

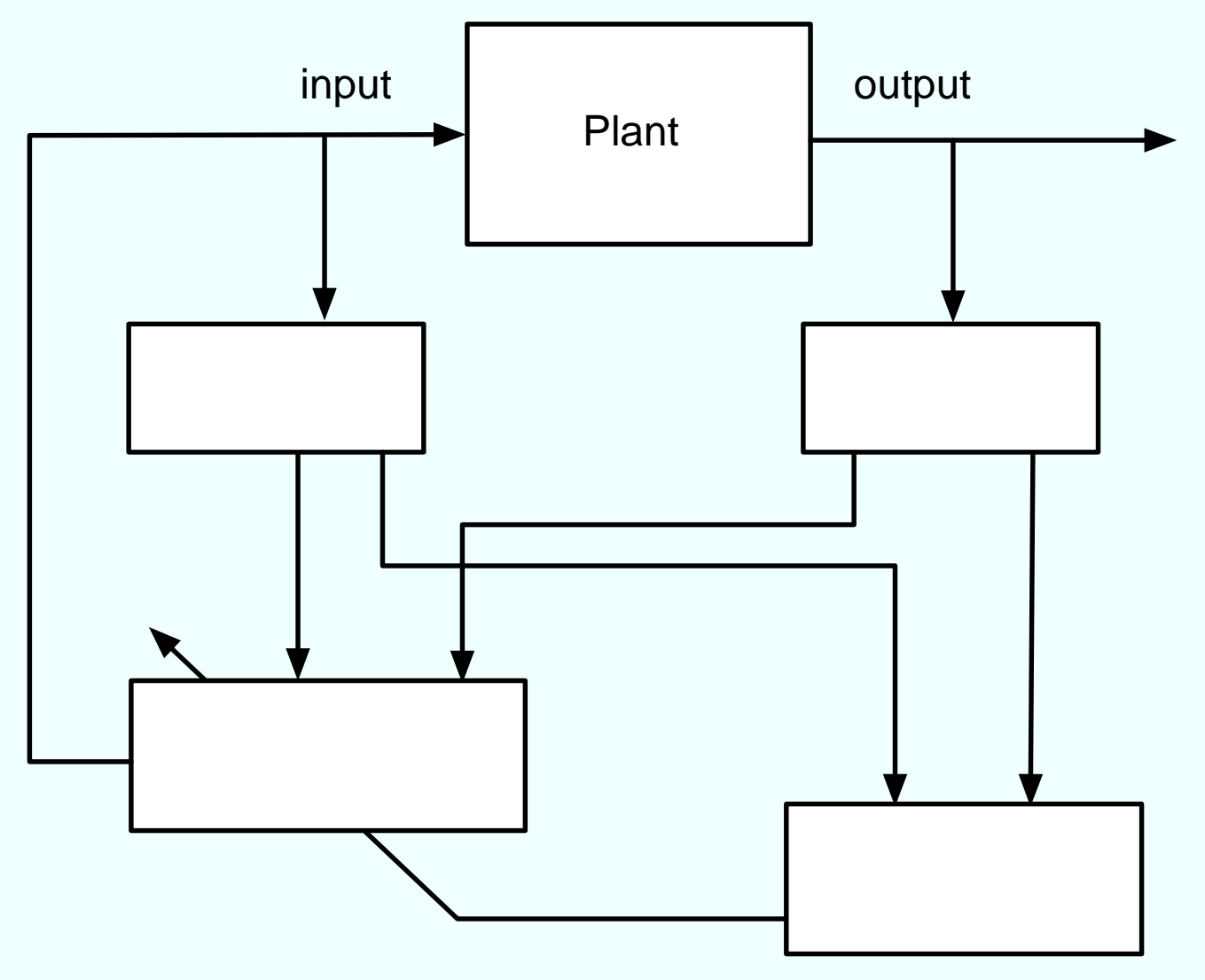
# **Adaptive Control of PDEs and Nonlinear Systems**

**Miroslav Krstic**

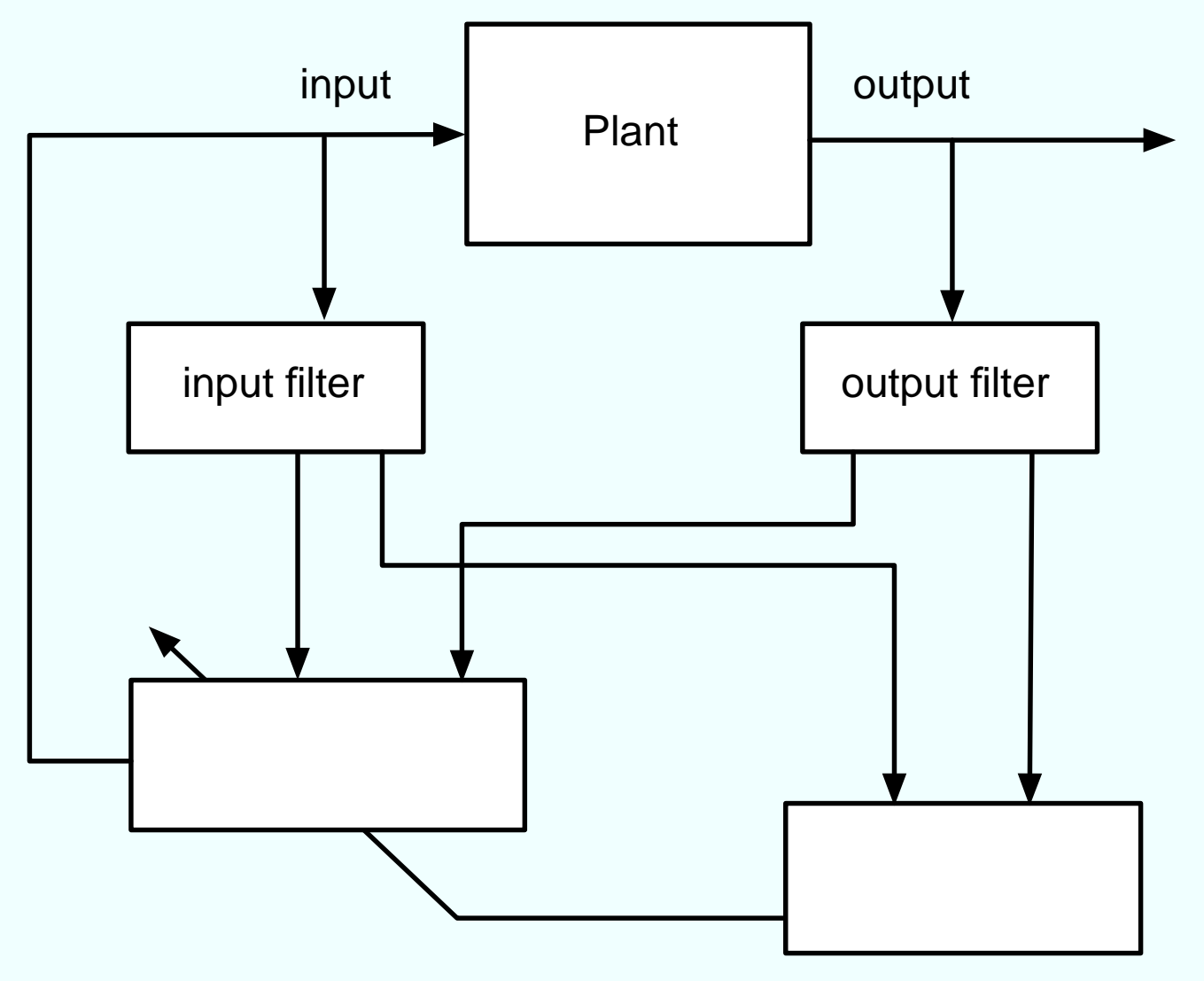
University of California, San Diego

NSF-DOE Fusion Control Workshop, General Atomics, 2006

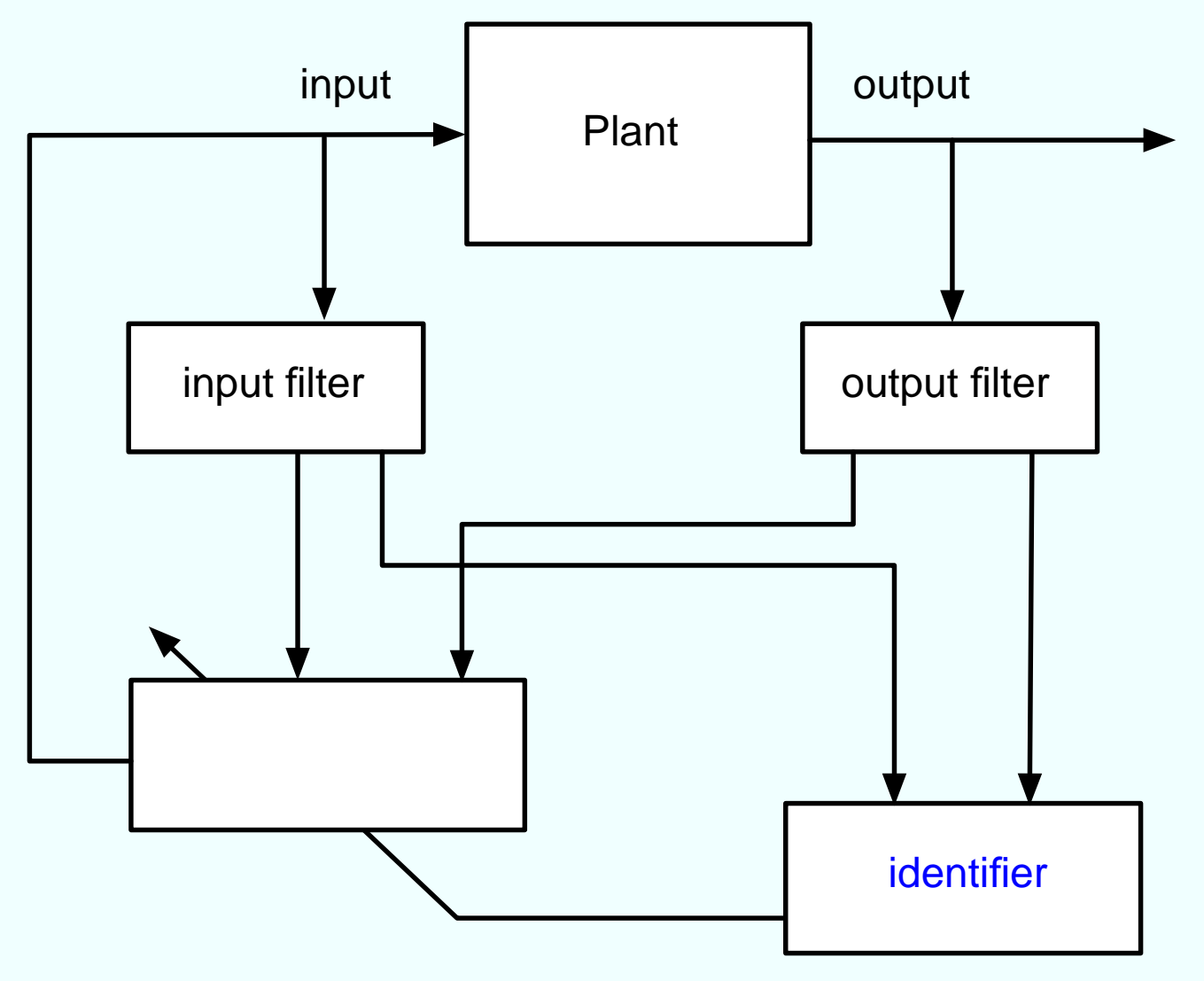
# Adaptive Control



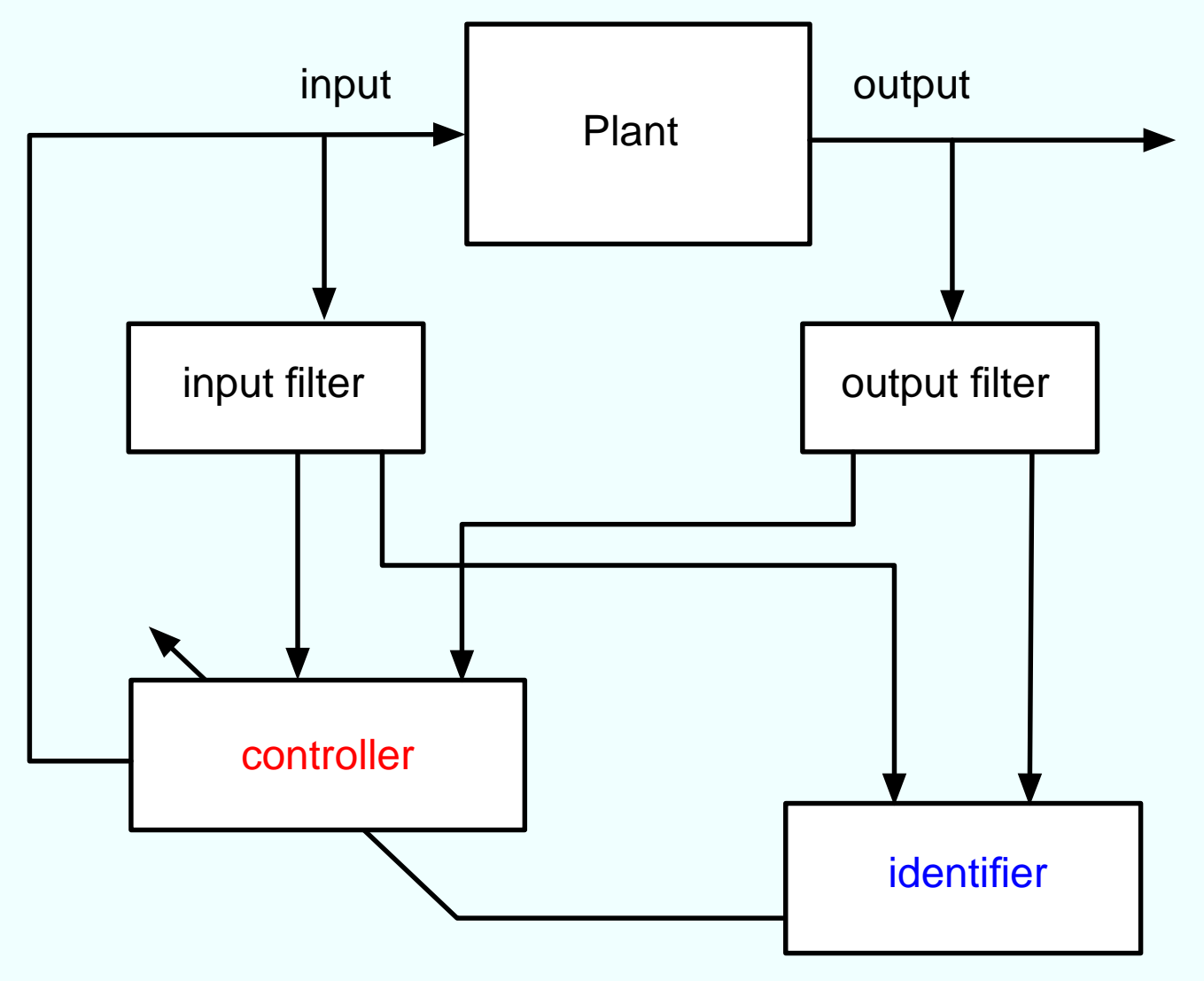
# Adaptive Control



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# Plant

$$y = \frac{B(s)}{A(s)}u$$

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Unknown parameter vector

$$\theta = [b_m \ b_{m-1} \ \dots \ b_1 \ b_0 \ a_n \ a_{n-1} \ \dots \ a_1 \ a_0]^T$$



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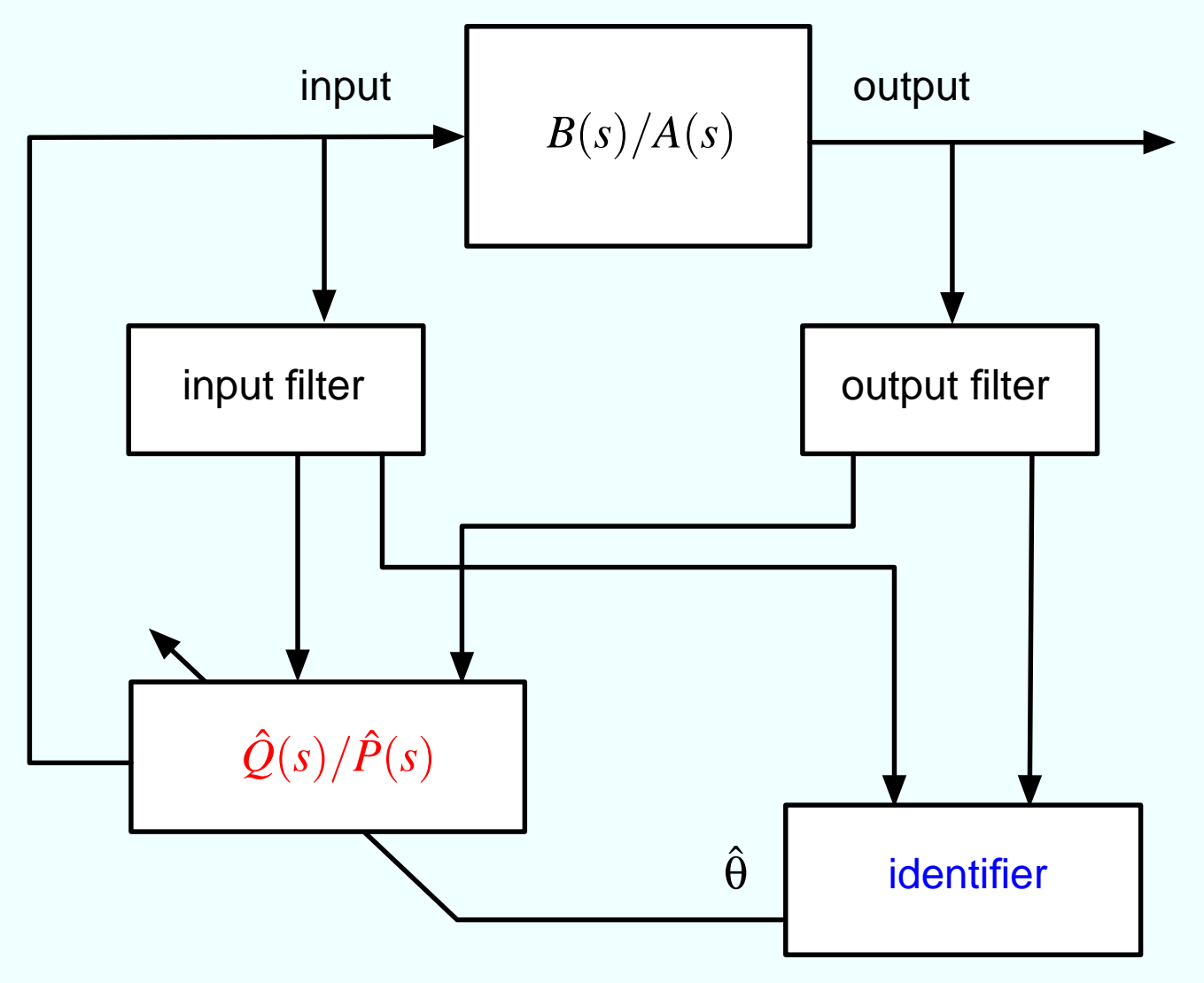
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So, the coefficients of  $P(s)$  and  $Q(s)$  at each time step are determined from the estimate  $\hat{\theta}(t)$  of  $\theta$  at each time step.

# Adaptive Control



# Approaches to identifier design

- Lyapunov

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- Lyapunov
- Estimation based/Certainty equivalence
  - with **passive** identifier (often called “observer-based” method)
  - with **swapping** identifier (often called the “gradient” method)

## PDE with unknown functional parameter

$$u_t = u_{xx} + \lambda(x)u$$



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Measurement:  $u(0)$

Control:  $u(1)$

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**Measurement:**  $u(0)$

**Control:**  $u(1)$

- **Unstable**
- “Infinitely many” unknown parameters / infinite-dimensional state
- **Scalar** input / **scalar** output

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and  $\theta(x)$ ,  $\theta_1$  are related to  $\lambda(x)$  through the solution of the PDE

$$p_{xx}(x, y) = p_{yy}(x, y) + \lambda(y)p(x, y)$$

$$p(1, y) = 0$$

$$p(x, x) = \frac{1}{2} \int_x^1 \lambda(y) dy$$

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$$\theta(x) = -p_y(x, 0)$$

$$\theta_1 = -p(0, 0)$$

**Compensator:**

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$$P(s) = \cosh \sqrt{s} - \int_0^1 k(1-y) \cosh(\sqrt{s}y) dy$$
$$Q(s) = \int_0^1 k(y) \left[ -\theta_1 \frac{\sinh(\sqrt{s}y)}{\sqrt{s}} + \frac{\sinh(\sqrt{s}y)}{\sqrt{s}} \int_0^{1-y} \theta(\xi) \cosh(\sqrt{s}\xi) d\xi \right] dy$$
$$+ \int_0^1 k(y) \cosh(\sqrt{s}(1-y)) \int_{1-y}^1 \theta(\xi) \frac{\sinh(\sqrt{s}(1-\xi))}{\sqrt{s}} d\xi dy$$



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$$+ \int_0^1 k(y) \cosh(\sqrt{s}(1-y)) \int_{1-y}^1 \theta(\xi) \frac{\sinh(\sqrt{s}(1-\xi))}{\sqrt{s}} d\xi dy$$

and

$$k(x) = \theta_1 - \int_0^x \theta(y) dy - \int_0^x \left[ \theta_1 - \int_0^{x-y} \theta(s) ds \right] k(y) dy$$

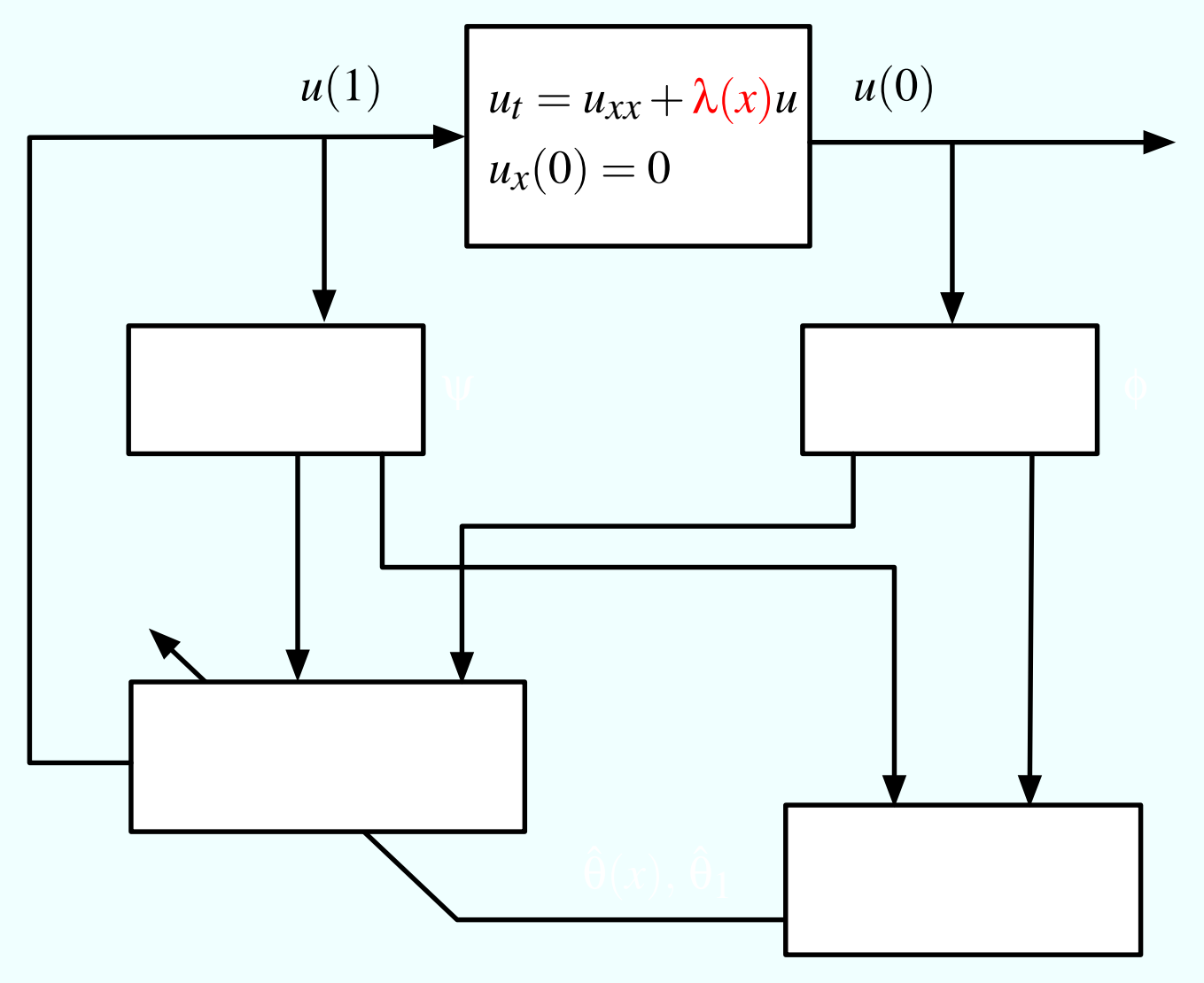
## Input filter

$$\begin{aligned}\psi_t &= \psi_{xx} \\ \psi_x(0) &= 0 \\ \psi(1) &= u(1)\end{aligned}$$

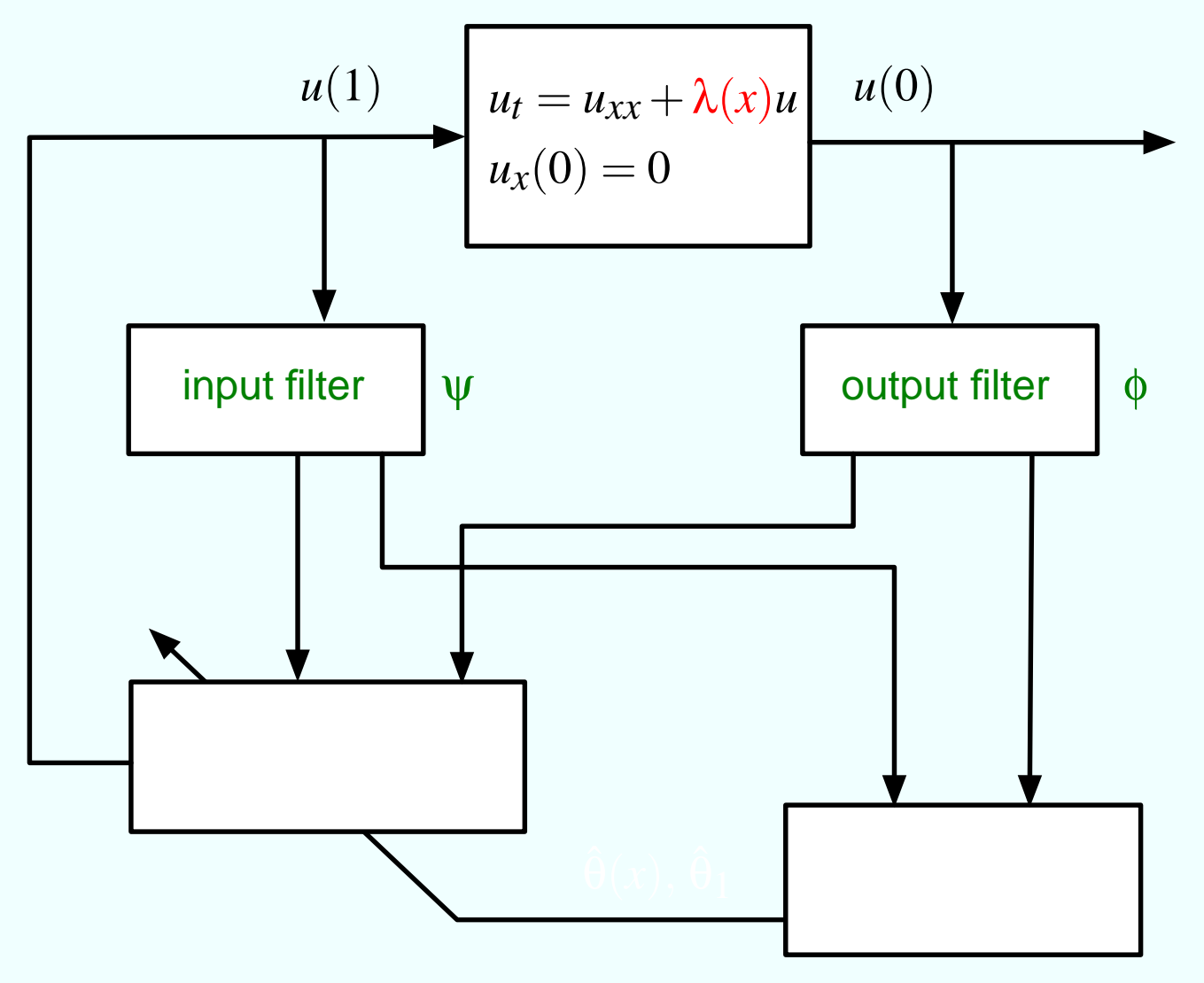
## Output filters

$$\begin{aligned}\phi_t &= \phi_{xx} \\ \phi_x(0) &= u(0) \\ \phi(1) &= 0\end{aligned}$$

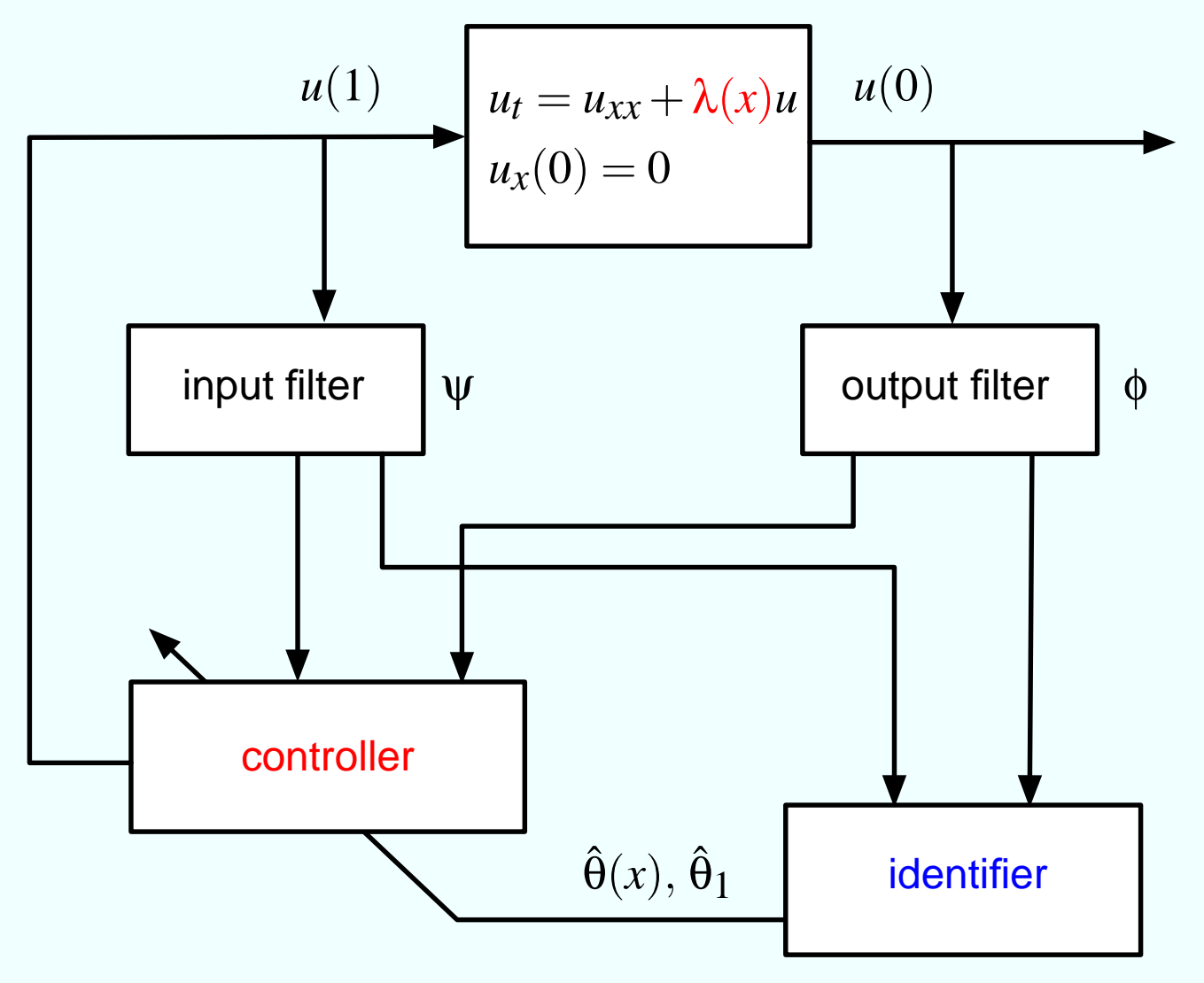
# Adaptive scheme



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## Update laws (least squares)

$$\hat{\theta}_t(\mathbf{x}, t) = \frac{\int_0^1 \gamma(\mathbf{x}, y, t) \phi(y) dy + \gamma_0(\mathbf{x}, t) \phi(0)}{1 + \|\phi\|^2 + \phi^2(0)} \left( v(0) - \psi(0) - \hat{\theta}_1 \phi(0) + \int_0^1 \hat{\theta}(\xi) \phi(\xi) d\xi \right)$$

$$\dot{\hat{\theta}}_1 = \frac{\int_0^1 \gamma_0(y, t) \phi(y) dy + \gamma_1(t) \phi(0)}{1 + \|\phi\|^2 + \phi^2(0)} \left( v(0) - \psi(0) - \hat{\theta}_1 \phi(0) + \int_0^1 \hat{\theta}(\xi) \phi(\xi) d\xi \right)$$

## Riccati adaptation gains

$$\gamma_t(\mathbf{x}, \mathbf{y}, t) = \frac{\int_0^1 \gamma(\mathbf{x}, s) \phi(s) ds \int_0^1 \gamma(\mathbf{y}, s) \phi(s) ds + \gamma_0(\mathbf{x}) \gamma_0(\mathbf{y}) \phi^2(0)}{1 + \|\phi\|^2 + \phi^2(0)} \\ - \frac{\phi(0) \gamma_0(\mathbf{y}) \int_0^1 \gamma(\mathbf{x}, s) \phi(s) ds + \phi(0) \gamma_0(\mathbf{x}) \int_0^1 \gamma(\mathbf{y}, s) \phi(s) ds}{1 + \|\phi\|^2 + \phi^2(0)}$$

$$\dot{\gamma}_0(\mathbf{x}) = \frac{\left( \int_0^1 \gamma(\mathbf{x}, s) \phi(s) ds + \gamma_0(\mathbf{x}) \phi(0) \right) \left( \int_0^1 \gamma_0(s) \phi(s) ds + \gamma_1 \phi(0) \right)}{1 + \|\phi\|^2 + \phi^2(0)}$$

$$\dot{\gamma}_1 = \frac{\left( \int_0^1 \gamma_0(s) \phi(s) ds + \gamma_1 \phi(0) \right)^2}{1 + \|\phi\|^2 + \phi^2(0)}$$

# Adaptive Controller

$$u(1) = \int_0^1 \hat{k}(1-y)\hat{u}(y)dy$$

where  $\hat{u}(y)$  is the “adaptive observer”

$$\hat{u}(y) = \psi(y) + \hat{\theta}_1 \phi(y) - 2 \sum_{n=0}^{\infty} \cos \frac{\pi(2n+1)y}{2} \left( \int_0^1 \cos \frac{\pi(2n+1)\xi}{2} \hat{\theta}(\xi) d\xi \right) \left( \int_0^1 \cos \frac{\pi(2n+1)\eta}{2} \phi(\eta) d\eta \right)$$

and  $\hat{k}(x)$  is the control gain given by the integral equation in **one** variable

$$\hat{k}(x) = \hat{\theta}_1 - \int_0^x \hat{\theta}(y) dy - \int_0^x \left[ \hat{\theta}_1 - \int_0^{x-y} \hat{\theta}(s) ds \right] \hat{k}(y) dy$$



## Simulation Example

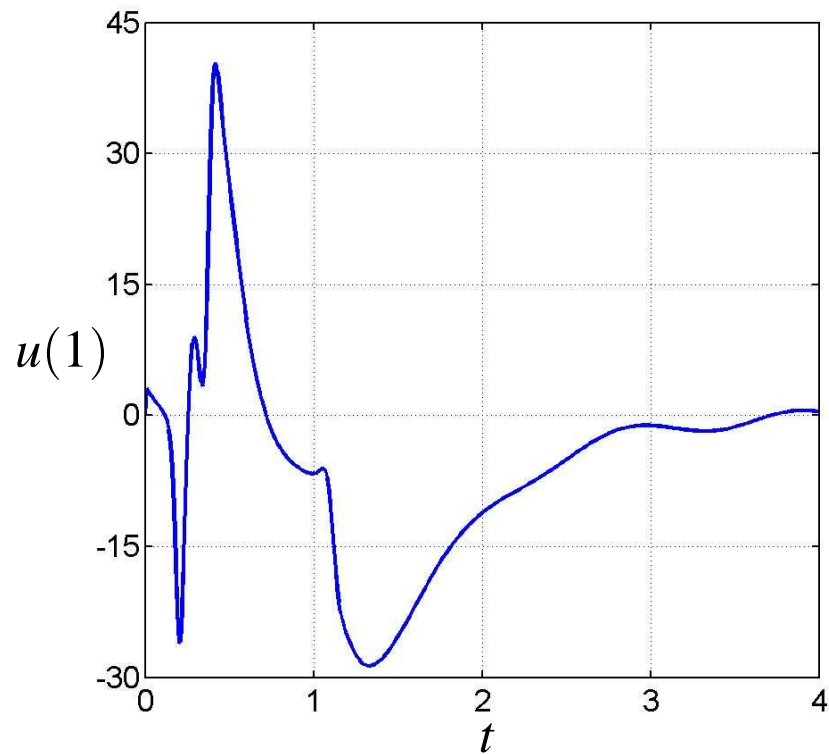
$$\begin{aligned}u_t &= u_{xx} + b(x)u_x + \lambda(x)u \\u_x(0) &= 0\end{aligned}$$

Reference signal:  $u^r(0, t) = 3 \sin 6t$      $b(x) = 3 - 2x^2$      $\lambda(x) = 16 + 3 \sin(2\pi x)$

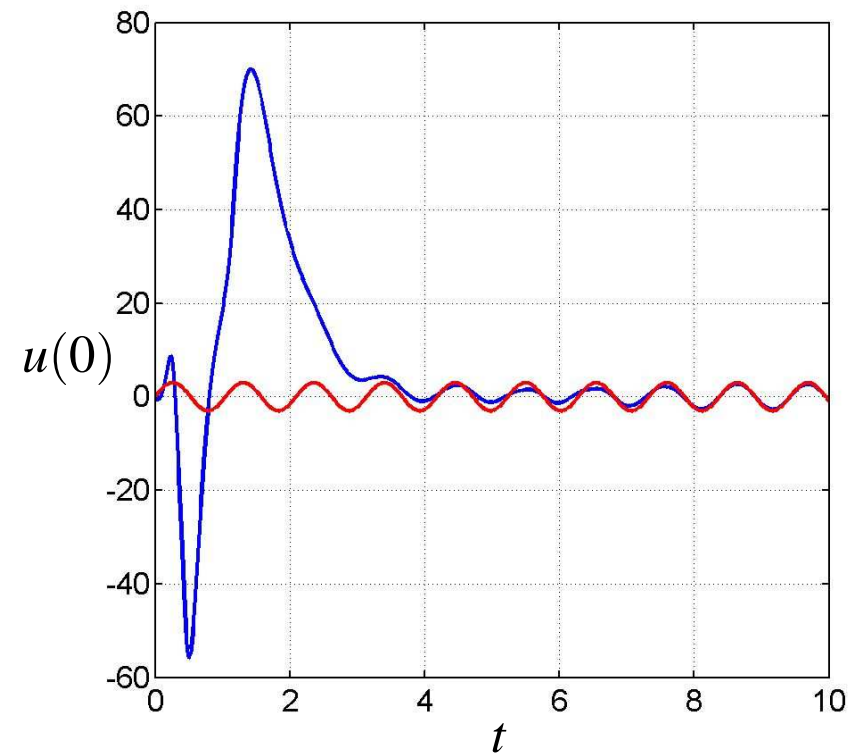
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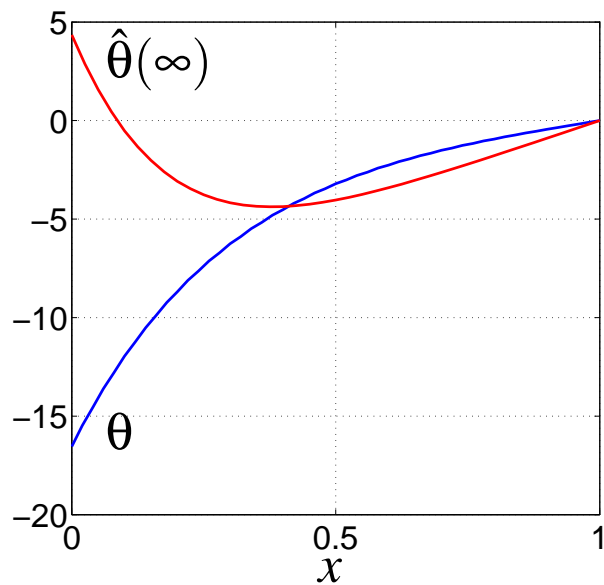
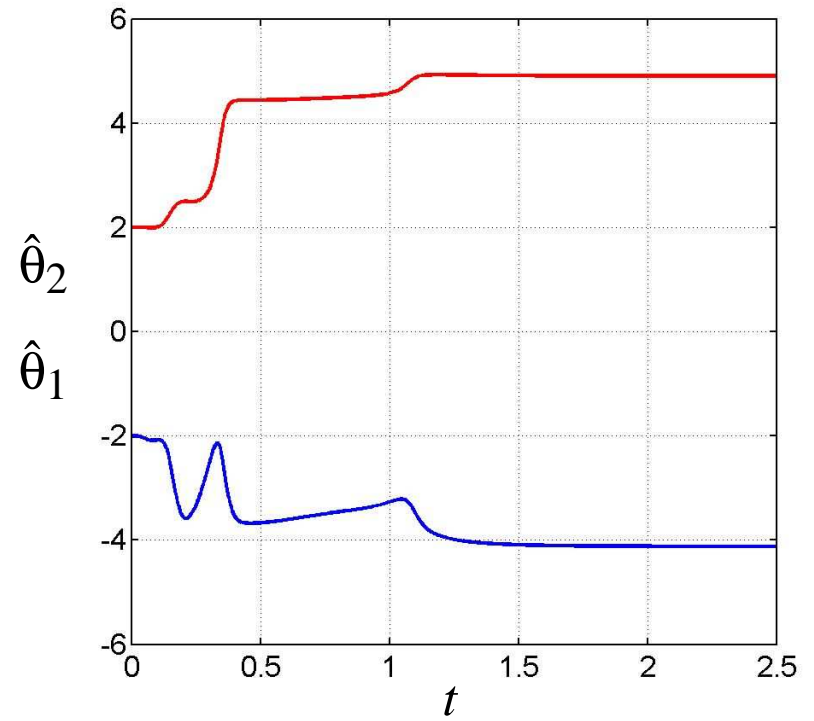
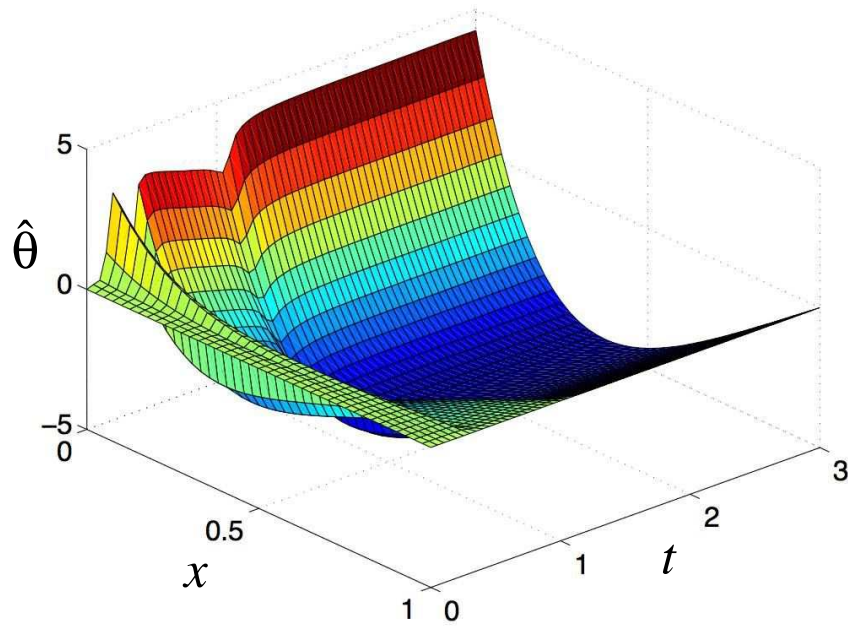


Control effort



Output evolution

# Simulation Results (parameter estimates)



# Adaptive Nonlinear Control—A Tutorial

- Backstepping
- Tuning Functions Design
- Modular Design
- Output Feedback
- Extensions
- A Stochastic Example
- Applications and Additional References

main source:

*Nonlinear and Adaptive Control Design* (Wiley, 1995)

M. Krstić, I. Kanellakopoulos and P. V. Kokotović

## Backstepping (nonadaptive)

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi(x_1)^T \theta, & \varphi(0) &= 0 \\ \dot{x}_2 &= u\end{aligned}$$

where  $\theta$  is **known** parameter vector and  $\varphi(x_1)$  is smooth nonlinear function.

**Goal:** stabilize the equilibrium  $x_1 = 0, x_2 = -\varphi(0)^T \theta = 0$ .

**virtual control** for the  $x_1$ -equation:

$$\alpha_1(x_1) = -c_1 x_1 - \varphi(x_1)^T \theta, \quad c_1 > 0$$

**error variables:**

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= x_2 - \alpha_1(x_1),\end{aligned}$$

System in error coordinates:

$$\begin{aligned}\dot{z}_1 &= \dot{x}_1 = x_2 + \boldsymbol{\varphi}^T \boldsymbol{\theta} = z_2 + \alpha_1 + \boldsymbol{\varphi}^T \boldsymbol{\theta} = -c_1 z_1 + z_2 \\ \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 = u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \boldsymbol{\varphi}^T \boldsymbol{\theta}).\end{aligned}$$

Need to design  $u = \alpha_2(x_1, x_2)$  to stabilize  $z_1 = z_2 = 0$ .

Choose Lyapunov function

$$V(x_1, x_2) = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2$$

we have

$$\begin{aligned}\dot{V} &= z_1 (-c_1 z_1 + z_2) + z_2 \left[ u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \boldsymbol{\varphi}^T \boldsymbol{\theta}) \right] \\ &= -c_1 z_1^2 + z_2 \underbrace{\left[ u + z_1 - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \boldsymbol{\varphi}^T \boldsymbol{\theta}) \right]}_{=-c_2 z_2} \\ &\Rightarrow \dot{V} = -c_1 z_1^2 - c_2 z_2^2\end{aligned}$$

$z = 0$  is globally asymptotically stable

invertible change of coordinates



$x = 0$  is globally asymptotically stable

The closed-loop system in  $z$ -coordinates is linear:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

## Tuning Functions Design

Introductory examples:

A	B	C
$\dot{x}_1 = u + \varphi(x_1)^T \theta$	$\dot{x}_1 = x_2 + \varphi(x_1)^T \theta$	$\dot{x}_1 = x_2 + \varphi(x_1)^T \theta$
	$\dot{x}_2 = u$	$\dot{x}_2 = x_3$
		$\dot{x}_3 = u$

where  $\theta$  is **unknown** parameter vector and  $\varphi(0) = 0$ .

**Design A.** Let  $\hat{\theta}$  be the estimate of  $\theta$  and  $\tilde{\theta} = \theta - \hat{\theta}$ ,

Using

$$u = -c_1 x_1 - \varphi(x_1)^T \hat{\theta}$$

gives

$$\dot{x}_1 = -c_1 x_1 + \varphi(x_1)^T \tilde{\theta}$$



To find update law for  $\hat{\theta}(t)$ , choose

$$V_1(x, \hat{\theta}) = \frac{1}{2}x_1^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta}$$

then

$$\begin{aligned}\dot{V}_1 &= -c_1x_1^2 + x_1\varphi(x_1)^T\tilde{\theta} - \tilde{\theta}^T\Gamma^{-1}\dot{\hat{\theta}} \\ &= -c_1x_1^2 + \underbrace{\tilde{\theta}^T\Gamma^{-1}\left(\Gamma\varphi(x_1)x_1 - \dot{\hat{\theta}}\right)}_{=0}\end{aligned}$$

Update law:

$$\dot{\hat{\theta}} = \Gamma\varphi(x_1)x_1, \quad \varphi(x_1)\text{—regressor}$$

gives

$$\dot{V}_1 = -c_1x_1^2 \leq 0.$$

By Lasalle's invariance theorem,  $x_1 = 0, \hat{\theta} = \theta$  is stable and

$$\lim_{t \rightarrow \infty} x_1(t) = 0$$

**Design B.** replace  $\theta$  by  $\hat{\theta}$  in the nonadaptive design:

$$z_2 = x_2 - \alpha_1(x_1, \hat{\theta}), \quad \alpha_1(x_1, \hat{\theta}) = -c_1 z_1 - \varphi^T \hat{\theta}$$

and strengthen the control law by  $v_2(x_1, x_2, \hat{\theta})$  (to be designed)

$$u = \alpha_2(x_1, x_2, \hat{\theta}) = -c_2 z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \hat{\theta}) + v_2(x_1, x_2, \hat{\theta})$$

error system

$$\begin{aligned} \dot{z}_1 &= \dot{x}_2 + \dot{\alpha}_1 + \varphi^T \dot{\theta} = -c_1 z_1 + z_2 + \varphi^T \tilde{\theta} \\ \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \varphi^T \theta) - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} \\ &= -z_1 - c_2 z_2 - \frac{\partial \alpha_1}{\partial x_1} \varphi^T \tilde{\theta} - \frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} + v_2(x_1, x_2, \hat{\theta}), \end{aligned}$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \end{bmatrix} \tilde{\theta} + \underbrace{\begin{bmatrix} 0 \\ -\frac{\partial \alpha_1}{\partial \hat{\theta}} \dot{\hat{\theta}} + v_2(x_1, x_2, \hat{\theta}) \end{bmatrix}}_{=0}$$

remaining: design adaptive law.

Choose

$$V_2(x_1, x_2, \hat{\theta}) = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

we have

$$\begin{aligned} \dot{V}_2 &= -c_1 z_1^2 - c_2 z_2^2 + [z_1, z_2] \begin{bmatrix} \phi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \phi^T \end{bmatrix} \tilde{\theta} - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \\ &= -c_1 z_1^2 - c_2 z_2^2 + \tilde{\theta}^T \Gamma^{-1} \left( \Gamma \begin{bmatrix} \phi, & -\frac{\partial \alpha_1}{\partial x_1} \phi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - \dot{\hat{\theta}} \right). \end{aligned}$$

The choice

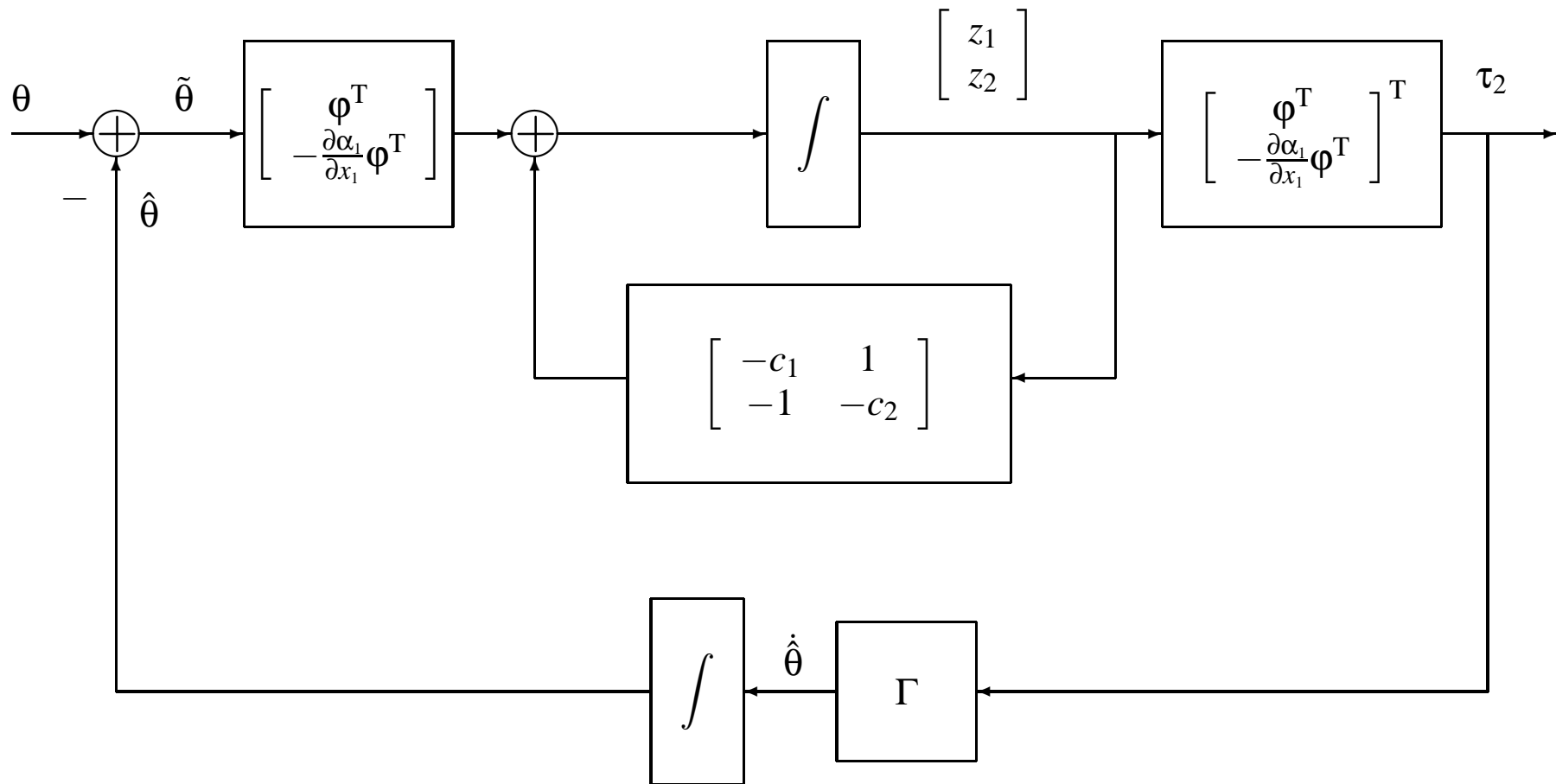
$$\dot{\hat{\theta}} = \Gamma \tau_2(x, \hat{\theta}) = \Gamma \begin{bmatrix} \phi, & -\frac{\partial \alpha_1}{\partial x_1} \phi \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \Gamma \underbrace{\left( \underbrace{\phi z_1}_{\tau_1} - \frac{\partial \alpha_1}{\partial x_1} \phi z_2 \right)}_{\tau_2}$$

( $\tau_1, \tau_2$  are called **tuning functions**)

makes

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2,$$

thus  $z = 0, \tilde{\theta} = 0$  is GS and  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .



The closed-loop adaptive system

### Design C.

We have one more integrator, so we define the third error coordinate and replace  $\hat{\theta}$  in design B by potential update law,

$$z_3 = x_3 - \alpha_2(x_1, x_2, \hat{\theta})$$

$$v_2(x_1, x_2, \hat{\theta}) = \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \tau_2(x_1, x_2, \hat{\theta}).$$

Now the  $z_1, z_2$ -system is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \phi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \phi^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ z_3 + \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) \end{bmatrix}$$

and

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) + \tilde{\theta}^T (\tau_2 - \Gamma^{-1} \dot{\hat{\theta}}).$$

$z_3$ -equation is given by

$$\begin{aligned}\dot{z}_3 &= u - \frac{\partial \alpha_2}{\partial x_1} \left( x_2 + \varphi^T \theta \right) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \\ &= u - \frac{\partial \alpha_2}{\partial x_1} \left( x_2 + \varphi^T \hat{\theta} \right) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_2}{\partial x_1} \varphi^T \tilde{\theta}.\end{aligned}$$

Choose

$$V_3(x, \hat{\theta}) = V_2 + \frac{1}{2} z_3^2 = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} z_3^2 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

we have

$$\begin{aligned}\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) \\ &\quad + z_3 \left[ z_2 + u - \frac{\partial \alpha_2}{\partial x_1} \left( x_2 + \varphi^T \hat{\theta} \right) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \right] \\ &\quad + \tilde{\theta}^T \left( \tau_2 - \frac{\partial \alpha_2}{\partial x_1} \varphi z_3 - \Gamma^{-1} \dot{\hat{\theta}} \right).\end{aligned}$$

Pick update law

$$\dot{\hat{\theta}} = \Gamma \tau_3(x_1, x_2, x_3, \hat{\theta}) = \Gamma \left( \tau_2 - \frac{\partial \alpha_2}{\partial x_1} \phi z_3 \right) = \Gamma \left[ \phi, \frac{\partial \alpha_1}{\partial x_1} \phi, -\frac{\partial \alpha_2}{\partial x_1} \phi \right] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

and control law

$$u = \alpha_3(x_1, x_2, x_3, \hat{\theta}) = -z_2 - c_3 z_3 + \frac{\partial \alpha_2}{\partial x_1} (x_2 + \phi^T \hat{\theta}) + \frac{\partial \alpha_2}{\partial x_2} x_3 + v_3,$$

results in

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_2 \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) + z_3 \left( v_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \dot{\hat{\theta}} \right).$$

Notice

$$\dot{\hat{\theta}} - \Gamma \tau_2 = \dot{\hat{\theta}} - \Gamma \tau_3 - \Gamma \frac{\partial \alpha_2}{\partial x_1} \phi z_3$$

we have

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 + z_3 \underbrace{\left( v_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \Gamma \tau_3 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_2}{\partial x_1} \phi z_2 \right)}_{=0}.$$

Stability and regulation of  $x$  to zero follows.

Further insight:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \\ -\frac{\partial \alpha_2}{\partial x_1} \varphi^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ \frac{\partial \alpha_1}{\partial \hat{\theta}} (\Gamma \tau_2 - \dot{\hat{\theta}}) \\ v_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \Gamma \tau_3 \end{bmatrix}.$$

$$\Downarrow \dot{\hat{\theta}} - \Gamma \tau_2 = \dot{\hat{\theta}} - \Gamma \tau_3 - \Gamma \frac{\partial \alpha_2}{\partial x_1} \varphi z_3$$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_2}{\partial x_1} \varphi \\ 0 & -1 & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \\ -\frac{\partial \alpha_2}{\partial x_1} \varphi^T \end{bmatrix} \tilde{\theta} + \begin{bmatrix} 0 \\ 0 \\ v_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}} \Gamma \tau_3 \end{bmatrix}$$

$\Downarrow$  selection of  $v_3$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 + \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_2}{\partial x_1} \varphi \\ 0 & -1 - \frac{\partial \alpha_1}{\partial \hat{\theta}} \Gamma \frac{\partial \alpha_2}{\partial x_1} \varphi & -c_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \varphi^T \\ -\frac{\partial \alpha_1}{\partial x_1} \varphi^T \\ -\frac{\partial \alpha_2}{\partial x_1} \varphi^T \end{bmatrix} \tilde{\theta}.$$



## General Recursive Design Procedure

parametric strict-feedback system:

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi_1(x_1)^T \theta \\ \dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T \theta \\ &\vdots \\ \dot{x}_{n-1} &= x_n + \varphi_{n-1}(x_1, \dots, x_{n-1})^T \theta \\ \dot{x}_n &= \beta(x)u + \varphi_n(x)^T \theta \\ y &= x_1\end{aligned}$$

where  $\beta$  and  $\varphi_i$  are smooth.

**Objective:** asymptotically track reference output  $y_r(t)$ , with  $y_r^{(i)}(t), i = 1, \dots, n$  known, bounded and piecewise continuous.

## Tuning functions design for tracking ( $z_0 \triangleq 0, \alpha_0 \triangleq 0, \tau_0 \triangleq 0$ )

$$\begin{aligned}
 z_i &= x_i - y_r^{(i-1)} - \alpha_{i-1} \\
 \alpha_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) &= -z_{i-1} - c_i z_i - w_i^T \hat{\theta} + \sum_{k=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) + v_i \\
 v_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) &= + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \Gamma \tau_i + \sum_{k=2}^{i-1} \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \Gamma w_i z_k \\
 \tau_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) &= \tau_{i-1} + w_i z_i \\
 w_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-2)}) &= \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k
 \end{aligned}$$

$i = 1, \dots, n$

$$\bar{x}_i = (x_1, \dots, x_i), \quad \bar{y}_r^{(i)} = (y_r, \dot{y}_r, \dots, y_r^{(i)})$$

Adaptive control law:

$$u = \frac{1}{\beta(x)} \left[ \alpha_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) + y_r^{(n)} \right]$$

Parameter update law:

$$\dot{\hat{\theta}} = \Gamma \tau_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) = \Gamma W z$$

Closed-loop system

$$\begin{aligned}\dot{z} &= A_z(z, \hat{\theta}, t)z + W(z, \hat{\theta}, t)^T \tilde{\theta} \\ \dot{\hat{\theta}} &= \Gamma W(z, \hat{\theta}, t)z,\end{aligned}$$

where

$$A_z(z, \hat{\theta}, t) = \begin{bmatrix} -c_1 & 1 & 0 & \cdots & 0 \\ -1 & -c_2 & 1 + \sigma_{23} & \cdots & \sigma_{2n} \\ 0 & -1 - \sigma_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 + \sigma_{n-1,n} \\ 0 & -\sigma_{2n} & \cdots & -1 - \sigma_{n-1,n} & -c_n \end{bmatrix}$$

$$\sigma_{jk}(x, \hat{\theta}) = -\frac{\partial \alpha_{j-1}}{\partial \hat{\theta}} \Gamma w_k$$

This structure ensures that the Lyapunov function

$$V_n = \frac{1}{2} z^T z + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

has derivative

$$\dot{V}_n = -\sum_{k=1}^n c_k z_k^2.$$

## Modular Design

**Motivation:** Controller can be combined with different identifiers. (No flexibility for update law in tuning function design)

**Naive idea:** connect a good identifier and a good controller.

Example: error system

$$\dot{x} = -x + \varphi(x)\tilde{\theta}$$

suppose  $\tilde{\theta}(t) = e^{-t}$  and  $\varphi(x) = x^3$ , we have

$$\dot{x} = -x + x^3 e^{-t}$$

**But,** when  $|x_0| > \sqrt{\frac{3}{2}}$ ,

$$x(t) \rightarrow \infty \quad \text{as} \quad t \rightarrow \frac{1}{3} \ln \frac{x_0^2}{x_0^2 - 3/2}$$

**Conclusion:** Need stronger controller.

## Controller Design. nonlinear damping

$$u = -x - \varphi(x)\hat{\theta} - \varphi(x)^2x$$

closed-loop system

$$\dot{x} = -x - \varphi(x)^2x + \varphi(x)\tilde{\theta}.$$

With  $V = \frac{1}{2}x^2$ , we have

$$\begin{aligned}\dot{V} &= -x^2 - \varphi(x)^2x^2 + x\varphi(x)\tilde{\theta} \\ &= -x^2 - \left[ \varphi(x)x - \frac{1}{2}\tilde{\theta} \right]^2 + \frac{1}{4}\tilde{\theta}^2 \\ &\leq -x^2 + \frac{1}{4}\tilde{\theta}^2.\end{aligned}$$

bounded  $\tilde{\theta}(t) \Rightarrow$  bounded  $x(t)$

For higher order system

$$\begin{aligned}\dot{x}_1 &= x_2 + \boldsymbol{\varphi}(x_1)^T \boldsymbol{\theta} \\ \dot{x}_2 &= u\end{aligned}$$

set

$$\alpha_1(x_1, \hat{\boldsymbol{\theta}}) = -c_1 x_1 - \boldsymbol{\varphi}(x_1)^T \hat{\boldsymbol{\theta}} - \kappa_1 |\boldsymbol{\varphi}(x_1)|^2 x_1, \quad c_1, \kappa_1 > 0$$

and define

$$z_2 = x_2 - \alpha_1(x_1, \hat{\boldsymbol{\theta}})$$

error system

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 - \kappa_1 |\boldsymbol{\varphi}|^2 z_1 + \boldsymbol{\varphi}^T \tilde{\boldsymbol{\theta}} + z_2 \\ \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - \frac{\partial \alpha_1}{\partial x_1} (x_2 + \boldsymbol{\varphi}^T \boldsymbol{\theta}) - \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}} \dot{\hat{\boldsymbol{\theta}}}.\end{aligned}$$

Consider

$$V_2 = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}|z|^2$$

we have

$$\begin{aligned}\dot{V}_2 &\leq -c_1z_1^2 + \frac{1}{4\kappa_1}|\tilde{\theta}|^2 + z_1z_2 + z_2 \left[ u - \frac{\partial\alpha_1}{\partial x_1} (x_2 + \varphi^T\theta) - \frac{\partial\alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \right] \\ &\leq -c_1z_1^2 + \frac{1}{4\kappa_1}|\tilde{\theta}|^2 + z_2 \left[ u + z_1 - \frac{\partial\alpha_1}{\partial x_1} (x_2 + \varphi^T\hat{\theta}) - \left( \frac{\partial\alpha_1}{\partial x_1}\varphi^T\tilde{\theta} + \frac{\partial\alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} \right) \right].\end{aligned}$$

controller

$$u = -z_1 - c_2z_2 - \kappa_2 \left| \frac{\partial\alpha_1}{\partial x_1}\varphi \right|^2 z_2 - g_2 \left| \frac{\partial\alpha_1}{\partial \hat{\theta}} \right|^2 z_2 + \frac{\partial\alpha_1}{\partial x_1} (x_2 + \varphi^T\hat{\theta}),$$

achieves

$$\dot{V}_2 \leq -c_1z_1^2 - c_2z_2^2 + \left( \frac{1}{4\kappa_1} + \frac{1}{4\kappa_2} \right) |\tilde{\theta}|^2 + \frac{1}{4g_2} |\dot{\hat{\theta}}|^2$$

bounded  $\tilde{\theta}$ , bounded  $\dot{\hat{\theta}}$  (or  $\in \mathcal{L}_2$ )  $\Rightarrow$  bounded  $x(t)$

## Controller design in the modular approach ( $z_0 \triangleq 0, \alpha_0 \triangleq 0$ )

$$\begin{aligned}
 z_i &= x_i - y_r^{(i-1)} - \alpha_{i-1} \\
 \alpha_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-1)}) &= -z_{i-1} - c_i z_i - w_i^T \hat{\theta} + \sum_{k=1}^{i-1} \left( \frac{\partial \alpha_{i-1}}{\partial x_k} x_{k+1} + \frac{\partial \alpha_{i-1}}{\partial y_r^{(k-1)}} y_r^{(k)} \right) - s_i z_i \\
 w_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-2)}) &= \varphi_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_k} \varphi_k \\
 s_i(\bar{x}_i, \hat{\theta}, \bar{y}_r^{(i-2)}) &= \kappa_i |w_i|^2 + g_i \left| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right|^2
 \end{aligned}$$

$i = 1, \dots, n$

$$\bar{x}_i = (x_1, \dots, x_i), \quad \bar{y}_r^{(i)} = (y_r, \dot{y}_r, \dots, y_r^{(i)})$$

Adaptive control law:

$$u = \frac{1}{\beta(x)} \left[ \alpha_n(x, \hat{\theta}, \bar{y}_r^{(n-1)}) + y_r^{(n)} \right]$$

Controller module guarantees:

$$\text{If } \tilde{\theta} \in \mathcal{L}_\infty \text{ and } \dot{\hat{\theta}} \in \mathcal{L}_2 \text{ or } \mathcal{L}_\infty \text{ then } x \in \mathcal{L}_\infty$$



## Requirement for identifier

error system

$$\dot{z} = A_z(z, \hat{\theta}, t)z + W(z, \hat{\theta}, t)^T \tilde{\theta} + Q(z, \hat{\theta}, t)^T \dot{\hat{\theta}}$$

where

$$A_z(z, \hat{\theta}, t) = \begin{bmatrix} -c_1 - s_1 & 1 & 0 & \dots & 0 \\ -1 & -c_2 - s_2 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -1 & -c_n - s_n \end{bmatrix}$$
$$W(z, \hat{\theta}, t)^T = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_n^T \end{bmatrix}, \quad Q(z, \hat{\theta}, t)^T = \begin{bmatrix} 0 \\ -\frac{\partial \alpha_1}{\partial \hat{\theta}} \\ \vdots \\ -\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \end{bmatrix}.$$

Since

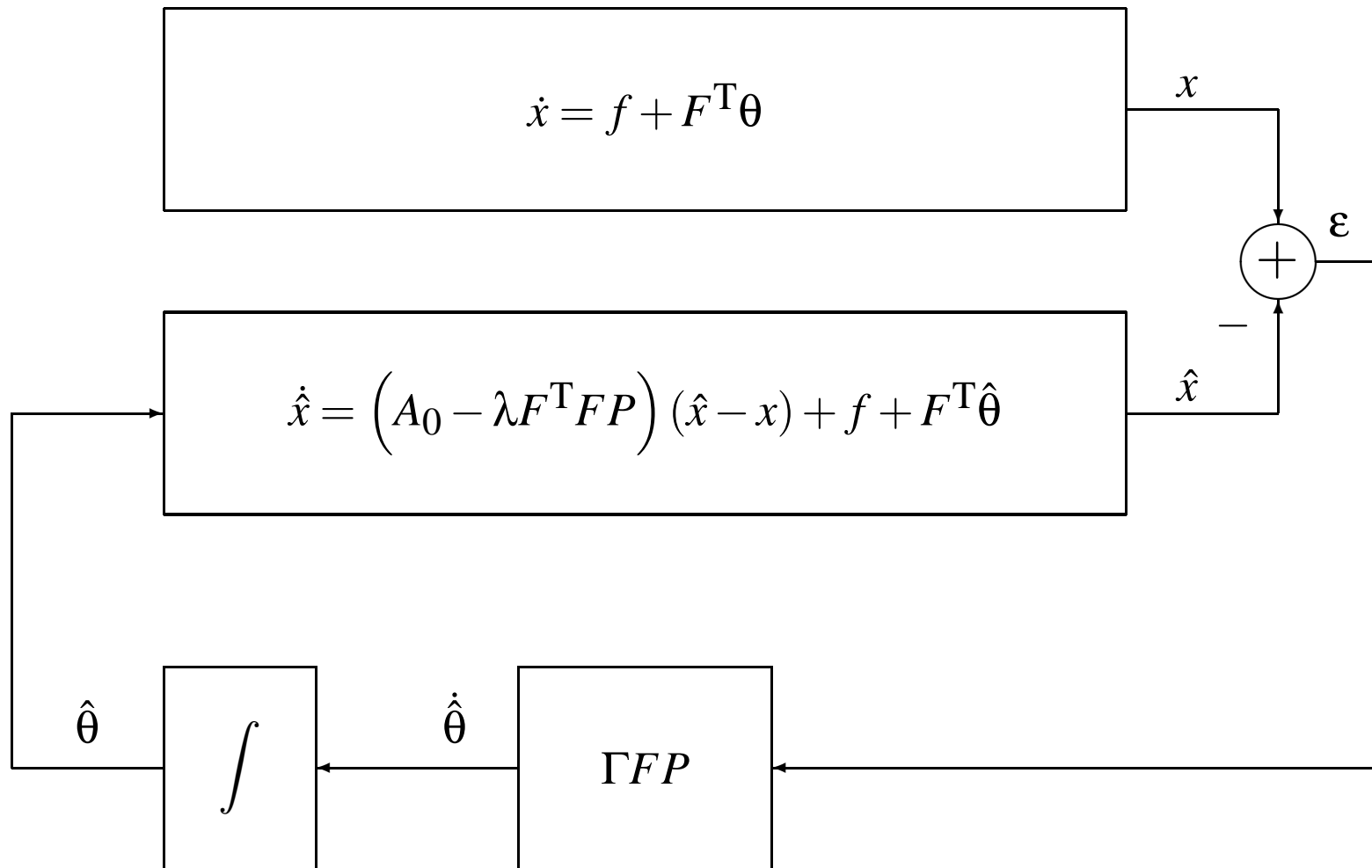
$$W(z, \hat{\theta}, t)^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\partial \alpha_1}{\partial x_1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\frac{\partial \alpha_{n-1}}{\partial x_1} & \cdots & -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} & 1 \end{bmatrix} F(x)^T \triangleq N(z, \hat{\theta}, t) F(x)^T.$$

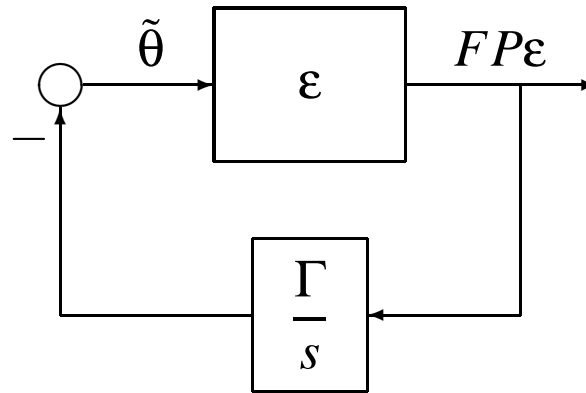
Identifier properties:

- (i)  $\tilde{\theta} \in \mathcal{L}_\infty$  and  $\dot{\hat{\theta}} \in \mathcal{L}_2$  or  $\mathcal{L}_\infty$ ,
- (ii) if  $x \in \mathcal{L}_\infty$  then  $F(x(t))^T \tilde{\theta}(t) \rightarrow 0$  and  $\dot{\hat{\theta}}(t) \rightarrow 0$ .

# Identifier Design

## Passive identifier





$$\dot{\epsilon} = \left[ A_0 - \lambda F(x, u)^T F(x, u) P \right] \epsilon + F(x, u)^T \tilde{\theta}$$

update law

$$\dot{\hat{\theta}} = \Gamma F(x, u) P \epsilon, \quad \Gamma = \Gamma^T > 0.$$

Use Lyapunov function

$$V = \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \epsilon^T P \epsilon$$

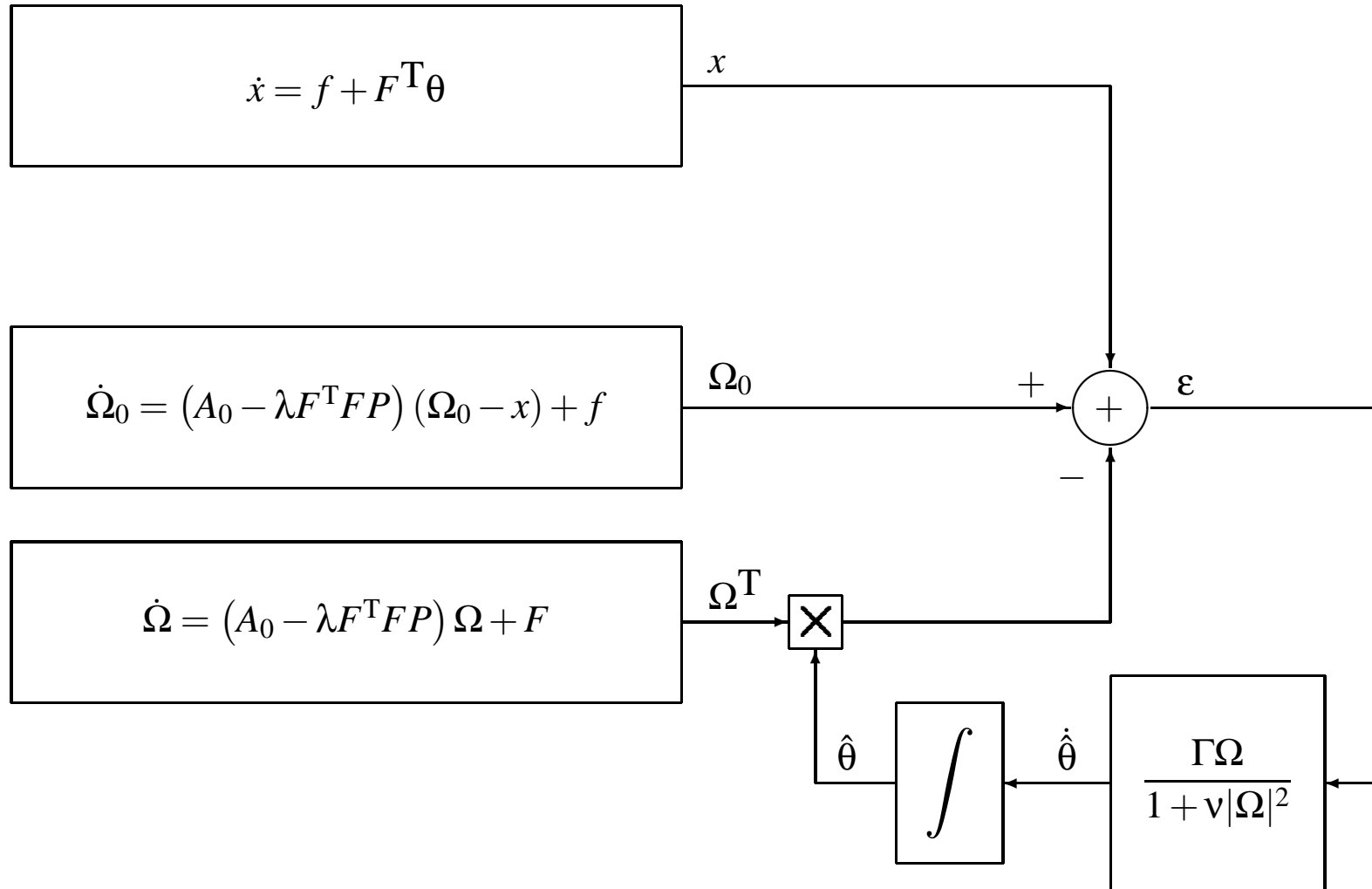
its derivative satisfies

$$\dot{V} \leq -\epsilon^T \epsilon - \frac{\lambda}{\bar{\lambda}(\Gamma)^2} |\dot{\hat{\theta}}|^2.$$

Thus, whenever  $x$  is bounded,  $F(x(t))^T \tilde{\theta}(t) \rightarrow 0$  and  $\dot{\hat{\theta}}(t) \rightarrow 0$ .

( $\dot{\epsilon}(t) \rightarrow 0$  because  $\int_0^\infty \dot{\epsilon}(\tau) d\tau = -\epsilon(0)$  exists, Barbalat's lemma...)

# Swapping identifier



define  $\tilde{\varepsilon} \triangleq x + \Omega_0 - \Omega^T \theta$ ,

$$\dot{\tilde{\varepsilon}} = \left[ A_0 - \lambda F(x, u)^T F(x, u) P \right] \tilde{\varepsilon}.$$

Choose

$$V = \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \tilde{\varepsilon} P \tilde{\varepsilon}$$

we have

$$\dot{V} \leq -\frac{3}{4} \frac{\varepsilon^T \varepsilon}{1 + \text{vtr}\{\Omega^T \Omega\}},$$

proves identifier properties.

## Output Feedback Adaptive Designs

$$\begin{aligned}\dot{x} &= Ax + \phi(y) + \Phi(y)a + \begin{bmatrix} 0 \\ b \end{bmatrix} \sigma(y)u, & x \in \mathbb{R}^n \\ y &= e_1^T x,\end{aligned}$$

$$A = \begin{bmatrix} 0 & & \\ \vdots & I_{n-1} & \\ 0 & \cdots & 0 \end{bmatrix},$$
$$\phi(y) = \begin{bmatrix} \varphi_{0,1}(y) \\ \vdots \\ \varphi_{0,n}(y) \end{bmatrix}, \quad \Phi(y) = \begin{bmatrix} \varphi_{1,1}(y) & \cdots & \varphi_{q,1}(y) \\ \vdots & & \vdots \\ \varphi_{1,n}(y) & \cdots & \varphi_{q,n}(y) \end{bmatrix},$$

unknown constant parameters:

$$a = [a_1, \dots, a_q]^T, \quad b = [b_m, \dots, b_0]^T.$$

## State estimation filters

Filters:

$$\dot{\xi} = A_0 \xi + ky + \phi(y)$$

$$\dot{\Xi} = A_0 \Xi + \Phi(y)$$

$$\dot{\lambda} = A_0 \lambda + e_n \sigma(y) u$$

$$v_j = A_0^j \lambda, \quad j = 0, \dots, m$$

$$\Omega^T = [v_m, \dots, v_1, v_0, \Xi]$$



Parameter-dependent state estimate

$$\hat{x} = \xi + \Omega^T \theta$$

The vector  $k = [k_1, \dots, k_n]^T$  chosen so that the matrix

$$A_0 = A - ke_1^T$$

is Hurwitz, that is,

$$PA_0 + A_0^T P = -I, \quad P = P^T > 0$$

The state estimation error

$$\varepsilon = x - \hat{x}$$

satisfies

$$\dot{\varepsilon} = A_0 \varepsilon$$

Parametric model for adaptation:

$$\begin{aligned} \dot{y} &= \omega_0 + \omega^T \theta + \varepsilon_2 \\ &= b_m v_{m,2} + \omega_0 + \bar{\omega}^T \theta + \varepsilon_2, \end{aligned}$$

where

$$\begin{aligned} \omega_0 &= \varphi_{0,1} + \xi_2 \\ \omega &= [v_{m,2}, v_{m-1,2}, \dots, v_{0,2}, \Phi_{(1)} + \Xi_{(2)}]^T \\ \bar{\omega} &= [0, v_{m-1,2}, \dots, v_{0,2}, \Phi_{(1)} + \Xi_{(2)}]^T. \end{aligned}$$

Since the states  $x_2, \dots, x_n$  are not measured, the backstepping design is applied to the system

$$\begin{aligned}\dot{y} &= b_m v_{m,2} + \omega_0 + \bar{\omega}^T \theta + \varepsilon_2 \\ \dot{v}_{m,i} &= v_{m,i+1} - k_i v_{m,1}, \quad i = 2, \dots, \rho - 1 \\ \dot{v}_{m,\rho} &= \sigma(y)u + v_{m,\rho+1} - k_\rho v_{m,1}.\end{aligned}$$

The order of this system is equal to the relative degree of the plant.

## Extensions

### Pure-feedback systems.

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \dots, x_{i+1})^T \theta, \quad i = 1, \dots, n-1 \\ \dot{x}_n &= \left( \beta_0(x) + \beta(x)^T \theta \right) u + \varphi_0(x) + \varphi_n(x)^T \theta,\end{aligned}$$

where  $\varphi_0(0) = 0$ ,  $\varphi_1(0) = \dots = \varphi_n(0) = 0$ ,  $\beta_0(0) \neq 0$ .

Because of the dependence of  $\varphi_i$  on  $x_{i+1}$ , the regulation or tracking for pure-feedback systems is, in general, not global, even when  $\theta$  is known.

## Unknown virtual control coefficients.

$$\begin{aligned}\dot{x}_i &= b_i x_{i+1} + \varphi_i(x_1, \dots, x_i)^T \theta, & i = 1, \dots, n-1 \\ \dot{x}_n &= b_n \beta(x) u + \varphi_n(x_1, \dots, x_n)^T \theta,\end{aligned}$$

where, in addition to the unknown vector  $\theta$ , the constant coefficients  $b_i$  are also unknown.

The unknown  $b_i$ -coefficients are frequent in applications ranging from electric motors to flight dynamics. The signs of  $b_i$ ,  $i = 1, \dots, n$ , are assumed to be known. In the tuning functions design, in addition to estimating  $b_i$ , we also estimate its inverse  $\rho_i = 1/b_i$ . In the modular design we assume that in addition to  $\text{sgn} b_i$ , a positive constant  $\zeta_i$  is known such that  $|b_i| \geq \zeta_i$ . Then, instead of estimating  $\rho_i = 1/b_i$ , we use the inverse of the estimate  $\hat{b}_i$ , i.e.,  $1/\hat{b}_i$ , where  $\hat{b}_i(t)$  is kept away from zero by using parameter projection.

## Multi-input systems.

$$\begin{aligned}\dot{X}_i &= B_i(\bar{X}_i)X_{i+1} + \Phi_i(\bar{X}_i)^T\theta, & i = 1, \dots, n-1 \\ \dot{X}_n &= B_n(X)u + \Phi_n(X)^T\theta,\end{aligned}$$

where  $X_i$  is a  $v_i$ -vector,  $v_1 \leq v_2 \leq \dots \leq v_n$ ,  $\bar{X}_i = [X_1^T, \dots, X_i^T]^T$ ,  $X = \bar{X}_n$ , and the matrices  $B_i(\bar{X}_i)$  have full rank for all  $\bar{X}_i \in \mathbb{R}^{\sum_{j=1}^i v_j}$ . The input  $u$  is a  $v_n$ -vector.

The matrices  $B_i$  can be allowed to be unknown provided they are constant and positive definite.

## Block strict-feedback systems.

$$\begin{aligned}\dot{x}_i &= x_{i+1} + \varphi_i(x_1, \dots, x_i, \zeta_1, \dots, \zeta_i)^T \theta, & i = 1, \dots, \rho - 1 \\ \dot{x}_\rho &= \beta(x, \zeta)u + \varphi_\rho(x, \zeta)^T \theta \\ \dot{\zeta}_i &= \Phi_{i,0}(\bar{x}_i, \bar{\zeta}_i) + \Phi_i(\bar{x}_i, \bar{\zeta}_i)^T \theta, & i = 1, \dots, \rho\end{aligned}$$

with the following notation:  $\bar{x}_i = [x_1, \dots, x_i]^T$ ,  $\bar{\zeta}_i = [\zeta_1^T, \dots, \zeta_i^T]^T$ ,  $x = \bar{x}_\rho$ , and  $\zeta = \bar{\zeta}_\rho$ .

Each  $\zeta_i$ -subsystem is assumed to be bounded-input bounded-state (BIBS) stable with respect to the input  $(\bar{x}_i, \bar{\zeta}_{i-1})$ . For this class of systems it is quite simple to modify the procedure in the tables. Because of the dependence of  $\varphi_i$  on  $\bar{\zeta}_i$ , the stabilizing function  $\alpha_i$  is augmented by the term  $+\sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \zeta_k} \Phi_{k,0}$ , and the regressor  $w_i$  is augmented by  $-\sum_{k=1}^{i-1} \Phi_i \left( \frac{\partial \alpha_{i-1}}{\partial \zeta_k} \right)^T$ .

**Partial state-feedback systems.** In many physical systems there are unmeasured states as in the output-feedback form, but there are also states other than the output  $y = x_1$  that are measured. An example of such a system is

$$\begin{aligned}\dot{x}_1 &= x_2 + \varphi_1(x_1)^T \theta \\ \dot{x}_2 &= x_3 + \varphi_2(x_1, x_2)^T \theta \\ \dot{x}_3 &= x_4 + \varphi_3(x_1, x_2)^T \theta \\ \dot{x}_4 &= x_5 + \varphi_4(x_1, x_2)^T \theta \\ \dot{x}_5 &= u + \varphi_5(x_1, x_2, x_5)^T \theta.\end{aligned}$$

The states  $x_3$  and  $x_4$  are assumed not to be measured. To apply the adaptive backstepping designs presented in this chapter, we combine the state-feedback techniques with the output-feedback techniques. The subsystem  $(x_2, x_3, x_4)$  is in the output-feedback form with  $x_2$  as a measured output, so we employ a state estimator for  $(x_2, x_3, x_4)$  using the filters introduced in the section on output feedback.



## Example of Adaptive Stabilization in the Presence of a Stochastic Disturbance

$$dx = udt + xdw$$

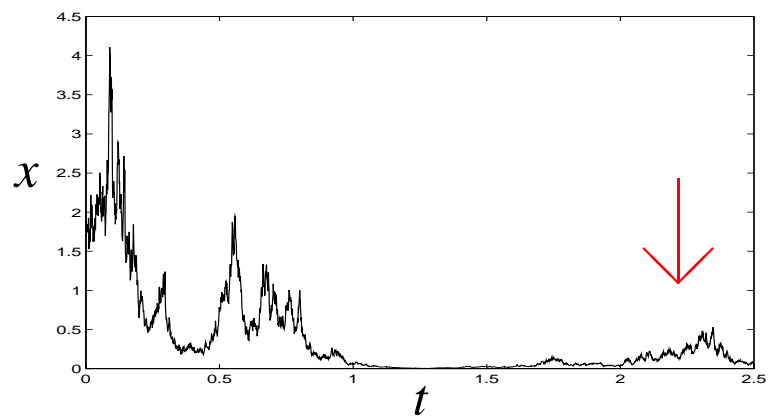
$w$ : Wiener process with  $E \{dw^2\} = \sigma(t)^2 dt$ , no a priori bound for  $\sigma$

Control laws:

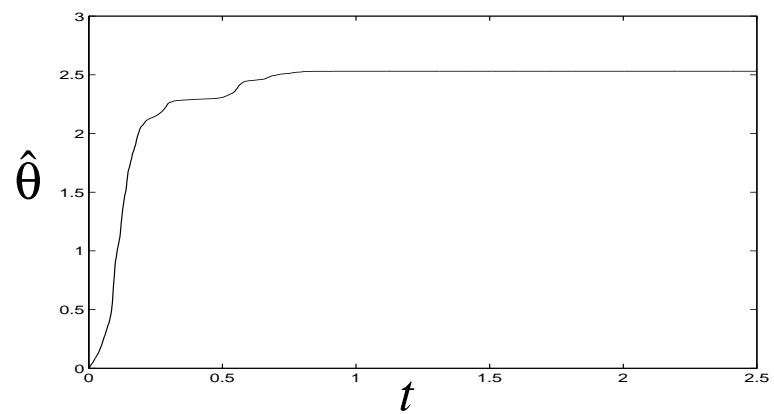
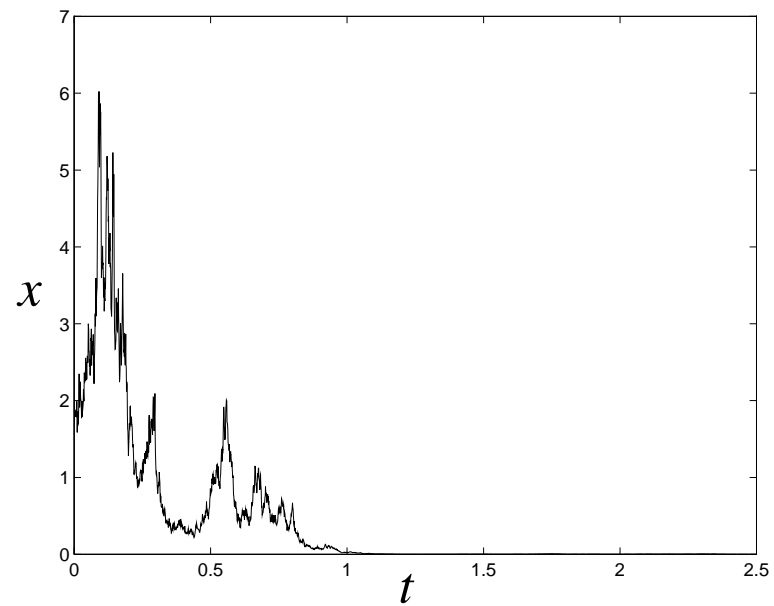
Disturbance Attenuation:  $u = -x - x^3$

Adaptive Stabilization:  $u = -x - \hat{\theta}x, \quad \dot{\hat{\theta}} = x^2$

## Disturbance Attenuation



## Adaptive Stabilization



# Major Applications of Adaptive Nonlinear Control

- **Electric Motors Actuating Robotic Loads**

*Nonlinear Control of Electric Machinery*, Dawson, Hu, Burg, 1998.

- **Marine Vehicles** (ships, UUVs; dynamic positioning, way point tracking, maneuvering)

*Marine Control Systems*, Fossen, 2002

- **Automotive Vehicles** (lateral and longitudinal control, traction, overall dynamics)

The groups of Tomizuka and Kanellakopoulos.

Dozens of other occasional applications, including: aircraft wing rock, compressor stall and surge, satellite attitude control.

## Other Books on Adaptive NL Control Theory Inspired by [KKK]

1. Marino and Tomei (1995),  
*Nonlinear Control Design: Geometric, Adaptive, and Robust*
2. Freeman and Kokotovic (1996),  
*Robust Nonlinear Control Design: State Space and Lyapunov Techniques*
3. Qu (1998),  
*Robust Control of Nonlinear Uncertain Systems*
4. Krstic and Deng (1998),  
*Stabilization of Nonlinear Uncertain Systems*
5. Ge, Hang, Lee, Zhang (2001),  
*Stable Adaptive Neural Network Control*
6. Spooner, Maggiore, Ordonez, and Passino (2002),  
*Stable Adaptive Control and Estimation for Nonlinear Systems: Neural and Fuzzy Approximation Techniques*
7. French, Szepesvari, Rogers (2003),  
*Performance of Nonlinear Approximate Adaptive Controllers*

## Adaptive NL Control/Backstepping Coverage in Major Texts

1. Khalil (1995/2002),  
*Nonlinear Systems*
2. Isidori (1995),  
*Nonlinear Control Systems*
3. Sastry (1999),  
*Nonlinear Systems: Analysis, Stability, and Control*
4. Astrom and Wittenmark (1995),  
*Adaptive Control*