Adaptive Control of PDEs and Nonlinear Systems

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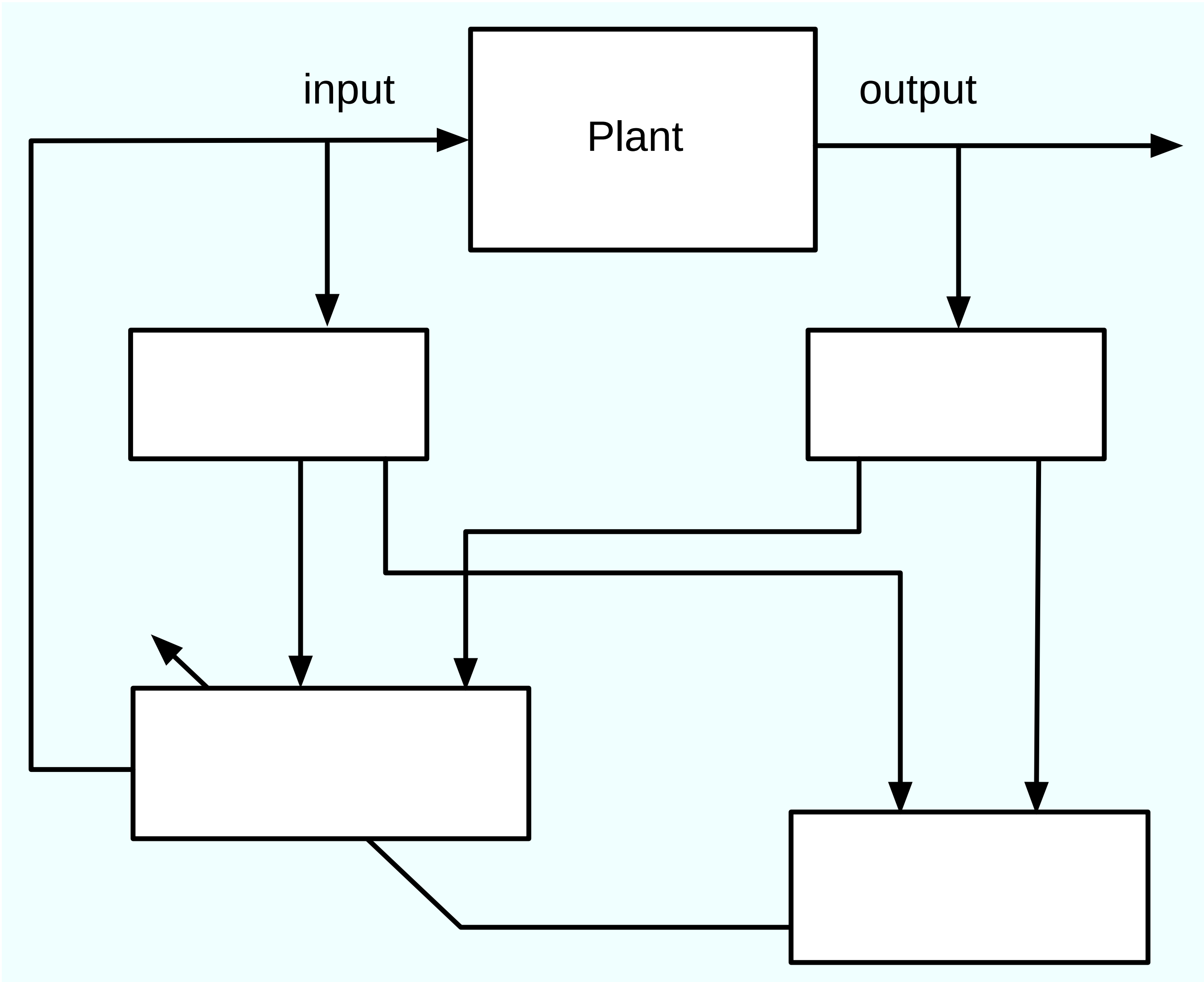
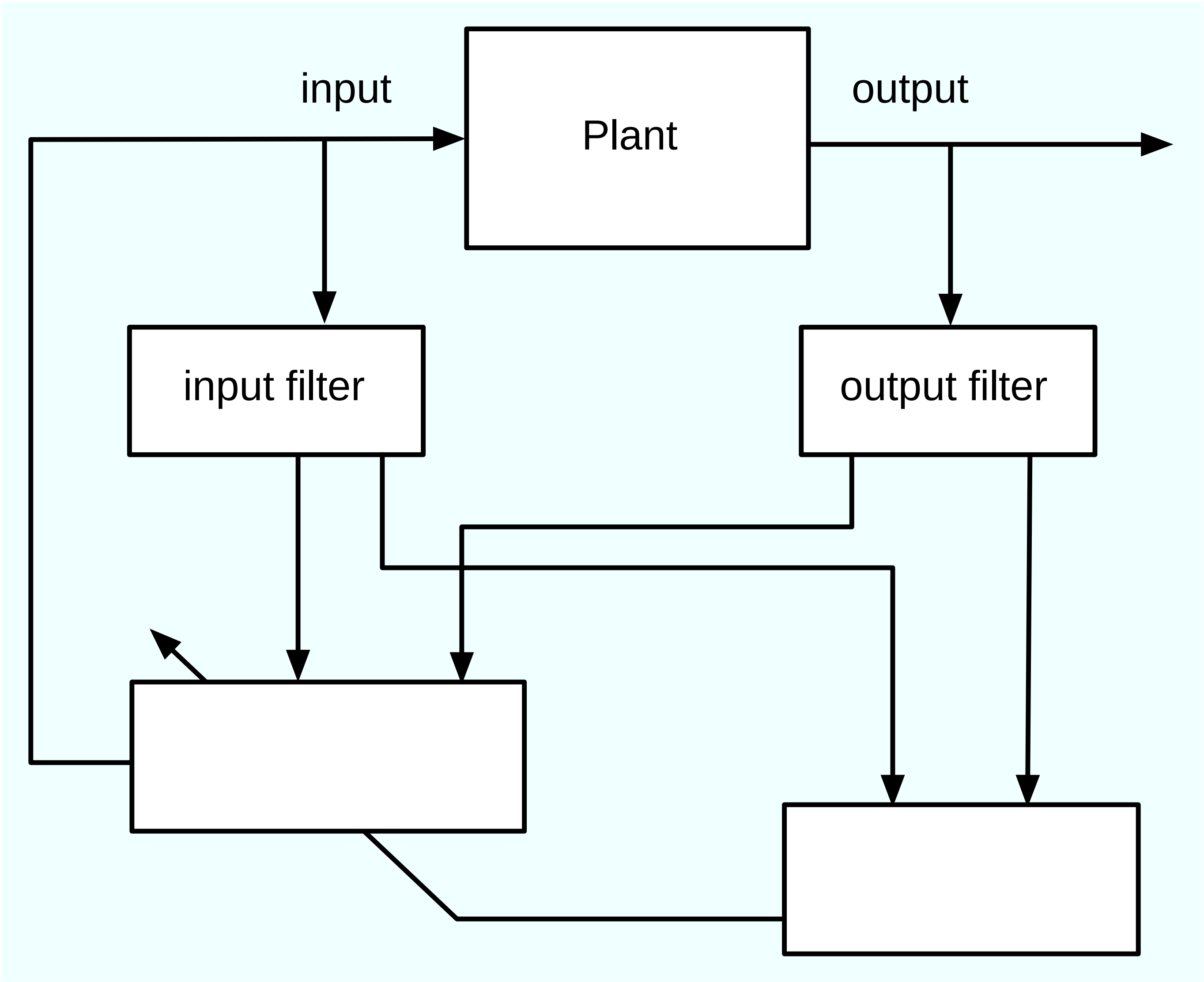
NSF-DOE Fusion Control Workshop, General Atomics, 2006

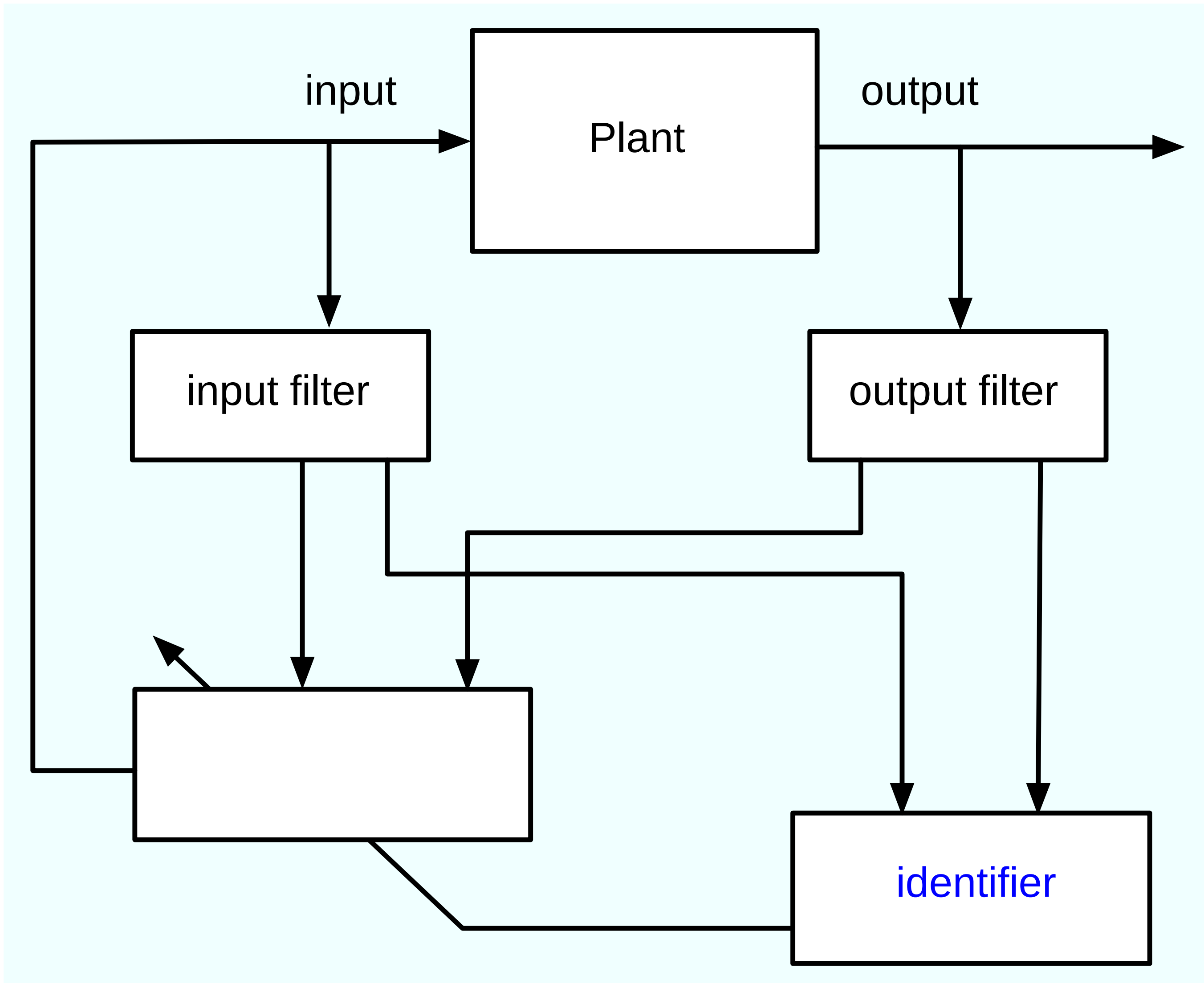
Adaptive Control



Adaptive Control



Adaptive Control



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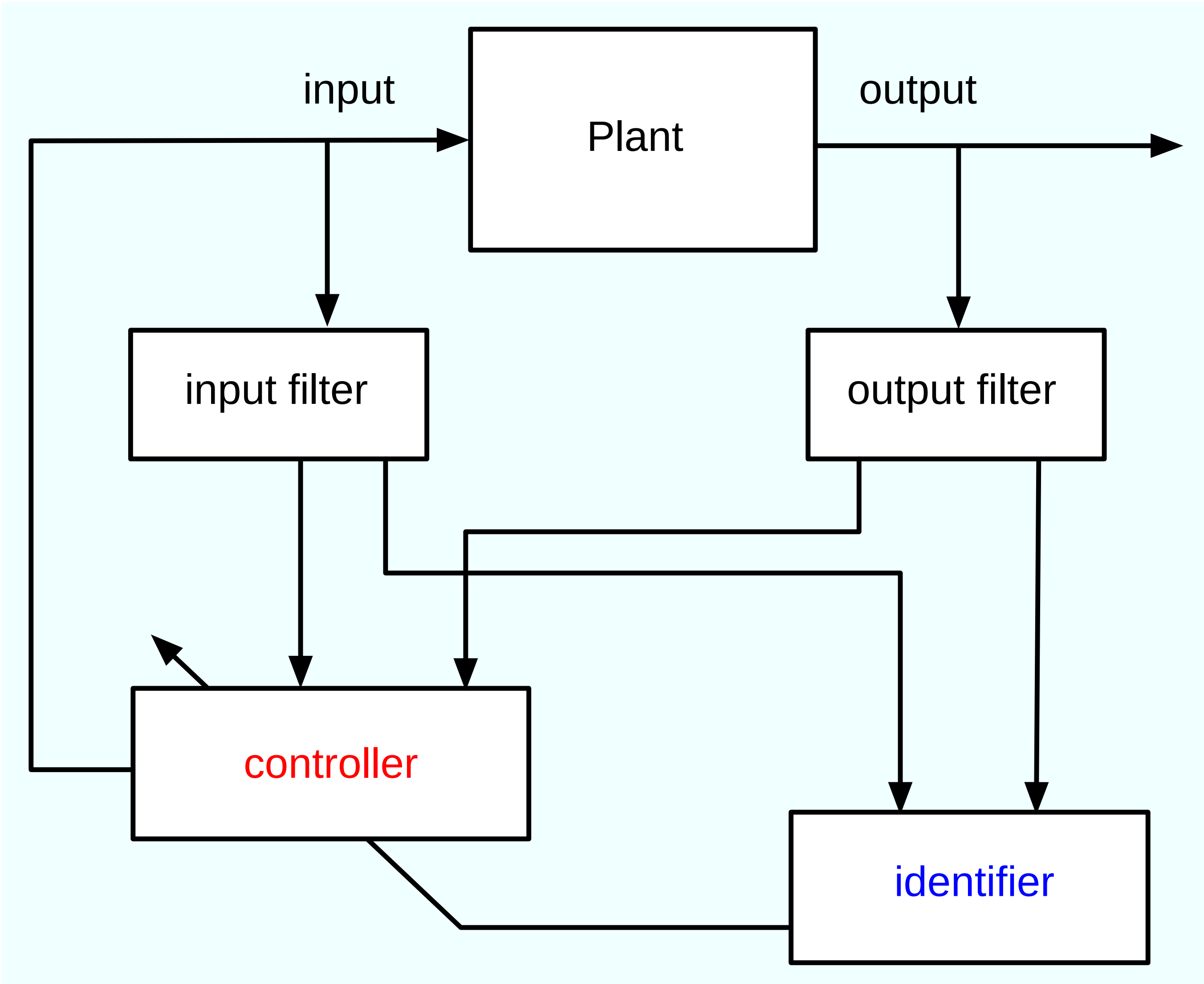

Plant

$$y = \frac{B(s)}{A(s)} u$$

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$$\begin{aligned} B(s) &= b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0 \\ A(s) &= a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \end{aligned}$$

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Unknown parameter vector

$$\theta = [b_m \;\; b_{m-1} \;\; \cdots \;\; b_1 \;\; b_0 \;\; a_n \;\; a_{n-1} \;\; \cdots \;\; a_1 \;\; a_0]^{\mathrm{T}}$$

Controller

$$u = \frac{Q(s)}{P(s)} y$$

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$P(s)$ and $Q(s)$ obtained by solving a “Bezout”-type polynomial equation involving $A(s)$ and $B(s)$ to satisfy some objective—for example, the placement of closed-loop poles.

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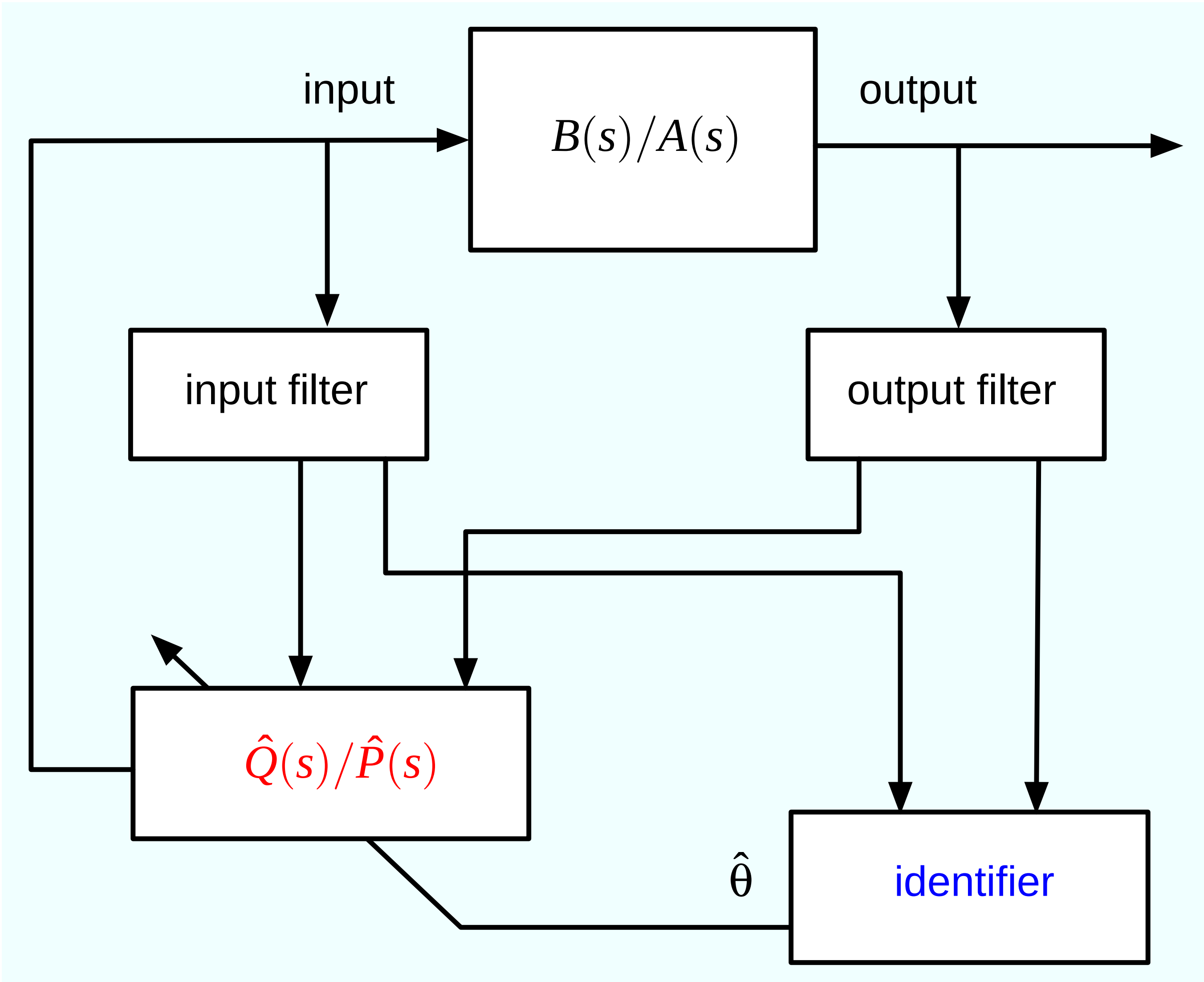
So, the coefficients of $P(s)$ and $Q(s)$ at each time step are determined from the estimate $\hat{\theta}(t)$ of θ at each time step.

Adaptive Control



Approaches to identifier design

- Lyapunov

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- Estimation based/Certainty equivalence

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- Estimation based/Certainty equivalence
  - with passive identifier (often called "observer-based" method)
  - with swapping identifier (often called the "gradient" method)

PDE with unknown functional parameter

$$u_t = u_{xx} + \lambda(x) u$$

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Measurement: $u(0)$

Control: $u(1)$

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Control: $u(1)$

- Unstable
- "Infinitely many" unknown parameters / infinite–dimensional state
- Scalar input / scalar output

Plant transfer function:

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$$\begin{aligned} B(s) &= 1 \\ A(s) &= \cosh\sqrt{s} + \left[\theta_1 \frac{\sinh\sqrt{s}}{\sqrt{s}} - \int_0^1 \theta(y) \frac{\sinh\sqrt{s}(1-y)}{\sqrt{s}}\, dy\right] \end{aligned}$$

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and $\theta(x)$, θ_1 are related to $\lambda(x)$ through the solution of the PDE

$$\begin{aligned} p_{xx}(x,y) &= p_{yy}(x,y) + \lambda(y) p(x,y) \\ p(1,y) &= 0 \\ p(x,x) &= \frac{1}{2}\int_x^1 \lambda(y)\, dy \end{aligned}$$

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$$\begin{aligned} \theta(x) &= -p_y(x,0) \\ \theta_1 &= -p(0,0) \end{aligned}$$

Compensator:

$$u(1,s) = \frac{Q(s)}{P(s)} u(0,s)$$

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P(s) &= \cosh\sqrt{s} - \int_0^1 k(1-y)\cosh\left(\sqrt{s}y\right) dy \\
Q(s) &= \int_0^1 k(y)\left[-\theta_1 \frac{\sinh\left(\sqrt{s}y\right)}{\sqrt{s}} + \frac{\sinh\left(\sqrt{s}y\right)}{\sqrt{s}} \int_0^{1-y} \theta(\xi)\cosh(\sqrt{s}\xi)\, d\xi\right] dy \\
&\quad + \int_0^1 k(y)\cosh\left(\sqrt{s}(1-y)\right) \int_{1-y}^1 \theta(\xi) \frac{\sinh\left(\sqrt{s}(1-\xi)\right)}{\sqrt{s}} d\xi dy
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\end{aligned}$$

and

$$k(x) = \theta_1 - \int_0^x \theta(y)\, dy - \int_0^x \left[\theta_1 - \int_0^{x-y} \theta(s)\, ds\right] k(y)\, dy$$

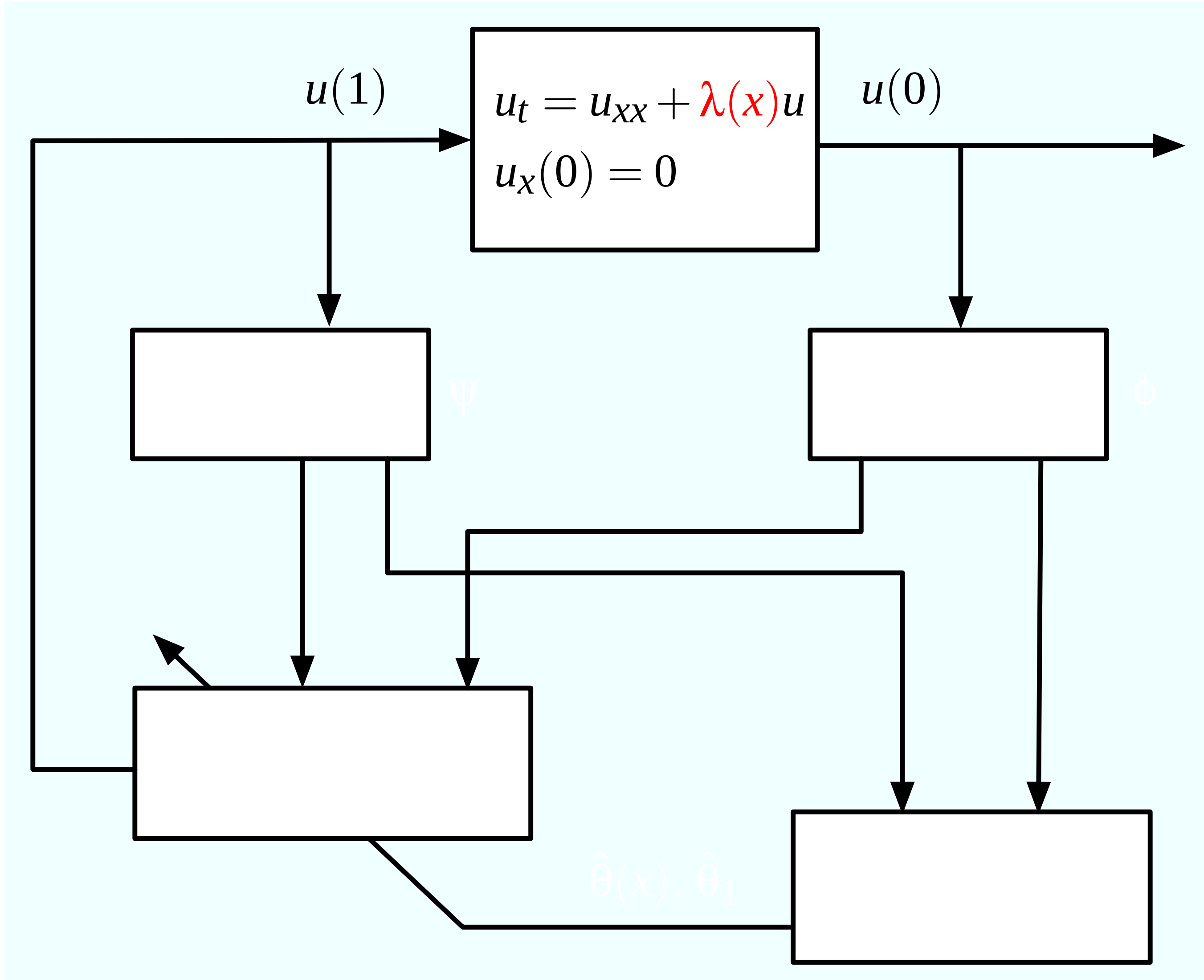
Input filter

$$
\begin{aligned}
\psi_t &= \psi_{xx} \\
\psi_x(0) &= 0 \\
\psi(1) &= u(1)
\end{aligned}
$$

Output filters

$$
\begin{aligned}
\phi_t &= \phi_{xx} \\
\phi_x(0) &= u(0) \\
\phi(1) &= 0
\end{aligned}
$$

Adaptive scheme



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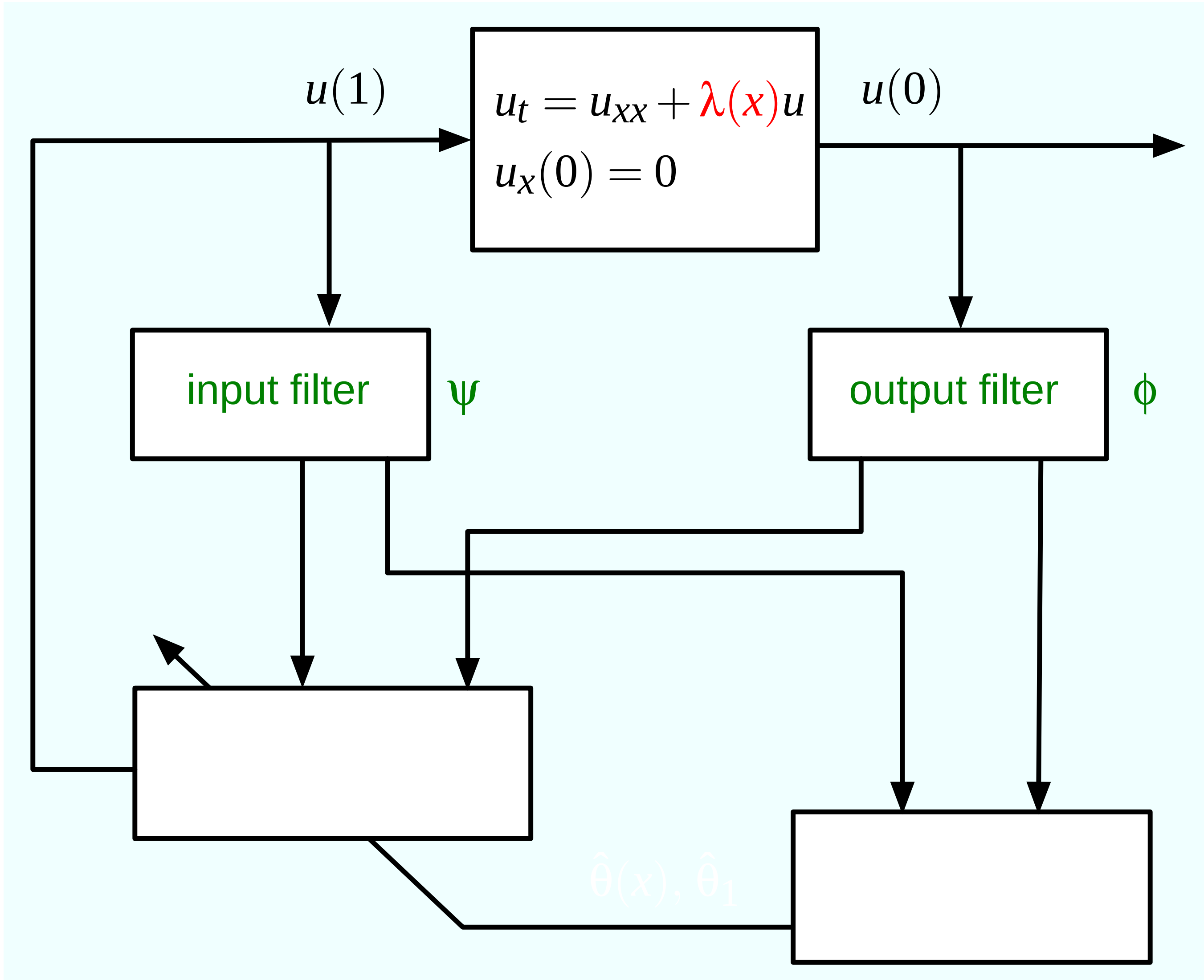

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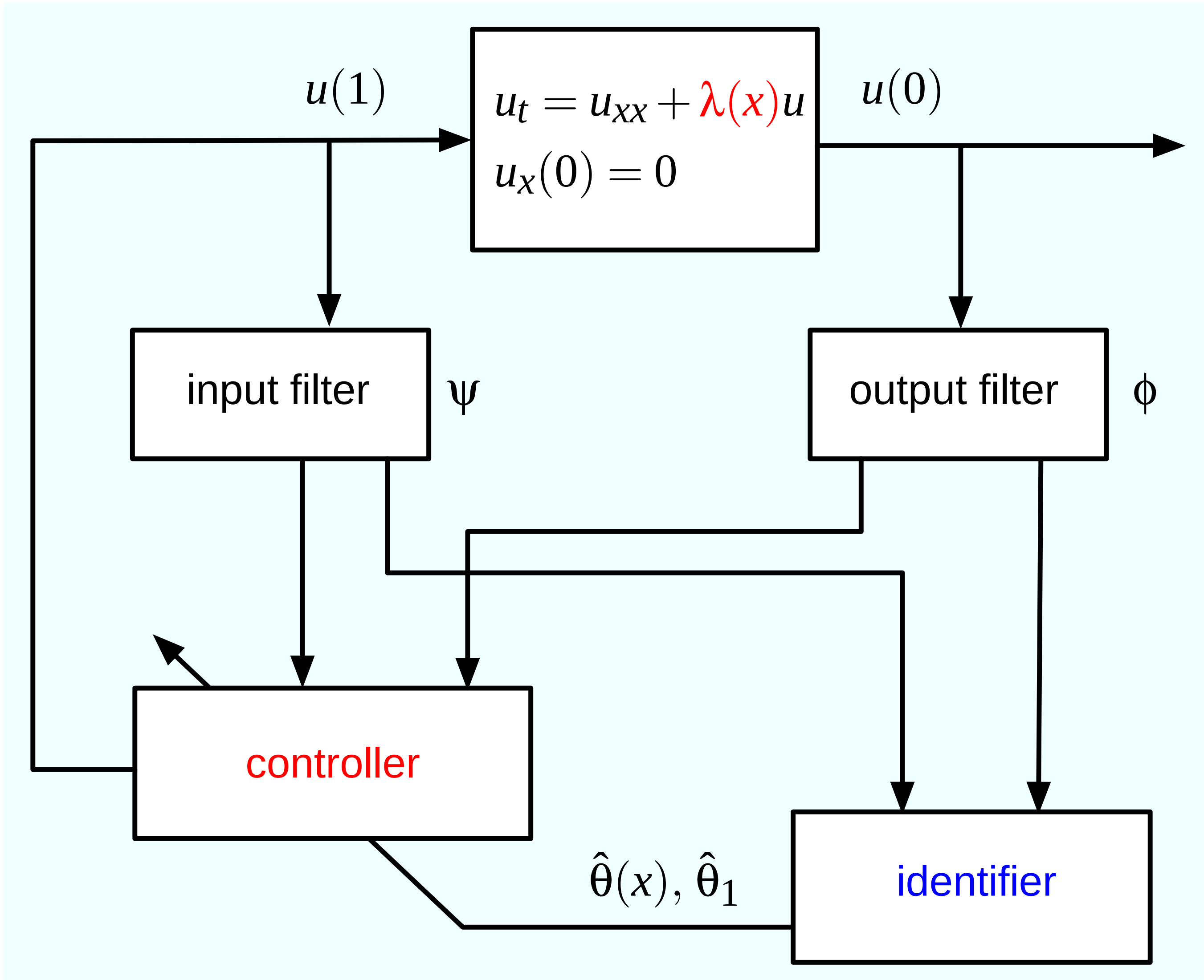

Update laws (least squares)

$$\hat{\theta}_t(x,t) = \frac{\int_0^1 \gamma(x,y,t)\phi(y)\,dy + \gamma_0(x,t)\phi(0)}{1+\|\phi\|^2+\phi^2(0)} \left(v(0) - \psi(0) - \hat{\theta}_1\phi(0) + \int_0^1 \hat{\theta}(\xi)\phi(\xi)\,d\xi \right)$$

$$\dot{\hat{\theta}}_1 = \frac{\int_0^1 \gamma_0(y,t)\phi(y)\,dy + \gamma_1(t)\phi(0)}{1+\|\phi\|^2+\phi^2(0)} \left(v(0) - \psi(0) - \hat{\theta}_1\phi(0) + \int_0^1 \hat{\theta}(\xi)\phi(\xi)\,d\xi \right)$$

Riccati adaptation gains

$$\gamma_t(x,y,t) = -\frac{\int_0^1 \gamma(x,s)\phi(s)\,ds \int_0^1 \gamma(y,s)\phi(s)\,ds + \gamma_0(x)\gamma_0(y)\phi^2(0)}{1+\|\phi\|^2+\phi^2(0)}$$

$$-\frac{\phi(0)\gamma_0(y)\int_0^1 \gamma(x,s)\phi(s)\,ds + \phi(0)\gamma_0(x)\int_0^1 \gamma(y,s)\phi(s)\,ds}{1+\|\phi\|^2+\phi^2(0)}$$

$$\dot{\gamma}_0(x) = -\frac{\left(\int_0^1 \gamma(x,s)\phi(s)\,ds + \gamma_0(x)\phi(0)\right)\left(\int_0^1 \gamma_0(s)\phi(s)\,ds + \gamma_1\phi(0)\right)}{1+\|\phi\|^2+\phi^2(0)}$$

$$\dot{\gamma}_1 = -\frac{\left(\int_0^1 \gamma_0(s)\phi(s)\,ds + \gamma_1\phi(0)\right)^2}{1+\|\phi\|^2+\phi^2(0)}$$

Adaptive Controller

$$u(1) = \int_0^1 \hat{k}(1-y)\hat{u}(y)dy$$

where $\hat{u}(y)$ is the “adaptive observer”

$$\begin{aligned}\hat{u}(y) &= \psi(y) + \hat{\theta}_1\phi(y) \\ &\quad -2\sum_{n=0}^{\infty}\cos\frac{\pi(2n+1)y}{2}\left(\int_0^1 \cos\frac{\pi(2n+1)\xi}{2}\hat{\theta}(\xi)d\xi\right)\left(\int_0^1 \cos\frac{\pi(2n+1)\eta}{2}\phi(\eta)\,d\eta\right)\end{aligned}$$

and $\hat{k}(x)$ is the control gain given by the integral equation in one variable

$$\hat{k}(x) = \hat{\theta}_1 - \int_0^x \hat{\theta}(y)\,dy - \int_0^x \left[\hat{\theta}_1 - \int_0^{x-y}\hat{\theta}(s)\,ds\right]\hat{k}(y)\,dy$$

Simulation Example

$$\begin{aligned} u_t &= u_{xx} + b(x)u_x + \lambda(x)u \\ u_x(0) &= 0 \end{aligned}$$

Reference signal: $u^r(0,t) = 3\sin 6t \qquad b(x) = 3 - 2x^2 \qquad \lambda(x) = 16 + 3\sin(2\pi x)$

Simulation Example

$$
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$$

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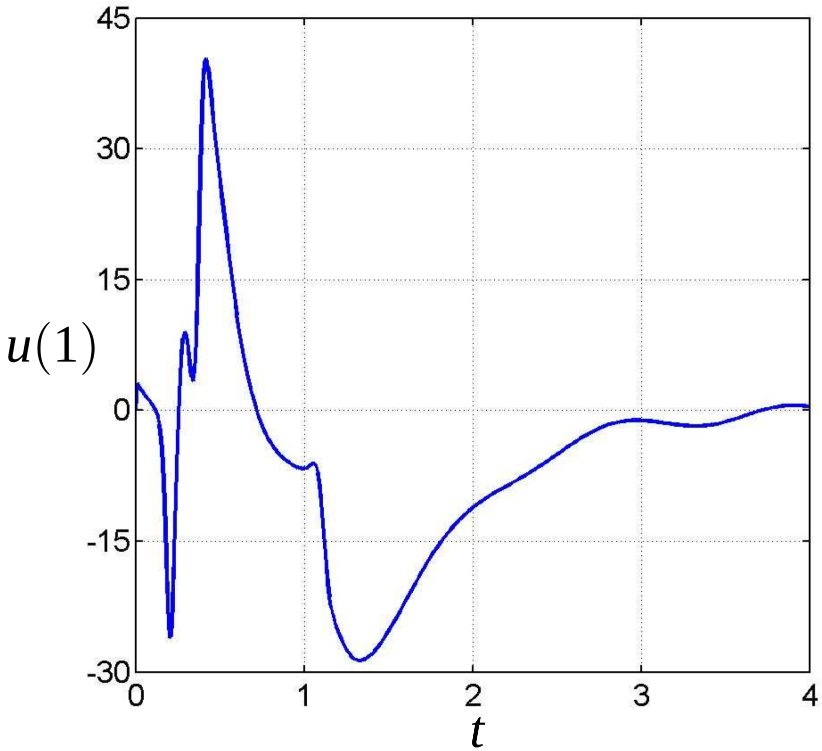
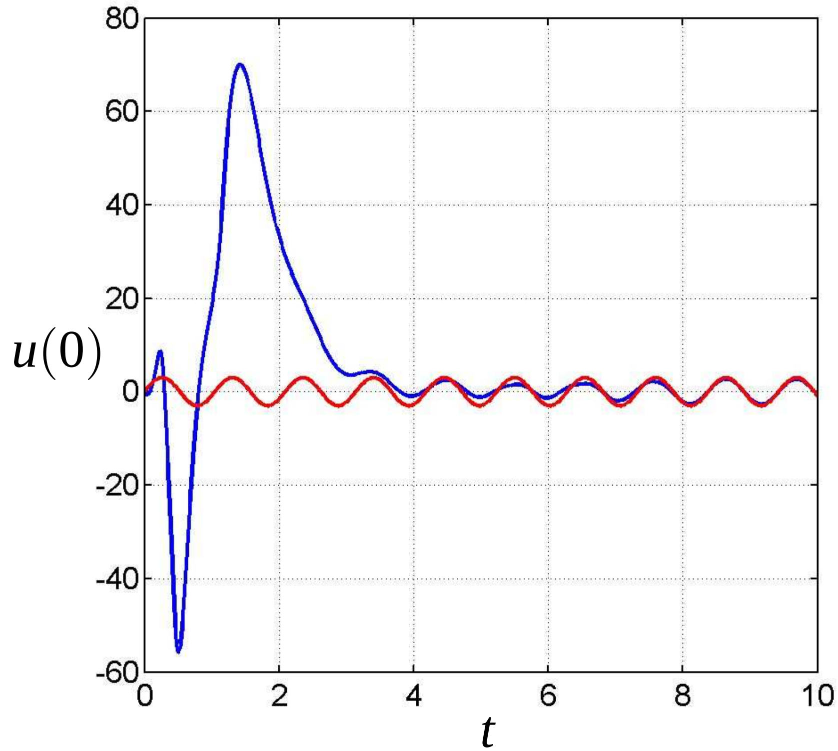
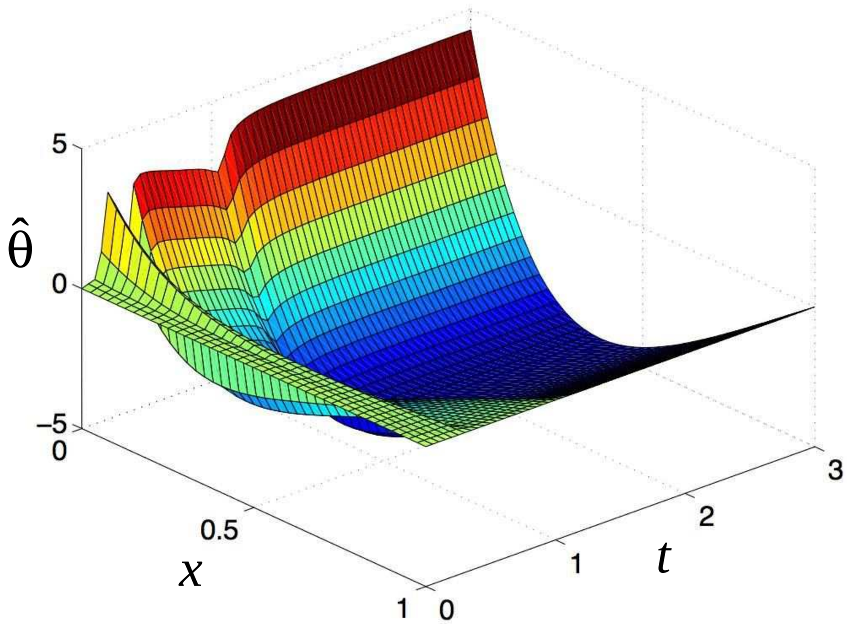


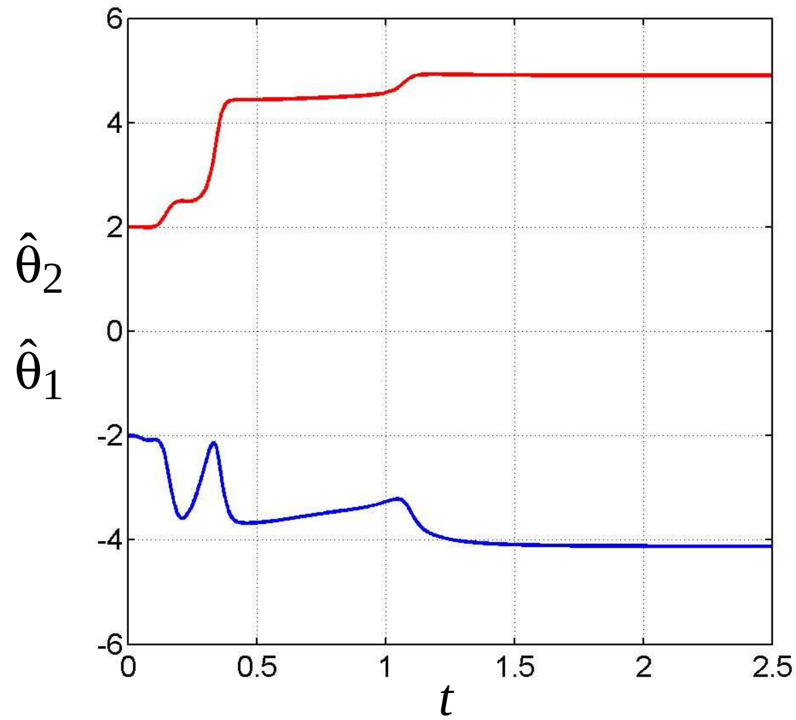
Control effort

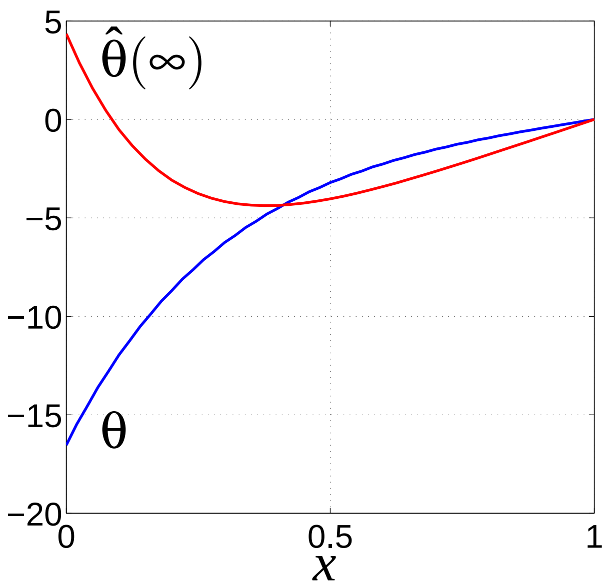


Output evolution

Simulation Results (parameter estimates)







Adaptive Nonlinear Control—A Tutorial

- Backstepping
- Tuning Functions Design
- Modular Design
- Output Feedback
- Extensions
- A Stochastic Example
- Applications and Additional References

main source:

Nonlinear and Adaptive Control Design (Wiley, 1995)

M. Krstić, I. Kanellakopoulos and P. V. Kokotović

Backstepping (nonadaptive)

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta\,, \qquad \varphi(0) = 0 \\
\dot{x}_2 &= u
\end{aligned}
$$

where θ is known parameter vector and $\varphi(x_1)$ is smooth nonlinear function.

Goal: stabilize the equilibrium $x_1 = 0$, $x_2 = -\varphi(0)^{\mathrm{T}}\theta = 0$.

virtual control for the x_1-equation:

$$
\alpha_1(x_1) = -c_1 x_1 - \varphi(x_1)^{\mathrm{T}}\theta\,, \qquad c_1 > 0
$$

error variables:

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 - \alpha_1(x_1)\,,
\end{aligned}
$$

System in error coordinates:

$$\begin{aligned}
\dot{z}_1 &= \dot{x}_1 = x_2 + \varphi^{\mathrm{T}}\theta = z_2 + \alpha_1 + \varphi^{\mathrm{T}}\theta = -c_1 z_1 + z_2 \\
\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 = u - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right).
\end{aligned}$$

Need to design $u = \alpha_2(x_1, x_2)$ to stabilize $z_1 = z_2 = 0$.

Choose Lyapunov function

$$V(x_1, x_2) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$$

we have

$$\begin{aligned}
\dot{V} &= z_1(-c_1 z_1 + z_2) + z_2\left[u - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right)\right] \\
&= -c_1 z_1^2 + z_2 \underbrace{\left[u + z_1 - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right)\right]}_{=-c_2 z_2}
\end{aligned}$$

$$\Rightarrow \dot{V} = -c_1 z_1^2 - c_2 z_2^2$$

$z = 0$ is globally asymptotically stable

invertible change of coordinates

$$\Downarrow$$

$x = 0$ is globally asymptotically stable

The closed-loop system in z-coordinates is linear:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Tuning Functions Design

Introductory examples:

A	**B**	**C**
$\dot{x}_1 = u + \varphi(x_1)^T\theta$	$\dot{x}_1 = x_2 + \varphi(x_1)^T\theta$ $\dot{x}_2 = u$	$\dot{x}_1 = x_2 + \varphi(x_1)^T\theta$ $\dot{x}_2 = x_3$ $\dot{x}_3 = u$

where θ is unknown parameter vector and $\varphi(0) = 0$.

Degin A. Let $\hat{\theta}$ be the estimate of θ and $\tilde{\theta} = \theta - \hat{\theta}$,

Using

$$u = -c_1 x_1 - \varphi(x_1)^T\hat{\theta}$$

gives

$$\dot{x}_1 = -c_1 x_1 + \varphi(x_1)^T\tilde{\theta}$$

To find update law for $\hat{\theta}(t)$, choose

$$V_1(x, \hat{\theta}) \;=\; \frac{1}{2}x_1^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}$$

then

$$\begin{aligned}\dot{V}_1 &= -c_1x_1^2 + x_1\varphi(x_1)^{\mathrm{T}}\tilde{\theta} - \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\hat{\theta}} \\ &= -c_1x_1^2 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\underbrace{\left(\Gamma\varphi(x_1)x_1 - \dot{\hat{\theta}}\right)}_{=0}\end{aligned}$$

Update law:

$$\dot{\hat{\theta}} \;=\; \Gamma\varphi(x_1)x_1, \qquad \varphi(x_1)\text{—regressor}$$

gives

$$\dot{V}_1 = -c_1x_1^2 \leq 0.$$

By Lasalle's invariance theorem, $x_1 = 0, \hat{\theta} = \theta$ is stable and

$$\lim_{t\to\infty} x_1(t) = 0$$

Design B. replace θ by $\hat{\theta}$ in the nonadaptive design:

$$z_2 = x_2 - \alpha_1(x_1,\hat{\theta}), \qquad \alpha_1(x_1,\hat{\theta}) = -c_1 z_1 - \varphi^T\hat{\theta}$$

and strengthen the control law by $\nu_2(x_1,x_2,\hat{\theta})$(to be designed)

$$u = \alpha_2(x_1,x_2,\hat{\theta}) = -c_2 z_2 - z_1 + \frac{\partial\alpha_1}{\partial x_1}\left(x_2 + \varphi^T\hat{\theta}\right) + \nu_2(x_1,x_2,\hat{\theta})$$

error system

$$\begin{aligned}
\dot{z}_1 &= z_2 + \alpha_1 + \varphi^T\theta = -c_1 z_1 + z_2 + \varphi^T\tilde{\theta} \\
\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial\alpha_1}{\partial x_1}\left(x_2 + \varphi^T\theta\right) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} \\
&= -z_1 - c_2 z_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi^T\tilde{\theta} - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} + \nu_2(x_1,x_2,\hat{\theta}),
\end{aligned}$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi^T \end{bmatrix}\tilde{\theta} + \underbrace{\begin{bmatrix} 0 \\ -\frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}} + \nu_2(x_1,x_2,\hat{\theta}) \end{bmatrix}}_{=0}$$

remaining: design adaptive law.

Choose

$$V_2(x_1, x_2, \hat{\theta}) = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}$$

we have

$$\begin{aligned}
\dot{V}_2 &= -c_1 z_1^2 - c_2 z_2^2 + [z_1,\ z_2]\begin{bmatrix} \varphi^{\mathrm{T}} \\ -\frac{\partial \alpha_1}{\partial x_1}\varphi^{\mathrm{T}} \end{bmatrix}\tilde{\theta} - \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\hat{\theta}} \\
&= -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\left(\Gamma\left[\varphi,\ -\frac{\partial \alpha_1}{\partial x_1}\varphi\right]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \dot{\hat{\theta}}\right).
\end{aligned}$$

The choice

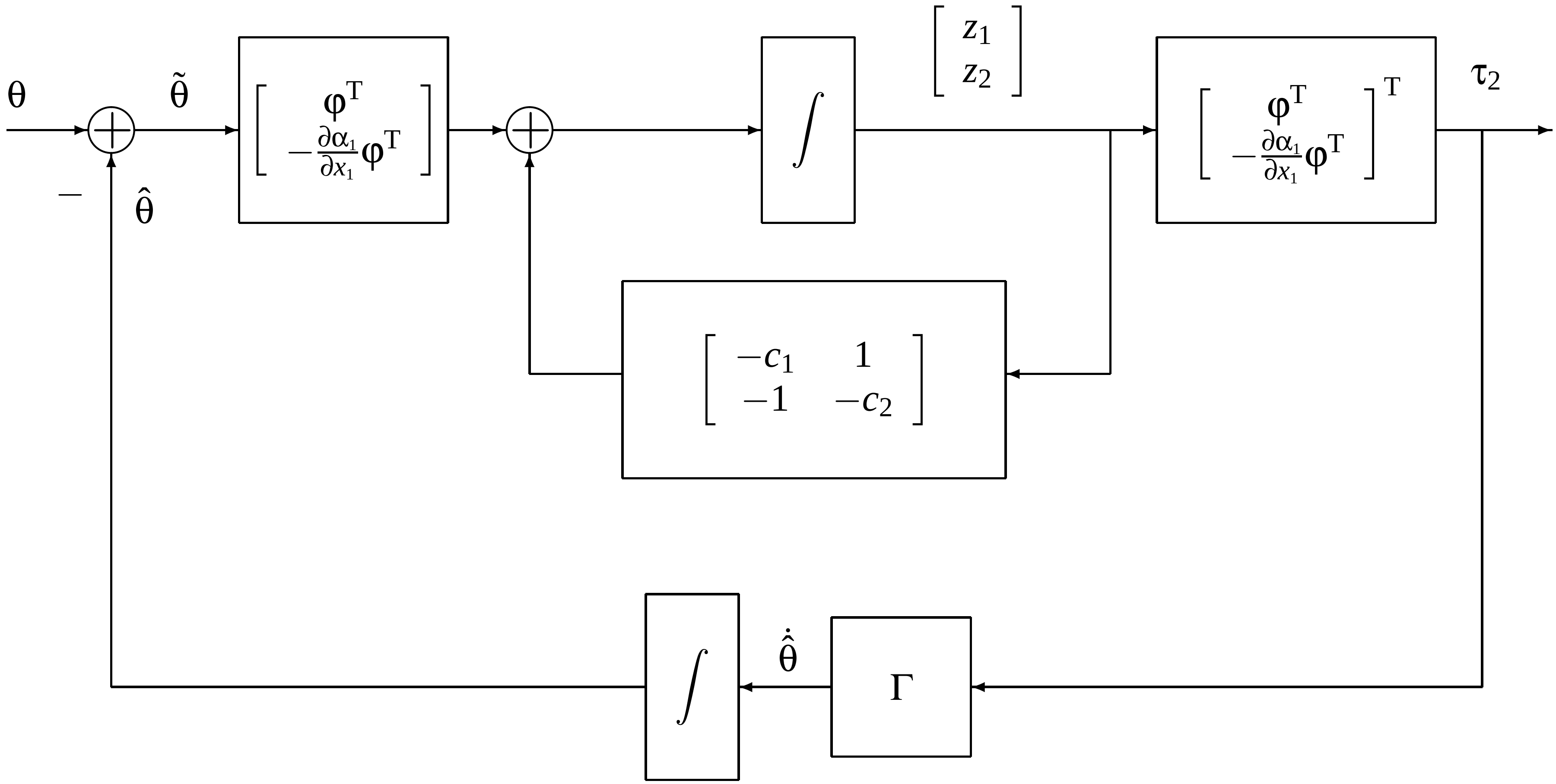
$$\dot{\hat{\theta}} = \Gamma\tau_2(x, \hat{\theta}) = \Gamma\left[\varphi,\ -\frac{\partial \alpha_1}{\partial x_1}\varphi\right]\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \Gamma\underbrace{\left(\overbrace{\varphi z_1}^{\tau_1} - \frac{\partial \alpha_1}{\partial x_1}\varphi z_2\right)}_{\tau_2}$$

($\tau_1,\ \tau_2$ are called tuning functions)

makes

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2,$$

thus $z = 0,\ \tilde{\theta} = 0$ is GS and $x(t) \to 0$ as $t \to \infty$.



The closed-loop adaptive system

Design C.

We have one more integrator, so we define the third error coordinate and replace $\dot{\hat{\theta}}$ in design B by potential update law,

$$z_3 \; = \; x_3 - \alpha_2(x_1, x_2, \hat{\theta})$$

$$\nu_2(x_1, x_2, \hat{\theta}) \; = \; \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2(x_1, x_2, \hat{\theta}).$$

Now the z_1, z_2-system is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ z_3 + \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) \end{bmatrix}$$

and

$$\dot{V}_2 \; = \; -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_2 - \Gamma^{-1} \dot{\hat{\theta}}).$$

z_3-equation is given by

$$\begin{aligned}\dot{z}_3 &= u - \frac{\partial\alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right) - \frac{\partial\alpha_2}{\partial x_2}x_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ &= u - \frac{\partial\alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right) - \frac{\partial\alpha_2}{\partial x_2}x_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_2}{\partial x_1}\varphi^{\mathrm{T}}\tilde{\theta}\,.\end{aligned}$$

Choose

$$V_3(x,\hat{\theta}) = V_2 + \frac{1}{2}z_3^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}$$

we have

$$\begin{aligned}\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) \\ &\quad + z_3\left[z_2 + u - \frac{\partial\alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right) - \frac{\partial\alpha_2}{\partial x_2}x_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}}\right] \\ &\quad + \tilde{\theta}^{\mathrm{T}}\left(\tau_2 - \frac{\partial\alpha_2}{\partial x_1}\varphi z_3 - \Gamma^{-1}\dot{\hat{\theta}}\right).\end{aligned}$$

Pick update law

$$\dot{\hat{\theta}} = \Gamma\tau_3(x_1,x_2,x_3,\hat{\theta}) \;=\; \Gamma\left(\tau_2 - \frac{\partial\alpha_2}{\partial x_1}\varphi z_3\right) = \Gamma\left[\varphi, \frac{\partial\alpha_1}{\partial x_1}\varphi,\; -\frac{\partial\alpha_2}{\partial x_1}\varphi\right]\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

and control law

$$u = \alpha_3(x_1,x_2,x_3,\hat{\theta}) \;=\; -z_2 - c_3 z_3 + \frac{\partial\alpha_2}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right) + \frac{\partial\alpha_2}{\partial x_2}x_3 + \nu_3,$$

results in

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_2\frac{\partial\alpha_1}{\partial\hat{\theta}}(\Gamma\tau_2 - \dot{\hat{\theta}}) + z_3\left(\nu_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\dot{\hat{\theta}}\right).$$

Notice

$$\dot{\hat{\theta}} - \Gamma\tau_2 = \dot{\hat{\theta}} - \Gamma\tau_3 - \Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi z_3$$

we have

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3\underbrace{\left(\nu_3 - \frac{\partial\alpha_2}{\partial\hat{\theta}}\Gamma\tau_3 + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi z_2\right)}_{=0}.$$

Stability and regulation of x to zero follows.

Further insight:

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi^T \\ -\frac{\partial\alpha_2}{\partial x_1}\varphi^T \end{bmatrix} \tilde\theta + \begin{bmatrix} 0 \\ \frac{\partial\alpha_1}{\partial\hat\theta}(\Gamma\tau_2 - \dot{\hat\theta}) \\ \nu_3 - \frac{\partial\alpha_2}{\partial\hat\theta}\Gamma\tau_3 \end{bmatrix}.$$

$$\Downarrow \dot{\hat\theta} - \Gamma\tau_2 = \dot{\hat\theta} - \Gamma\tau_3 - \Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi z_3$$

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 + \frac{\partial\alpha_1}{\partial\hat\theta}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi^T \\ -\frac{\partial\alpha_2}{\partial x_1}\varphi^T \end{bmatrix} \tilde\theta + \begin{bmatrix} 0 \\ 0 \\ \nu_3 - \frac{\partial\alpha_2}{\partial\hat\theta}\Gamma\tau_3 \end{bmatrix}$$

$\Downarrow$ seletion of ν_3

$$\begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 + \frac{\partial\alpha_1}{\partial\hat\theta}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi \\ 0 & -1 - \frac{\partial\alpha_1}{\partial\hat\theta}\Gamma\frac{\partial\alpha_2}{\partial x_1}\varphi & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial\alpha_1}{\partial x_1}\varphi^T \\ -\frac{\partial\alpha_2}{\partial x_1}\varphi^T \end{bmatrix} \tilde\theta.$$

General Recursive Design Procedure

parametric strict-feedback system:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1(x_1)^{\mathrm{T}}\theta \\
\dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^{\mathrm{T}}\theta \\
&\vdots \\
\dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \ldots, x_{n-1})^{\mathrm{T}}\theta \\
\dot{x}_n &= \beta(x)u + \varphi_n(x)^{\mathrm{T}}\theta \\
y &= x_1
\end{aligned}$$

where β and φ_i are smooth.

Objective: asymptotically track reference output $y_r(t)$, with $y_r^{(i)}(t), i = 1, \cdots, n$ known, bounded and piecewise continuous.

Tuning functions design for tracking ($z_0 \triangleq 0$, $\alpha_0 \triangleq 0$, $\tau_0 \triangleq 0$)

$$
\begin{aligned}
z_i &= x_i - y_{\mathrm{r}}^{(i-1)} - \alpha_{i-1} \\
\alpha_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-1)}) &= -z_{i-1} - c_i z_i - w_i^{\mathrm{T}}\hat{\theta} + \sum_{k=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial x_k}x_{k+1} + \frac{\partial\alpha_{i-1}}{\partial y_{\mathrm{r}}^{(k-1)}}y_{\mathrm{r}}^{(k)}\right) + \nu_i \\
\nu_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-1)}) &= +\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\Gamma\tau_i + \sum_{k=2}^{i-1}\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\Gamma w_i z_k \\
\tau_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-1)}) &= \tau_{i-1} + w_i z_i \\
w_i(\bar{x}_i, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(i-2)}) &= \varphi_i - \sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_k}\varphi_k
\end{aligned}
$$

$$i = 1, \ldots, n$$

$$\bar{x}_i = (x_1, \ldots, x_i), \qquad \bar{y}_{\mathrm{r}}^{(i)} = (y_{\mathrm{r}}, \dot{y}_{\mathrm{r}}, \ldots, y_{\mathrm{r}}^{(i)})$$

Adaptive control law:

$$u = \frac{1}{\beta(x)}\left[\alpha_n(x, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(n-1)}) + y_{\mathrm{r}}^{(n)}\right]$$

Parameter update law:

$$\dot{\hat{\theta}} = \Gamma\tau_n(x, \hat{\theta}, \bar{y}_{\mathrm{r}}^{(n-1)}) = \Gamma W z$$

Closed-loop system

$$\begin{aligned} \dot{z} &= A_z(z,\hat{\theta},t)z + W(z,\hat{\theta},t)^{\mathrm{T}}\tilde{\theta} \\ \dot{\hat{\theta}} &= \Gamma W(z,\hat{\theta},t)z\,, \end{aligned}$$

where

$$A_z(z,\hat{\theta},t) = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1+\sigma_{23} & \cdots & \sigma_{2n} \\ 0 & -1-\sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1+\sigma_{n-1,n} \\ 0 & -\sigma_{2n} & \cdots & -1-\sigma_{n-1,n} & -c_n \end{bmatrix}$$

$$\sigma_{jk}(x,\hat{\theta}) = -\frac{\partial\alpha_{j-1}}{\partial\hat{\theta}}\Gamma w_k$$

This structure ensures that the Lyapunov function

$$V_n = \frac{1}{2}z^{\mathrm{T}}z + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}$$

has derivative

$$\dot{V}_n = -\sum_{k=1}^{n} c_k z_k^2.$$

Modular Design

Motivation: Controller can be combined with different identifiers. (No flexibility for update law in tuning function design)

Naive idea: connect a good identifier and a good controller.

Example: error system

$$\dot{x} = -x + \varphi(x)\tilde{\theta}$$

suppose $\tilde{\theta}(t) = \mathrm{e}^{-t}$ and $\varphi(x) = x^3$, we have

$$\dot{x} = -x + x^3 \mathrm{e}^{-t}$$

But, when $|x_0| > \sqrt{\frac{3}{2}}$,

$$x(t) \to \infty \quad \text{as} \quad t \to \frac{1}{3} \ln \frac{x_0^2}{x_0^2 - 3/2}$$

Conclusion: Need stronger controller.

Controller Design. nonlinear damping

$$u = -x - \varphi(x)\hat{\theta} - \varphi(x)^2 x$$

closed-loop system

$$\dot{x} = -x - \varphi(x)^2 x + \varphi(x)\tilde{\theta}\,.$$

With $V = \frac{1}{2}x^2$, we have

$$\begin{aligned} \dot{V} &= -x^2 - \varphi(x)^2 x^2 + x\varphi(x)\tilde{\theta} \\ &= -x^2 - \left[\varphi(x)x - \frac{1}{2}\tilde{\theta}\right]^2 + \frac{1}{4}\tilde{\theta}^2 \\ &\le -x^2 + \frac{1}{4}\tilde{\theta}^2\,. \end{aligned}$$

bounded $\tilde{\theta}(t) \Rightarrow$ bounded $x(t)$

For higher order system

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi(x_1)^{\mathrm{T}}\theta \\ \dot{x}_2 &= u\end{aligned}$$

set

$$\alpha_1(x_1, \hat{\theta}) = -c_1 x_1 - \varphi(x_1)^{\mathrm{T}}\hat{\theta} - \kappa_1 |\varphi(x_1)|^2 x_1, \quad c_1, \kappa_1 > 0$$

and define

$$z_2 = x_2 - \alpha_1(x_1, \hat{\theta})$$

error system

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 - \kappa_1 |\varphi|^2 z_1 + \varphi^{\mathrm{T}}\tilde{\theta} + z_2 \\ \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right) - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}\,.\end{aligned}$$

Consider

$$V_2 = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}|z|^2$$

we have

$$\begin{aligned}
\dot{V}_2 &\le -c_1 z_1^2 + \frac{1}{4\kappa_1}|\tilde{\theta}|^2 + z_1 z_2 + z_2\left[u - \frac{\partial\alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\theta\right) - \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\right] \\
&\le -c_1 z_1^2 + \frac{1}{4\kappa_1}|\tilde{\theta}|^2 + z_2\left[u + z_1 - \frac{\partial\alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right) - \left(\frac{\partial\alpha_1}{\partial x_1}\varphi^{\mathrm{T}}\tilde{\theta} + \frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}\right)\right].
\end{aligned}$$

controller

$$u = -z_1 - c_2 z_2 - \kappa_2\left|\frac{\partial\alpha_1}{\partial x_1}\varphi\right|^2 z_2 - g_2\left|\frac{\partial\alpha_1}{\partial\hat{\theta}}^{\mathrm{T}}\right|^2 z_2 + \frac{\partial\alpha_1}{\partial x_1}\left(x_2 + \varphi^{\mathrm{T}}\hat{\theta}\right),$$

achieves

$$\dot{V}_2 \le -c_1 z_1^2 - c_2 z_2^2 + \left(\frac{1}{4\kappa_1} + \frac{1}{4\kappa_2}\right)|\tilde{\theta}|^2 + \frac{1}{4g_2}|\dot{\hat{\theta}}|^2$$

bounded $\tilde{\theta}$, bounded $\dot{\hat{\theta}}$(or $\in \mathcal{L}_2$) $\Rightarrow$ bounded $x(t)$

Controller design in the modular approach ($z_0 \triangleq 0$, $\alpha_0 \triangleq 0$)

$$
\begin{aligned}
z_i &= x_i - y_r^{(i-1)} - \alpha_{i-1} \\
\alpha_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) &= -z_{i-1} - c_i z_i - w_i^T \hat{\theta} + \sum_{k=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) - s_i z_i \\
w_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-2)}) &= \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k \\
s_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-2)}) &= \kappa_i |w_i|^2 + g_i \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}^T \right|^2
\end{aligned}
$$

$$i = 1, \ldots, n$$

$$\bar{x}_i = (x_1, \ldots, x_i), \qquad \bar{y}_r^{(i)} = (y_r, \dot{y}_r, \ldots, y_r^{(i)})$$

Adaptive control law:

$$u = \frac{1}{\beta(x)} \left[\alpha_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) + y_r^{(n)} \right]$$

Controller module guarantees:

If $\tilde{\theta} \in \mathcal{L}_\infty$ and $\dot{\hat{\theta}} \in \mathcal{L}_2$ or $\mathcal{L}_\infty$ then $x \in \mathcal{L}_\infty$

Requirement for identifier

error system

$$\dot{z} = A_z(z,\hat{\theta},t)z + W(z,\hat{\theta},t)^{\mathrm{T}}\tilde{\theta} + Q(z,\hat{\theta},t)^{\mathrm{T}}\dot{\hat{\theta}}$$

where

$$A_z(z,\hat{\theta},t) = \begin{bmatrix} -c_1-s_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2-s_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & -c_n-s_n \end{bmatrix}$$

$$W(z,\hat{\theta},t)^{\mathrm{T}} = \begin{bmatrix} w_1^{\mathrm{T}} \\ w_2^{\mathrm{T}} \\ \vdots \\ w_n^{\mathrm{T}} \end{bmatrix}, \qquad Q(z,\hat{\theta},t)^{\mathrm{T}} = \begin{bmatrix} 0 \\ -\frac{\partial\alpha_1}{\partial\hat{\theta}} \\ \vdots \\ -\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}} \end{bmatrix}.$$

Since

$$W(z,\hat{\theta},t)^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\partial\alpha_1}{\partial x_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\frac{\partial\alpha_{n-1}}{\partial x_1} & \cdots & -\frac{\partial\alpha_{n-1}}{\partial x_{n-1}} & 1 \end{bmatrix} F(x)^{\mathrm{T}} \stackrel{\triangle}{=} N(z,\hat{\theta},t)F(x)^{\mathrm{T}}.$$

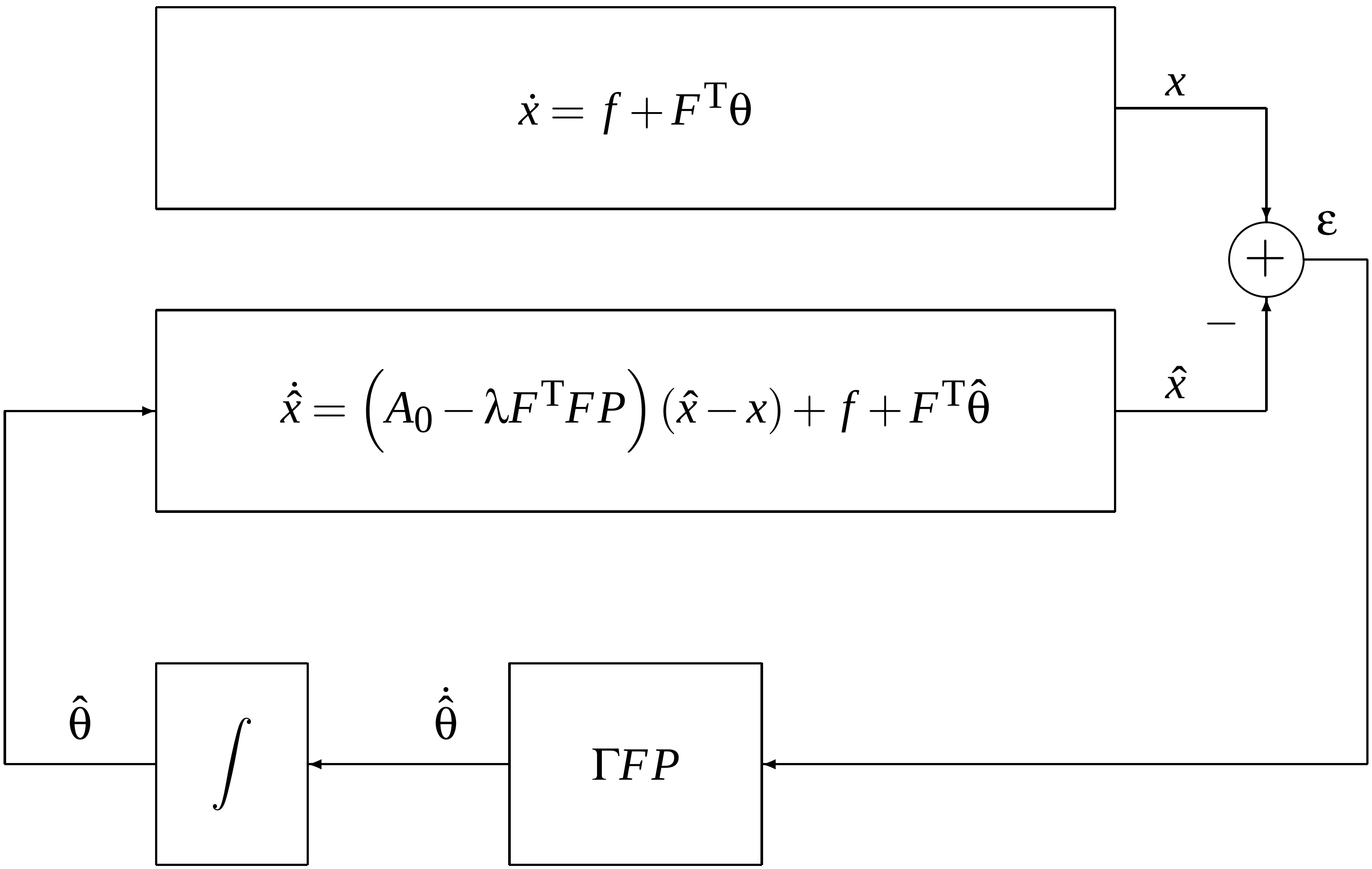
Identifier properties:

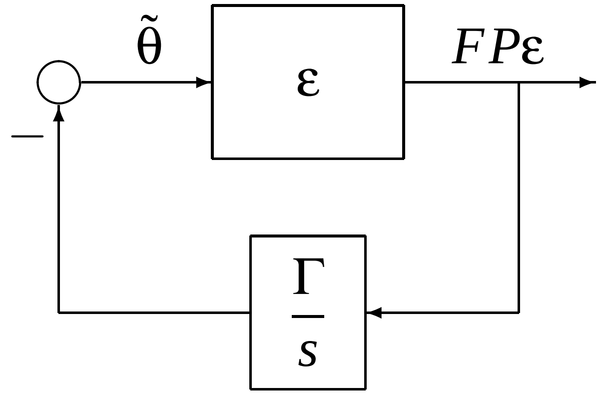
(i) $\tilde{\theta} \in \mathcal{L}_\infty$ and $\dot{\hat{\theta}} \in \mathcal{L}_2$ or $\mathcal{L}_\infty$,

(ii) if $x \in \mathcal{L}_\infty$ then $F(x(t))^{\mathrm{T}}\tilde{\theta}(t) \to 0$ and $\dot{\hat{\theta}}(t) \to 0$.

Identifier Design

Passive identifier





$$\dot{\varepsilon} = \left[A_0 - \lambda F(x,u)^{\mathrm{T}} F(x,u) P\right] \varepsilon + F(x,u)^{\mathrm{T}} \tilde{\theta}$$

update law

$$\dot{\hat{\theta}} = \Gamma F(x,u) P \varepsilon, \qquad \Gamma = \Gamma^{\mathrm{T}} > 0\,.$$

Use Lyapunov function

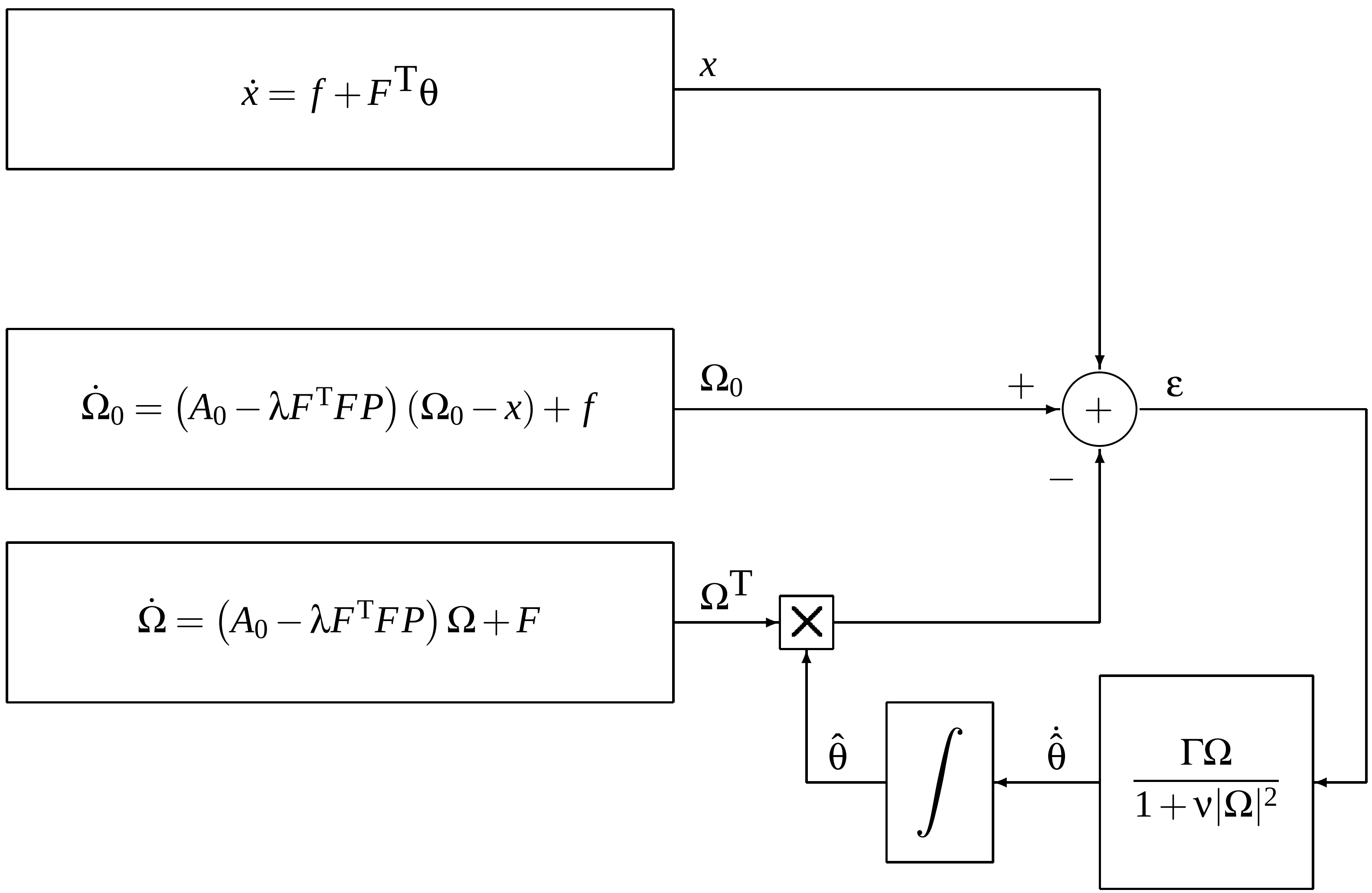
$$V = \tilde{\theta}^{\mathrm{T}} \Gamma^{-1} \tilde{\theta} + \varepsilon^{\mathrm{T}} P \varepsilon$$

its derivative satisfies

$$\dot{V} \leq -\varepsilon^{\mathrm{T}} \varepsilon - \frac{\lambda}{\lambda(\Gamma)^2} |\dot{\hat{\theta}}|^2\,.$$

Thus, whenever x is bounded, $F(x(t))^{\mathrm{T}} \tilde{\theta}(t) \to 0$ and $\dot{\hat{\theta}}(t) \to 0$.
($\dot{\varepsilon}(t) \to 0$ because $\int_0^\infty \dot{\varepsilon}(\tau) d\tau = -\varepsilon(0)$ exists, Barbalat's lemma...)

Swapping identifier



define $\tilde{\varepsilon} \stackrel{\triangle}{=} x + \Omega_0 - \Omega^{\mathrm{T}}\theta$,

$$\dot{\tilde{\varepsilon}} = \left[A_0 - \lambda F(x,u)^{\mathrm{T}} F(x,u) P\right] \tilde{\varepsilon}\,.$$

Choose

$$V = \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \tilde{\varepsilon}P\tilde{\varepsilon}$$

we have

$$\dot{V} \leq -\frac{3}{4}\frac{\varepsilon^{\mathrm{T}}\varepsilon}{1+\nu\mathrm{tr}\{\Omega^{\mathrm{T}}\Omega\}},$$

proves identifier properties.

Output Feedback Adaptive Designs

$$\begin{aligned} \dot{x} &= Ax + \phi(y) + \Phi(y)a + \begin{bmatrix} 0 \\ b \end{bmatrix} \sigma(y)u, \qquad x \in \mathbb{R}^n \\ y &= e_1^{\mathrm{T}} x, \end{aligned}$$

$$A = \begin{bmatrix} 0 & \\ \vdots & I_{n-1} \\ 0 & \cdots \; 0 \end{bmatrix},$$

$$\phi(y) = \begin{bmatrix} \varphi_{0,1}(y) \\ \vdots \\ \varphi_{0,n}(y) \end{bmatrix}, \qquad \Phi(y) = \begin{bmatrix} \varphi_{1,1}(y) & \cdots & \varphi_{q,1}(y) \\ \vdots & & \vdots \\ \varphi_{1,n}(y) & \cdots & \varphi_{q,n}(y) \end{bmatrix},$$

unknown constant parameters:

$$a = [a_1, \ldots, a_q]^{\mathrm{T}}, \quad b = [b_m, \ldots, b_0]^{\mathrm{T}}.$$

State estimation filters

Filters:

$$
\begin{aligned}
\dot{\xi} &= A_0\xi + ky + \phi(y) \\
\dot{\Xi} &= A_0\Xi + \Phi(y) \\
\dot{\lambda} &= A_0\lambda + e_n\sigma(y)u
\end{aligned}
$$

$$
\begin{aligned}
v_j &= A_0^j\lambda, \qquad j = 0,\ldots,m \\
\Omega^{\mathrm{T}} &= [v_m,\ldots,v_1,v_0,\ \Xi]
\end{aligned}
$$

Parameter-dependent state estimate

$$\hat{x} = \xi + \Omega^{\mathrm{T}} \theta$$

The vector $k = [k_1, \ldots, k_n]^{\mathrm{T}}$ chosen so that the matrix

$$A_0 = A - k e_1^{\mathrm{T}}$$

is Hurwitz, that is,

$$P A_0 + A_0^{\mathrm{T}} P = -I, \quad P = P^{\mathrm{T}} > 0$$

The state estimation error

$$\varepsilon = x - \hat{x}$$

satisfies

$$\dot{\varepsilon} = A_0 \varepsilon$$

Parametric model for adaptation:

$$\begin{aligned}\dot{y} &= \omega_0 + \omega^{\mathrm{T}}\theta + \varepsilon_2 \\ &= b_m v_{m,2} + \omega_0 + \bar{\omega}^{\mathrm{T}}\theta + \varepsilon_2\,,\end{aligned}$$

where

$$\begin{aligned}\omega_0 &= \varphi_{0,1} + \xi_2 \\ \omega &= \left[v_{m,2}, v_{m-1,2}, \ldots, v_{0,2},\ \Phi_{(1)} + \Xi_{(2)}\right]^{\mathrm{T}} \\ \bar{\omega} &= \left[0, v_{m-1,2}, \ldots, v_{0,2},\ \Phi_{(1)} + \Xi_{(2)}\right]^{\mathrm{T}}.\end{aligned}$$

Since the states $x_2, \ldots, x_n$ are not measured, the backstepping design is applied to the system

$$
\begin{aligned}
\dot{y} &= b_m v_{m,2} + \omega_0 + \bar{\omega}^{\mathrm{T}}\theta + \varepsilon_2 \\
\dot{v}_{m,i} &= v_{m,i+1} - k_i v_{m,1}\,, & i = 2, \ldots, \rho - 1 \\
\dot{v}_{m,\rho} &= \sigma(y)u + v_{m,\rho+1} - k_\rho v_{m,1}\,.
\end{aligned}
$$

The order of this system is equal to the relative degree of the plant.

Extensions

Pure-feedback systems.

$$
\begin{aligned}
\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \ldots, x_{i+1})^{\mathrm{T}}\theta, \qquad i = 1, \ldots, n-1 \\
\dot{x}_n &= \Big(\beta_0(x) + \beta(x)^{\mathrm{T}}\theta\Big) u + \varphi_0(x) + \varphi_n(x)^{\mathrm{T}}\theta\,,
\end{aligned}
$$

where $\varphi_0(0) = 0,\ \varphi_1(0) = \cdots = \varphi_n(0) = 0,\ \beta_0(0) \neq 0$.

Because of the dependence of φ_i on x_{i+1}, the regulation or tracking for pure-feedback systems is, in general, not global, even when θ is known.

Unknown virtual control coefficients.

$$\begin{aligned}\dot{x}_i &= b_i x_{i+1} + \varphi_i(x_1,\ldots,x_i)^{\mathrm{T}}\theta, \qquad i = 1,\ldots,n-1 \\ \dot{x}_n &= b_n \beta(x) u + \varphi_n(x_1,\ldots,x_n)^{\mathrm{T}}\theta\,,\end{aligned}$$

where, in addition to the unknown vector θ, the constant coefficients b_i are also unknown.

The unknown b_i-coefficients are frequent in applications ranging from electric motors to flight dynamics. The signs of $b_i,\ i = 1,\ldots,n$, are assumed to be known. In the tuning functions design, in addition to estimating b_i, we also estimate its inverse $\rho_i = 1/b_i$. In the modular design we assume that in addition to $\mathrm{sgn} b_i$, a positive constant ς_i is known such that $|b_i| \geq \varsigma_i$. Then, instead of estimating $\rho_i = 1/b_i$, we use the inverse of the estimate $\hat{b}_i$, i.e., $1/\hat{b}_i$, where $\hat{b}_i(t)$ is kept away from zero by using parameter projection.

Multi-input systems.

$$\begin{aligned}\dot{X}_i &= B_i(\overline{X}_i)X_{i+1} + \Phi_i(\overline{X}_i)^{\mathrm{T}}\theta, \qquad i = 1,\ldots,n-1 \\ \dot{X}_n &= B_n(X)u + \Phi_n(X)^{\mathrm{T}}\theta,\end{aligned}$$

where X_i is a ν_i-vector, $\nu_1 \le \nu_2 \le \cdots \le \nu_n$, $\overline{X}_i = \left[X_1^{\mathrm{T}},\ldots,X_i^{\mathrm{T}}\right]^{\mathrm{T}}$, $X = \overline{X}_n$, and the matrices $B_i(\overline{X}_i)$ have full rank for all $\overline{X}_i \in \mathbb{R}^{\sum_{j=1}^{i}\nu_j}$. The input u is a ν_n-vector.

The matrices B_i can be allowed to be unknown provided they are constant and positive definite.

Block strict-feedback systems.

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1,\ldots,x_i,\zeta_1,\ldots,\zeta_i)^{\mathrm{T}}\theta, \qquad i = 1,\ldots,\rho-1 \\ \dot{x}_\rho &= \beta(x,\zeta)u + \varphi_\rho(x,\zeta)^{\mathrm{T}}\theta \\ \dot{\zeta}_i &= \Phi_{i,0}(\bar{x}_i,\bar{\zeta}_i) + \Phi_i(\bar{x}_i,\bar{\zeta}_i)^{\mathrm{T}}\theta, \qquad i = 1,\ldots,\rho\end{aligned}$$

with the following notation: $\bar{x}_i = [x_1,\ldots,x_i]^{\mathrm{T}}$, $\bar{\zeta}_i = \left[\zeta_1^{\mathrm{T}},\ldots,\zeta_i^{\mathrm{T}}\right]^{\mathrm{T}}$, $x = \bar{x}_\rho$, and $\zeta = \bar{\zeta}_\rho$.

Each ζ_i-subsystem is assumed to be bounded-input bounded-state (BIBS) stable with respect to the input $(\bar{x}_i,\bar{\zeta}_{i-1})$. For this class of systems it is quite simple to modify the procedure in the tables. Because of the dependence of φ_i on $\bar{\zeta}_i$, the stabilizing function α_i is augmented by the term $+\sum_{k=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\zeta_k}\Phi_{k,0}$, and the regressor w_i is augmented by $-\sum_{k=1}^{i-1}\Phi_i\left(\frac{\partial\alpha_{i-1}}{\partial\zeta_k}\right)^{\mathrm{T}}$.

Partial state-feedback systems. In many physical systems there are unmeasured states as in the output-feedback form, but there are also states other than the output $y = x_1$ that are measured. An example of such a system is

$$\begin{aligned}
\dot{x}_1 &= x_2 + \varphi_1(x_1)^{\mathrm{T}}\theta \\
\dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^{\mathrm{T}}\theta \\
\dot{x}_3 &= x_4 + \varphi_3(x_1, x_2)^{\mathrm{T}}\theta \\
\dot{x}_4 &= x_5 + \varphi_4(x_1, x_2)^{\mathrm{T}}\theta \\
\dot{x}_5 &= u + \varphi_5(x_1, x_2, x_5)^{\mathrm{T}}\theta \,.
\end{aligned}$$

The states x_3 and x_4 are assumed not to be measured. To apply the adaptive backstepping designs presented in this chapter, we combine the state-feedback techniques with the output-feedback techniques. The subsystem (x_2, x_3, x_4) is in the output-feedback form with x_2 as a measured output, so we employ a state estimator for (x_2, x_3, x_4) using the filters introduced in the section on output feedback.

Example of Adaptive Stabilization in the Presence of a Stochastic Disturbance

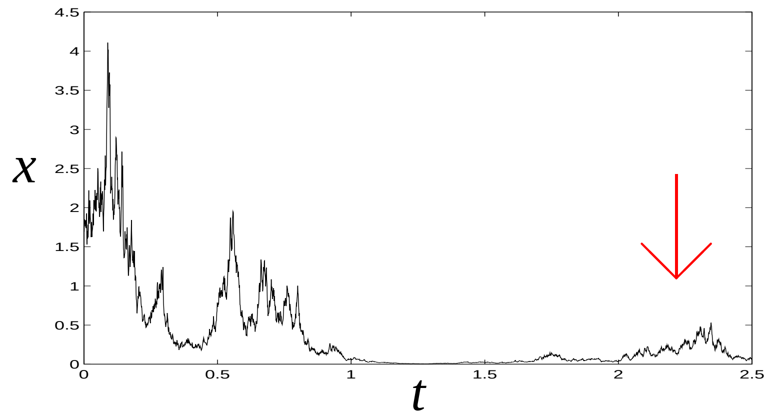
$$dx = udt + xdw$$

w: Wiener process with $E\left\{dw^2\right\} = \sigma(t)^2 dt$, no a priori bound for σ

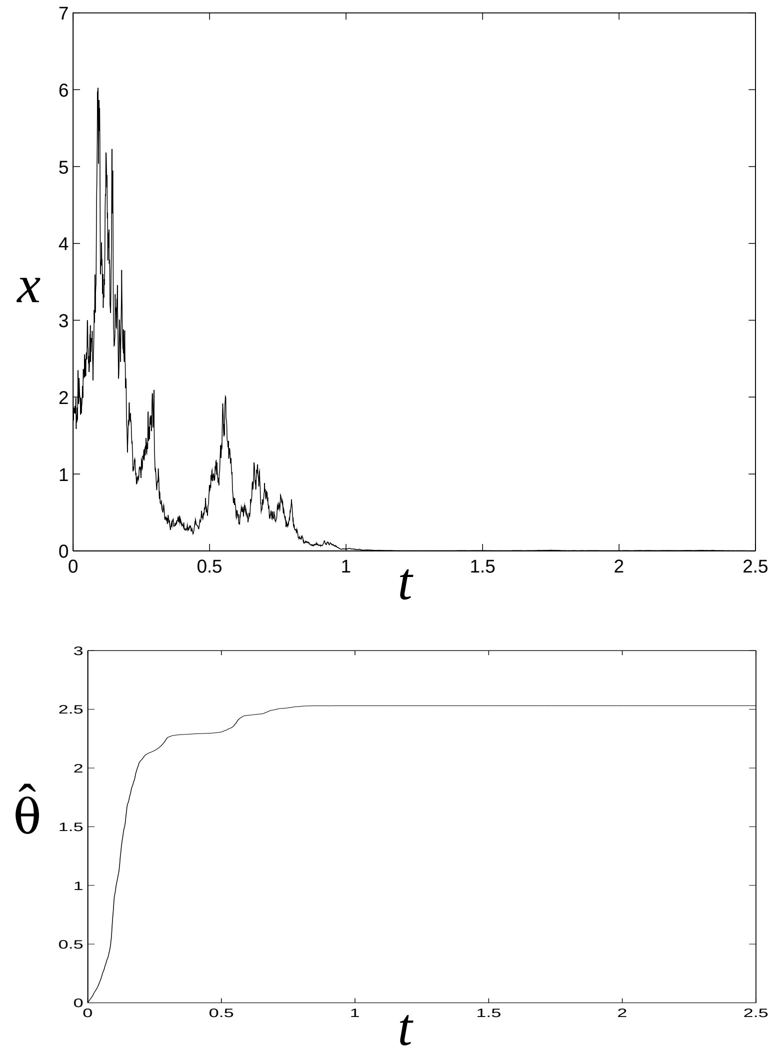
Control laws:

$$\begin{aligned} \text{Disturbance Attenuation:} \quad u &= -x - x^3 \\ \text{Adaptive Stabilization:} \quad u &= -x - \hat{\theta}x, \qquad \dot{\hat{\theta}} = x^2 \end{aligned}$$

Disturbance Attenuation



Adaptive Stabilization



Major Applications of Adaptive Nonlinear Control

- **Electric Motors Actuating Robotic Loads**

  Nonlinear Control of Electric Machinery, Dawson, Hu, Burg, 1998.
- **Marine Vehicles** (ships, UUVs; dynamic positioning, way point tracking, maneuvering)

  Marine Control Systems, Fossen, 2002
- **Automotive Vehicles** (lateral and longitudinal control, traction, overall dynamics)

  The groups of Tomizuka and Kanellakopoulos.

Dozens of other occasional applications, including: aircraft wing rock, compressor stall and surge, satellite attitude control.

Other Books on Adaptive NL Control Theory Inspired by [KKK]

1. Marino and Tomei (1995),
   Nonlinear Control Design: Geometric, Adaptive, and Robust

2. Freeman and Kokotovic (1996),
   Robust Nonlinear Control Design: State Space and Lyapunov Techniques

3. Qu (1998),
   Robust Control of Nonlinear Uncertain Systems

4. Krstic and Deng (1998),
   Stabilization of Nonlinear Uncertain Systems

5. Ge, Hang, Lee, Zhang (2001),
   Stable Adaptive Neural Network Control

6. Spooner, Maggiore, Ordonez, and Passino (2002),
   Stable Adaptive Control and Estimation for Nonlinear Systems: Neural and Fuzzy Approximation Techniques

7. French, Szepesvari, Rogers (2003),
   Performance of Nonlinear Approximate Adaptive Controllers

Adaptive NL Control/Backstepping Coverage in Major Texts

1. Khalil (1995/2002),
   Nonlinear Systems

2. Isidori (1995),
   Nonlinear Control Systems

3. Sastry (1999),
   Nonlinear Systems: Analysis, Stability, and Control

4. Astrom and Wittenmark (1995),
   Adaptive Control