) &= \hat{\theta}_1
\end{aligned}$$

This PDE can be simplified to the integro-differential equation by setting $\hat{k}(x,y) = f(x-y)$

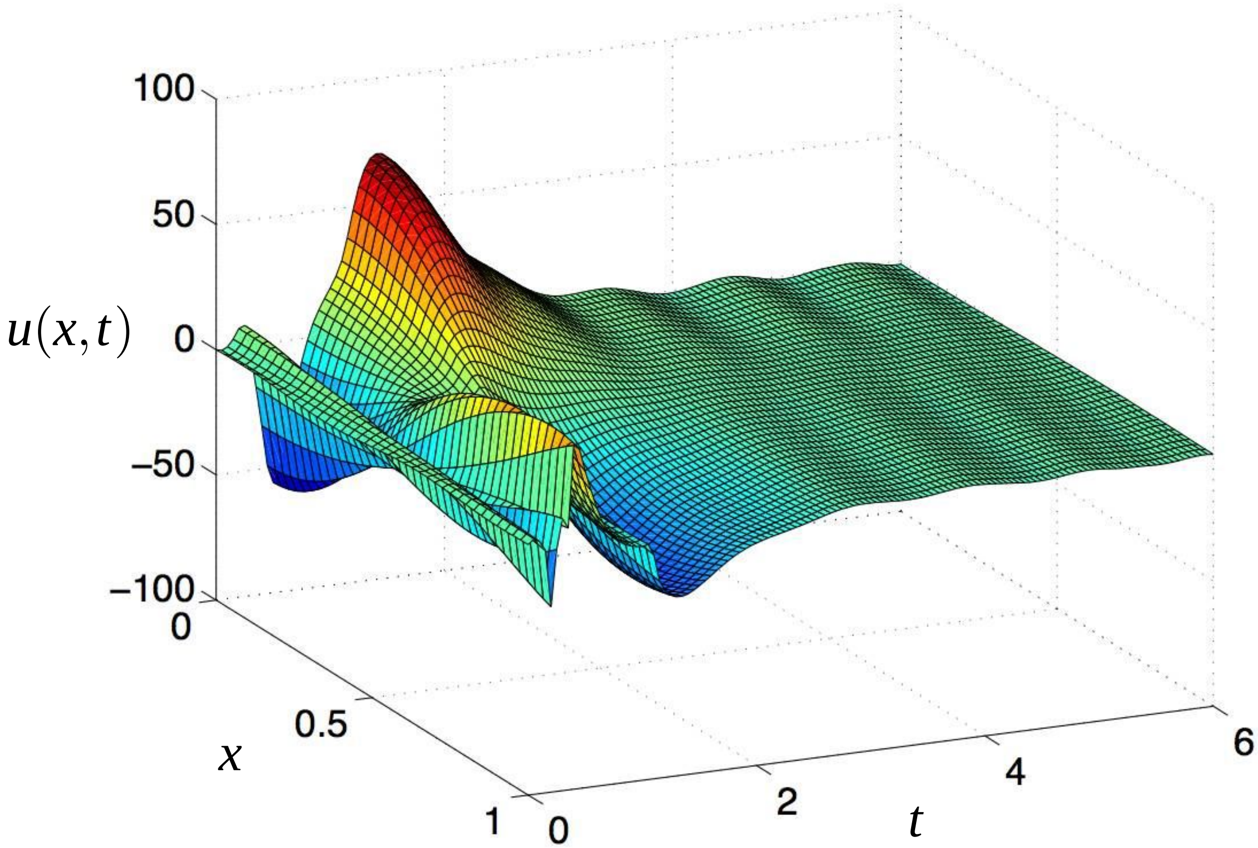
$$\begin{aligned}
f'(x) &= -\hat{\theta}_1 f(x) - \hat{\theta}(x) + \int_0^x f(x-y)\hat{\theta}(y)\,dy \\
f(0) &= \hat{\theta}_1
\end{aligned}$$

This equation is solved at each time step.

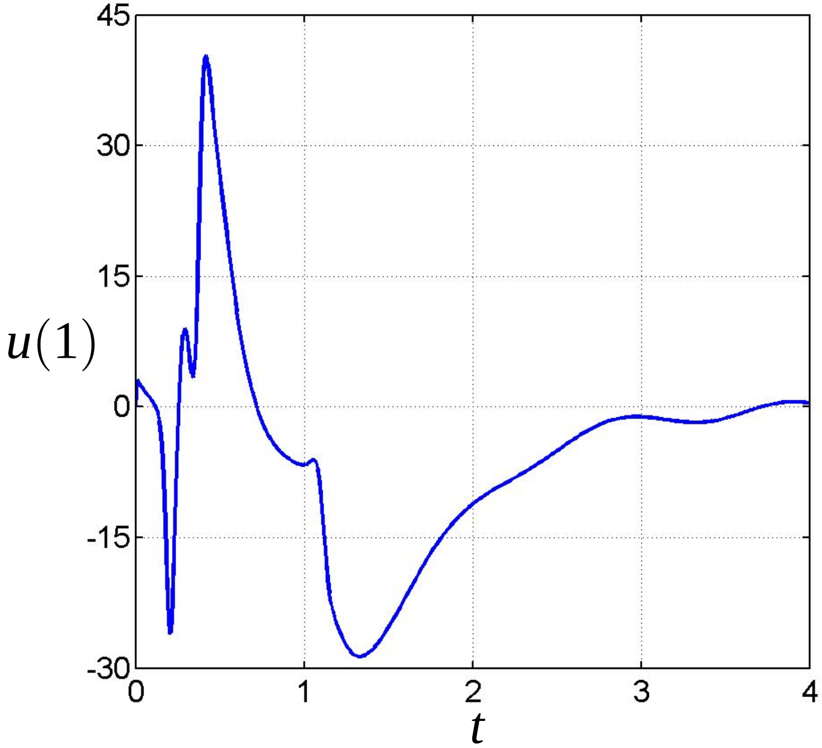
Simulation Example

$$\begin{aligned} u_t &= u_{xx} + b(x)u_x + \lambda(x)u \\ u_x(0) &= 0 \end{aligned}$$

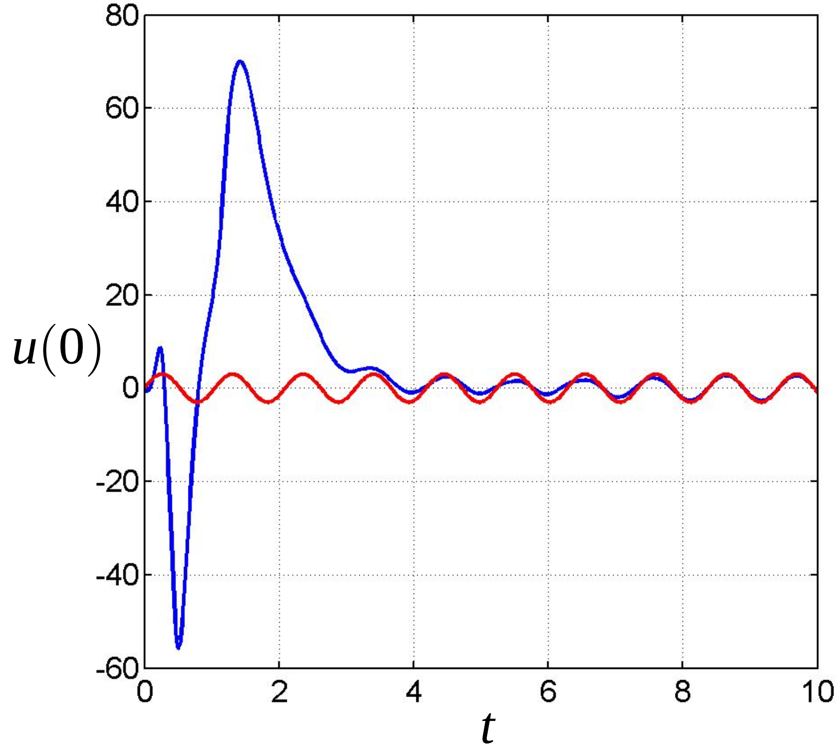
Reference signal: $u^r(0,t) = 3\sin 6t$ $\quad b(x) = 3 - 2x^2$ $\quad \lambda(x) = 16 + 3\sin(2\pi x)$



Simulation Results

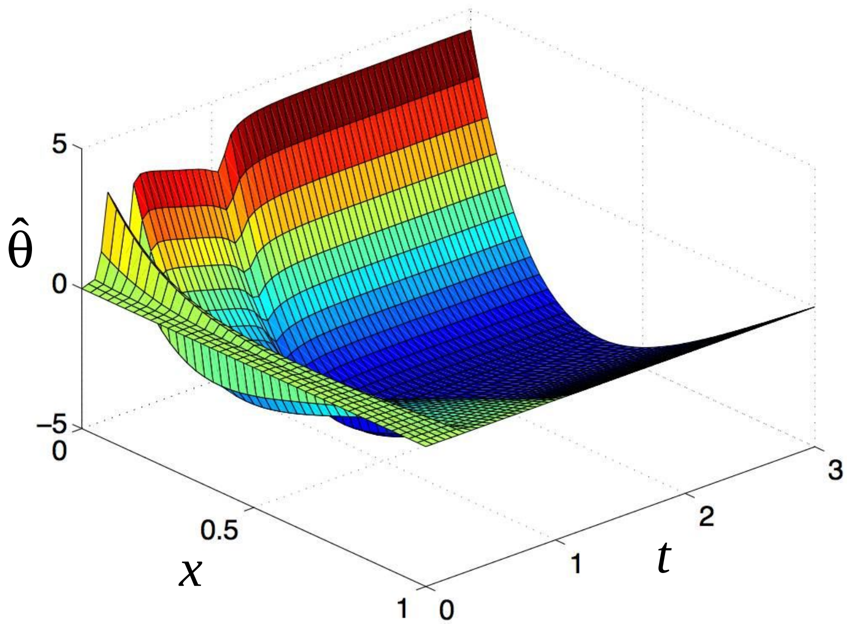


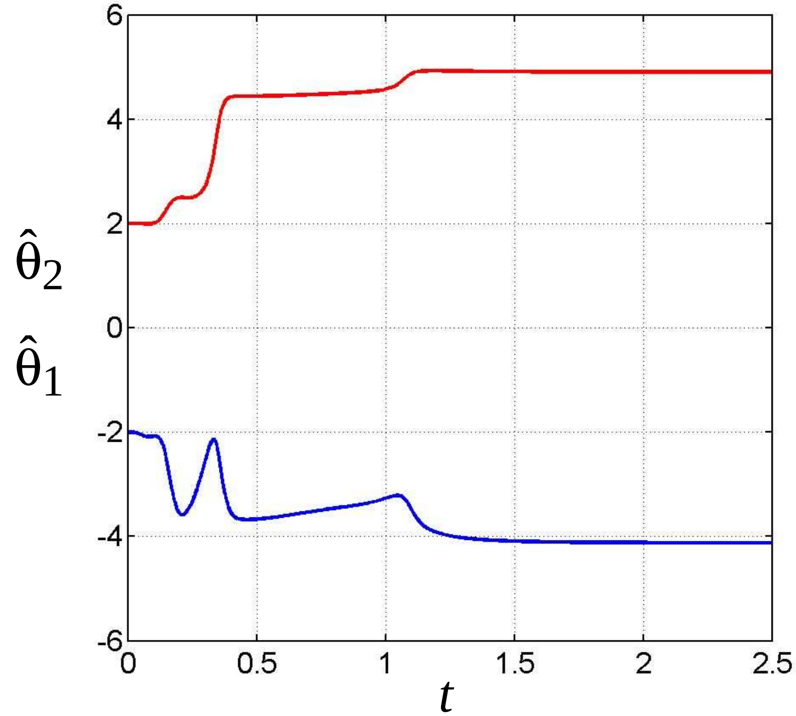
Control effort

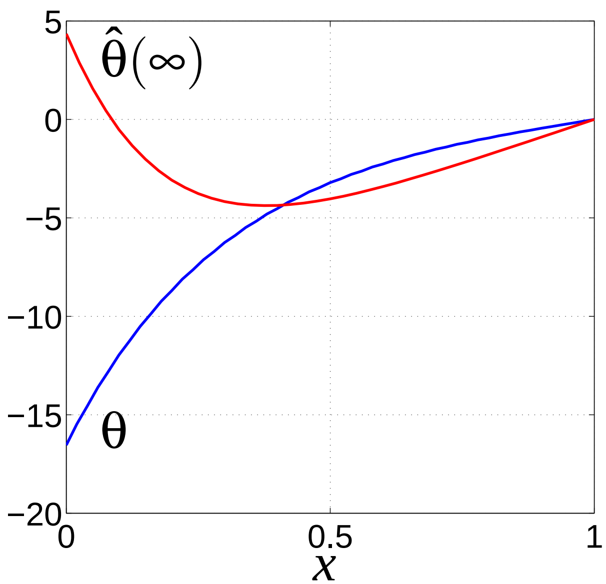


Output evolution

Simulation Results (parameter estimates)







Motion Planning and Trajectory Tracking for PDEs

Miroslav Krstic

mini-course at UCSB, 2006

Introduction

- Initiated by the French school of "differential flatness:" Pierre Rouchon, Nicolas Petit, Phillippe Martin, Michel Fliess. Since late 1990s.

- Motion planning for *arbitrary functions* is mainly an analysis exercise, like studies of controllability. Best pursued with Gevrey functions.

- We focus on trajectories that can be generated as combinations of sinusoids, polynomials, and exponentials—anything that can be produced by an LTI system.

Trajectory Generation

We present the ideas through several examples.

Consider the heat equation

$$
\begin{aligned}
u_t &= u_{xx} \\
u_x(0) &= 0 \\
u(0) &= \text{system output} \\
u(1) &= \text{system input}
\end{aligned}
$$

The objective is to follow the **output** trajectory

$$u^r(0,t) = 2-(t-1)^2 = 1+2t-t^2$$

The first step is to construct the full state trajectory $u^r(x,t)$ which satisfies the PDE. Then add a feedback law that stabilizes that solution.

We postulate the state trajectory in the form:

$$u^r(x,t) = \sum_{k=0}^{\infty} a_k(t) \frac{x^k}{k!}.$$

This is a Taylor series in x with time varying coefficients $a_k(t)$ that will be determined from the PDE, boundary condition, and desired trajectory.

From the desired trajectory we get

$$u^r(0,t) = a_0(t) = 1 + 2t - t^2.$$

The boundary condition gives

$$u_x^r(0,t) = a_1(t) = 0.$$

Substituting $u^r(x,t)$ into the PDE we get:

$$\begin{aligned}
\sum_{k=0}^{\infty} \dot{a}_k(t)\frac{x^k}{k!} &= \frac{\partial^2}{\partial x^2}\sum_{k=0}^{\infty} a_k(t)\frac{x^k}{k!} \\
&= \sum_{k=2}^{\infty} a_k(t)\frac{k(k-1)x^{k-2}}{k!} \\
&= \sum_{k=2}^{\infty} a_k(t)\frac{x^{k-2}}{(k-2)!} \\
&= \sum_{k=0}^{\infty} a_{k+2}(t)\frac{x^k}{k!}.
\end{aligned}$$

We get the recursive relationship

$$a_{k+2}(t) = \dot{a}_k(t)$$

The recursion yields

$$\begin{array}{lcl} a_0 & = & 1+2t-t^2, \qquad\qquad a_1 = 0 \\ a_2 & = & 2-2t, \qquad\qquad a_3 = 0 \\ a_4 & = & -2, \qquad\qquad a_5 = 0 \\ a_6 & = & 0, \qquad\qquad a_i = 0 \quad \text{for } i > 6. \end{array}$$

This gives the reference trajectory

$$u^r(x,t) = 1+2t+t^2+(1-t)x^2 - \frac{1}{12}x^4,$$

and the input signal

$$u^r(1,t) = \frac{23}{12} + t - t^2.$$

Remark. Perfect trajectory obtained only if the initial condition of the plant agrees with the initial condition of the trajectory, that is, $u(x,0) = 1+x^2-\frac{1}{12}x^4$.

Example 2. Reaction-diffusion equation

$$
\begin{aligned}
u_t &= u_{xx} + \lambda u \\
u_x(0) &= 0
\end{aligned}
$$

Desired output reference signal

$$
u^r(0,t) = \mathrm{e}^{\alpha t}.
$$

From the boundary condition we have

$$
a_1(t) = 0,
$$

and from the PDE

$$
a_{k+2}(t) = \dot{a}_k(t) + \lambda a_k(t).
$$

These conditions give

$$
\begin{aligned}
a_{2k+1} &= 0 \\
a_{2k+2} &= \dot{a}_{2k} - \lambda a_{2k} \\
a_2 &= (\alpha - \lambda)\mathrm{e}^{\alpha t} \\
a_4 &= (\alpha - \lambda)^2 \mathrm{e}^{\alpha t} \\
a_{2k} &= (\alpha - \lambda)^k \mathrm{e}^{\alpha t}
\end{aligned}
$$

The state trajectory is

$$
\begin{aligned}
u^r(x,t) &= \sum_{k=0}^{\infty} (\alpha-\lambda)^k \mathrm{e}^{\alpha t} \frac{x^{2k}}{2k!} \\
&= \mathrm{e}^{\alpha t} \sum_{k=0}^{\infty} \frac{(\sqrt{\alpha-\lambda}x)^{2k}}{2k!} \\
&= \mathrm{e}^{\alpha t} \begin{cases} \cosh(\sqrt{\alpha-\lambda}x) & \alpha \geq \lambda \\ \cos(\sqrt{\alpha-\lambda}x) & \alpha < \lambda \end{cases}
\end{aligned}
$$

The reference input is

$$
u^r(1,t) = \mathrm{e}^{\alpha t} \cosh(\sqrt{\alpha-\lambda}) \, .
$$

Useful formulae when calculating trajectories for sinusoidal outputs:

$$
\begin{aligned}
\cosh(a) &= \sum_{k=0}^{\infty} \frac{a^{2k}}{(2k)!}, \qquad \sinh(a) = \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)!} \\
\cosh(ja) &= \cos(a), \qquad \sinh(ja) = j\sin(a), \qquad \sin(ja) = j\sinh(a).
\end{aligned}
$$

Example 3.

$$\begin{aligned} u_t &= u_{xx} \\ u_x(0) &= 0 \\ u^r(0,t) &= a_0(t) = \sin(\omega t) = \mathrm{Im}\{e^{j\omega t}\} \end{aligned}$$

Set $\lambda = 0$ and $\alpha = j\omega$ to get

$$u^r(x,t) = \mathrm{Im}\left\{\cosh\left(\sqrt{j\omega}x\right) e^{j\omega t}\right\}$$

Example 4.

$$\begin{aligned} u_t &= u_{xx} \\ u(0) &= 0 \\ u_x(1) &= \text{output} \\ u(1) &= \text{input} \end{aligned}$$

As before, we get $a_{i+2} = \dot{a}_i$ and from the BC

$$\begin{aligned} a_{2k} &= 0 \\ a_{2k+1} &= a_1^{(k)} \qquad (k^{\text{th}} \text{ derivative}) \end{aligned}$$

The state trajectory becomes

$$\begin{aligned} u^r(x,t) &= \sum_{k=0}^{\infty} a_1^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!} \\ u_x^r(x,t) &= \sum_{k=0}^{\infty} a_1^{(k)}(t) \frac{x^{2k}}{(2k)!} \end{aligned}$$

The output reference is

$$u_x^r(1,t) = \sum_{k=0}^{\infty} \frac{a_1^{(k)}(t)}{(2k)!}$$

and we want

$$u_x^r(1,t) = \sin(\omega t) = \mathrm{Im}\left\{\mathrm{e}^{j\omega t}\right\}.$$

Suppose that

$$a_1(t) = \mathrm{Im}\left\{A\mathrm{e}^{j\omega t}\right\},$$

then

$$a_1^{(k)}(t) = \mathrm{Im}\left\{A(j\omega)^k \mathrm{e}^{j\omega t}\right\}.$$

From the output reference

$$\begin{aligned} u_x^r(1,t) &= \mathrm{Im}\left\{A\mathrm{e}^{j\omega t}\sum_{k=0}^{\infty}\frac{\sqrt{j\omega}^{2k}}{(2k)!}\right\} \\ &= \mathrm{Im}\left\{A\mathrm{e}^{j\omega t}\cosh\left(\sqrt{j\omega}\right)\right\} \quad \Rightarrow \quad A = \frac{1}{\cosh\left(\sqrt{j\omega}\right)}. \end{aligned}$$

The state trajectory is now

$$
\begin{aligned}
u^r(x,t) &= \mathrm{Im}\left\{ A \sum_{k=0}^{\infty} (j\omega)^k \frac{x^{2k+1}}{(2k+1)!} \mathrm{e}^{j\omega t} \right\} \\
&= \mathrm{Im}\left\{ \frac{A}{\sqrt{j\omega}} \sum_{k=0}^{\infty} \frac{(\sqrt{j\omega}x)^{2k+1}}{(2k+1)!} \mathrm{e}^{j\omega t} \right\} \\
&= \mathrm{Im}\left\{ \frac{\sinh(\sqrt{j\omega}x)}{\sqrt{j\omega}\cosh(\sqrt{j\omega})} \mathrm{e}^{j\omega t} \right\},
\end{aligned}
$$

which yields the reference input

$$
u^r(1,t) = \mathrm{Im}\left\{ \frac{\tanh(\sqrt{j\omega})}{\sqrt{j\omega}} \mathrm{e}^{j\omega t} \right\}.
$$

Or if the output reference is exponential

$$u_x^r(1,t) = \mathrm{e}^{\alpha t}, \quad \alpha \in \mathcal{R}$$

the input reference is

$$u^r(1,t) = \frac{\mathrm{e}^{\alpha t}}{\sqrt{|\alpha|}} \begin{cases} \tanh(\sqrt{|\alpha|}), & \alpha > 0 \\ \tan(\sqrt{|\alpha|}), & \alpha < 0. \end{cases}$$

Example 5. Wave equation

$$
\begin{aligned}
u_{tt} &= u_{xx} \\
u(0) &= 0 \\
u_x^r(0,t) &= \sin(\omega t)
\end{aligned}
$$

From previous examples we see that the PDE and the BCs give

$$
\begin{aligned}
a_0 &= 0 \\
a_1(t) &= \sin(\omega t) = \mathrm{Im}\{\mathrm{e}^{j\omega t}\} \\
a_{i+2} &= \ddot{a}_i(t) \\
a_{2k} &= 0 \\
a_{2k+1}(t) &= (j\omega)^{2k} a_1(t)
\end{aligned}
$$

State reference

$$\begin{aligned}
u^r(x,t) &= \mathrm{Im}\left\{\sum_{k=0}^{\infty} a_1(t)(j\omega)^{2k}\frac{x^{2k+1}}{(2k+1)!}\right\} \\
&= \mathrm{Im}\left\{\frac{\mathrm{e}^{j\omega t}}{j\omega}\sum_{k=0}^{\infty}\frac{(j\omega x)^{2k+1}}{(2k+1)!}\right\} \\
&= \mathrm{Im}\left\{\frac{\mathrm{e}^{j\omega t}}{j\omega}\sinh(j\omega x)\right\} \\
&= \mathrm{Im}\left\{\frac{\mathrm{e}^{j\omega t}}{\omega}\sin(\omega x)\right\} \\
&= \frac{1}{\omega}\sin(\omega x)\sin(\omega t)
\end{aligned}$$

Separable in x and t! Not so for heat equation.

Input reference

$$u^r(1,t) = \frac{\sin(\omega)}{\omega}\sin(\omega t)$$

Example 6. String with Kelvin-Voigt Damping

$$\begin{aligned} \varepsilon u_{tt} &= (1+\alpha\partial_t)u_{xx} \\ u_x(0) &= 0 \\ u^r(0,t) &= \sin(\omega t) \end{aligned}$$

Without derivation, but following previous examples, the reference state trajectory is

$$u^r(x,t) = \frac{1}{2}\left[e^{\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2\alpha^2}-1}}{\sqrt{2}\sqrt{1+\omega^2\alpha^2}}x} \sin\left(\omega\left(t+\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2\alpha^2}-1}}{\sqrt{2}\sqrt{1+\omega^2\alpha^2}}x\right)\right)\right.$$
$$\left. + e^{\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2\alpha^2}-1}}{\sqrt{2}\sqrt{1+\omega^2\alpha^2}}x} \sin\left(\omega\left(t-\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2\alpha^2}-1}}{\sqrt{2}\sqrt{1+\omega^2\alpha^2}}x\right)\right)\right].$$

Amplitude and phase are complicated functions of $\varepsilon, \alpha, \omega$, and x.

Example 7. Euler Beam

$$
\begin{aligned}
u_{tt} + u_{xxxx} &= 0 \\
u_{xx} = u_{xxx} &= 0 \qquad (\text{free end})
\end{aligned}
$$

4th order PDE $\longrightarrow$ requires two BCs on each end $\longrightarrow$ we are free to impose both $u^r(0,t)$ and $u^r_x(0,t)$.

Let

$$
\begin{aligned}
u^r(0,t) &= \sin(\omega t) = \mathrm{Im}\left\{ \mathrm{e}^{j\omega t} \right\} \\
u^r_x(0,t) &= 0.
\end{aligned}
$$

Using the PDE and the series expansion for the reference we get

$$
a_{i+4} = -\ddot{a}_i
$$

From the BCs we get $a_2(t) = a_3(t) = 0$. Thus,

$$
\begin{aligned}
a_{4k}(t) &= (-1)^k a_0^{(2k)}(t) \\
a_{4k+1}(t) &= (-1)^k a_1^{(2k)}(t)
\end{aligned}
$$

State reference:

$$u^r(x,t) = \sum_{k=0}^{\infty}(-1)^k a_0^{(2k)}\frac{x^{4k}}{(4k)!} + \sum_{k=0}^{\infty}(-1)^k a_1^{(2k)}\frac{x^{4k+1}}{(4k+1)!}$$

However, $a_1(t) = u_x^r(0,t) = 0$, so we get

$$\begin{aligned} u^r(x,t) &= \sum_{k=0}^{\infty}(-1)^k a_0^{(2k)}\frac{x^{4k}}{(4k)!} \\ &= \mathrm{Im}\left\{ \mathrm{e}^{j\omega t}\sum_{k=0}^{\infty}\omega^{2k}\frac{x^{4k}}{(4k)!} \right\} \\ &= \mathrm{Im}\left\{ \mathrm{e}^{j\omega t}\sum_{k=0}^{\infty}\frac{\left(\sqrt{\omega}x\right)^{4k}}{(4k)!} \right\} \\ &= \mathrm{Im}\left\{ \mathrm{e}^{j\omega t}\frac{1}{2}\left[\cosh\left(\sqrt{\omega}x\right) + \cos\left(\sqrt{\omega}x\right)\right] \right\} \end{aligned}$$

$$u^r(x,t) = \frac{1}{2}\left[\cosh\left(\sqrt{\omega}x\right) + \cos\left(\sqrt{\omega}x\right)\right]\sin(\omega t)$$

Input references

$$
\begin{aligned}
u^r(1,t) &= \frac{1}{2}\left[\cosh\left(\sqrt{\omega}\right)+\cos\left(\sqrt{\omega}\right)\right]\sin(\omega t)\\
u_x^r(1,t) &= \frac{\sqrt{\omega}}{2}\left[\sinh\left(\sqrt{\omega}\right)-\sin\left(\sqrt{\omega}\right)\right]\sin(\omega t)
\end{aligned}
$$

Example 8. First order hyperbolic PDE

$$\begin{aligned} u_t &= u_x + g u(0) \\ u^r(1,t) &= \text{control} \\ u^r(0,t) &= \sin(\omega t) = \operatorname{Im}\left\{ \mathrm{e}^{j\omega t} \right\} = a_0(t) \end{aligned}$$

Since this is a first order PDE, the only boundary condition is the one that is available for control.

From the PDE we get

$$\begin{aligned} a_1 &= \dot{a}_0 - g a_0 \\ &= \operatorname{Im}\left\{ (j\omega - g)\mathrm{e}^{j\omega t} \right\} \end{aligned}$$

$$\begin{aligned} a_{i+1} &= \dot{a}_i \\ a_k(t) &= \operatorname{Im}\left\{ (j\omega - g)(j\omega)^{k-1}\mathrm{e}^{j\omega t} \right\} \\ &= \operatorname{Im}\left\{ \left(1 - \frac{g}{j\omega}\right)(j\omega)^k \mathrm{e}^{j\omega t} \right\} \end{aligned}$$

State reference

$$
\begin{aligned}
u^r(x,t) &= \mathrm{Im}\left\{\left[1+\left(1-\frac{g}{j\omega}\right)\sum_{k=1}^{\infty}\frac{(j\omega x)^k}{k!}\right]\mathrm{e}^{j\omega t}\right\} \\
&\quad \text{add and subtract } \frac{g}{j\omega} \\
&= \mathrm{Im}\left\{\left[\frac{g}{j\omega}+\left(1-\frac{g}{j\omega}\right)\mathrm{e}^{j\omega x}\right]\mathrm{e}^{j\omega t}\right\} \\
&= \mathrm{Im}\left\{\frac{g}{j\omega}\mathrm{e}^{j\omega t}+\frac{j\omega-g}{j\omega}\mathrm{e}^{j\omega(t+x)}\right\}
\end{aligned}
$$

$$
u^r(x,t) = -\frac{g}{\omega}[\cos(\omega t)-\cos(\omega(t+x))]+\sin(\omega(t+x))
$$

Input reference

$$
u^r(\color{red}{1},t) = \frac{g}{\omega}[\cos(\omega(t+1))-\cos(\omega t)]+\sin(\omega(t+1)).
$$

Trajectory Tracking

Trajectory Tracking = stabilization of reference trajectory with feedback control.

Consider the previous example

$$
\begin{array}{rrcl}
\text{plant:} & u_t & = & u_x + gu(0) \\
\text{trajectory:} & u^r(x,t) & = & \frac{g}{\omega}[\cos(\omega(t+x)) - \cos(\omega t)] + \sin(\omega(t+x))
\end{array}
$$

Stabilizing controller

$$
u(1,t) - u^r(1,t) = \int_0^1 k(1,y)[u(y,t) - u^r(y,t)]dy
$$

$$
\begin{array}{rrcll}
\text{transformation:} & w(x,t) & = & u(x,t) - u^r(x,t) - \int_0^x k(x,y)[u(y,t) - u^r(y,t)]dy & \\
\text{kernel:} & k(x,y) & = & -ge^{g(x-y)} & \\
\text{target system:} & w_t & = & w_x & \text{(unit delay with zero input)} \\
 & w(1) & = & 0 &
\end{array}
$$

Stabilizing controller

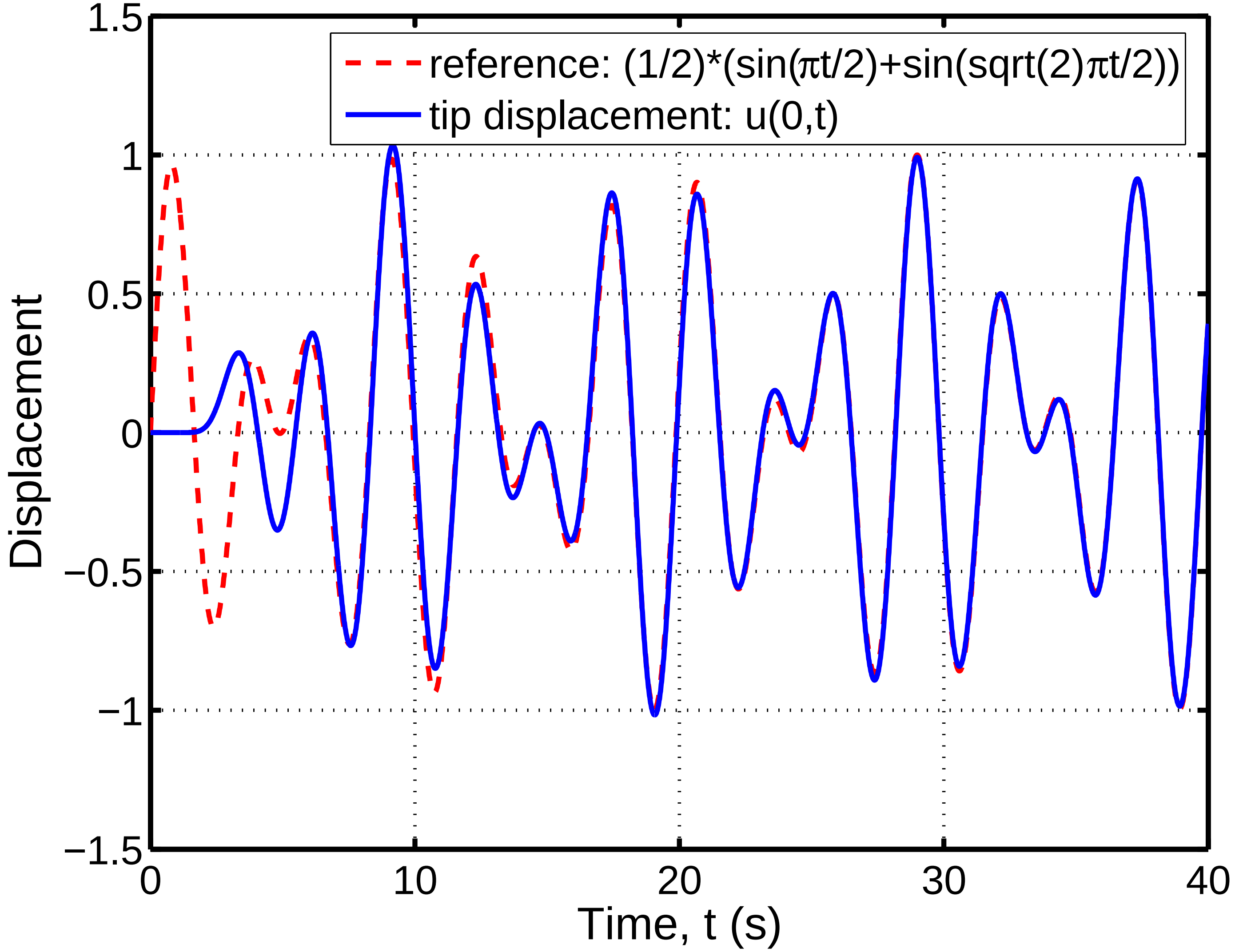
$$
\begin{aligned}
u(1,t) - u^r(1,t) &= \int_0^1 k(1,y)[u(y,t) - u^r(y,t)]dy \\
u(1,t) &= u^r(1,t) + \int_0^1 k(1,y)[u(y,t) - u^r(y,t)]dy \\
&= \underbrace{u^r(1,t) - \int_0^1 k(1,y)u^r(y,t)dy}_{\text{feedforward (fcn of } t\text{)}} + \underbrace{\int_0^1 k(1,y)u(y,t)dy}_{\text{feedback}}
\end{aligned}
$$

$$
\begin{aligned}
\text{Ffwd} &= \frac{g}{\omega}[\cos(\omega(t+1)) - \cos(\omega t)] + \sin(\omega(t+1)) \\
&\quad + \int_0^1 g\mathrm{e}^{g(1-y)} \left\{ \frac{g}{\omega}[\cos(\omega(t+y)) - \cos(\omega t)] + \sin(\omega(t+x)) \right\} dy \\
&= \frac{g}{\omega}[\cos(\omega(t+1)) - \cos(\omega t)] + \sin(\omega(t+1)) - \frac{g}{\omega}[\cos(\omega(t+1)) - \cos(\omega t)] \\
&= \sin(\omega(t+1))
\end{aligned}
$$

which is $u^r(0,t) = \sin(\omega t)$ advanced by one time unit!

Thus, it suffices to determine the reference trajectory for the target system (rather than for the complicated original system). This is true in general.

Example 9. String with Kelving-Voigt damping (Example 6)



Open Problems

Miroslav Krstic

Accessible Open Problems

- from beams to plates and shells
- beams with inhomogeneities (due to design or damage)
- Navier-Stokes with thermal convection
- more on delays and 1st order hyperbolic PDEs
- more on tracking
- networks of PDEs
- adaptive control of hyperbolic PDEs
- always good to start from an application

Hard Open Problems

- extensions to 2D and 3D in “odd-shaped” domains
- point actuation in 2D domains; 1D actuation in 3D domains
- control of flows around bluff bodies and airfoils
- nonlinear PDEs
- coupled PDEs with different “diffusion” coefficients or “wave speed” coefficients
- extension to PDEs with in-domain actuation (few actuators)
- compressible flows

Slides downloadable from

http://flyingv.ucsd.edu/pde.pdf

http://flyingv.ucsd.edu/movies.zip