Extremum Seeking for Motion Optimization: From Bacteria to Nonholonomic Vehicles

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Outline

- Extremum seeking - the basics
- Source seeking
  - 2D - full actuation
  - 2D - nonholonomic vehicle, tuning of speed
  - 2D - nonholonomic vehicle, tuning of heading
  - 3D
  - Fish locomotion models
  - Stochastic source seeking - bacterial locomotion
Extremum Seeking - Basics
Example of Single-Parameter Maximum Seeking

\[ f(\theta) = f^* + \frac{f''}{2}(\theta - \theta^*)^2 \]

\[ \theta \]

\[ f^* \quad \theta^* \]

\[ \text{Plant} \]

\[ \theta \]

\[ \hat{\theta} \]

\[ a \sin \omega t \]

\[ k \]

\[ s \]

\[ \xi \]

\[ \sin \omega t \]

\[ \frac{s}{s + h} \]
Example of Single-Parameter Maximum Seeking

\[ f(\theta(t)) \]

\[ f^* - \text{unknown!} \]
History

- **Leblanc (1922)** - electric railways
- **Early Russian literature (1940’s)** - many papers
- **Drapper and Li (1951)** - application to IC engine spark timing tuning
- **Tsien (1954)** - a chapter in his book on Engineering Cybernetics
- **Feldbaum (1959)** - book *Computers in Automatic Control Systems*
- **Blackman (1962 chap. in book by Westcott)** - nice intuitive presentation of ES
- **Meerkov (1967, 1968)** - papers with averaging analysis
- **Sternby (1980)** - survey
- **Astrom and Wittenmark (1995 book)** - rates ES as one of the most promising areas for adaptive control
Developments Over Last Decade

- Krstic and Wang - stability proof for single-parameter general dynamic nonlinear plants
- Krstic, Ariyur, Choi, Wang - discrete-time, limit cycle minimization, IMC for parameter tracking, dynamic compensators, etc.
- Rotea; Walsh; Ariyur-Krstic - multi-parameter ES
- Ariyur, Krstic - slope seeking
- Tan, Nesic, Mareels (2005) - semi-global stability of ES
Recent Applications

- Compressor instabilities in jet engines
- Combustion instabilities and thermoacoustic coolers
- Formation flight
- Fusion reflected RF power
- Beam matching in particle accelerators
- Flow control: diffusers, airfoils, “minivan” shaped bluff bodies,” cavity flow
- Internal combustion (HCCI) engine fuel consumption minimization
- PID tuning
Basic Extremum Seeking - Static Map

\[ f(\theta) = f^* + \frac{f''}{2}(\theta - \theta^*)^2 \]

- \( y = \) output to be minimized
- \( f^* = \) minimum of the map
- \( f'' = \) second derivative (positive - \( f(\theta) \) has a min.)
- \( \theta^* = \) unknown parameter
- \( \hat{\theta} = \) estimate of \( \theta^* \)
- \( a \sin \omega t \)
- \( k \)
- \( k = \) adaptation gain (positive) of the integrator \( \frac{1}{s} \)
- \( a = \) amplitude of the probing signal
- \( \omega = \) frequency of the probing signal
- \( \xi = \) \( \sin \omega t \)
- \( s \)
- \( s + h \)
- \( h = \) cut-off frequency of the "washout filter" \( \frac{s}{s + h} \)

\(+/\times = \) modulation/demodulation
Stability Analysis by Averaging

\[ f(\theta) = f^* + \frac{f''}{2}(\theta - \theta^*)^2 \]

\[ \tilde{\theta} = \theta^* - \hat{\theta} \]

\[ e = f^* - \frac{h}{s + h}[y] \]

\[ \tau = \omega t \]

Full nonlinear time-varying model:

\[ \frac{d}{d\tau} \tilde{\theta} = \frac{k}{\omega} \left( \frac{f''}{2} \left( \tilde{\theta} - a \sin \tau \right)^2 - e \right) \sin \tau \]

\[ \frac{d}{d\tau} e = \frac{h}{\omega} \left( -e - \frac{f''}{2} \left( \tilde{\theta} - a \sin \tau \right)^2 \right) \]
Stability Analysis by Averaging

\[ f(\theta) = f^* + \frac{f''}{2}(\theta - \theta^*)^2 \]

\[ \tilde{\theta} = \theta^* - \hat{\theta} \]

\[ e = f^* - \frac{h}{s + h}[y] \]

\[ \tau = \omega t \]

Average system:

\[ \frac{d}{d\tau} \tilde{\theta}_{av} = -\frac{k a f''}{2\omega} \tilde{\theta}_{av} \]

\[ \frac{d}{d\tau} e_{av} = \frac{h}{\omega} \left( -e_{av} - \frac{f''}{2} \left( \frac{\tilde{\theta}_{av}^2 + \frac{a^2}{2}}{2} \right) \right) \]

Average equilibrium:

\[ \tilde{\theta}_{av} = 0 \]

\[ e_{av} = -\frac{a^2 f''}{4} \]
Stability Analysis by Averaging

\[ f(\theta) = f^* + \frac{f''}{2}(\theta - \theta^*)^2 \]

\[ \dot{\theta} = \theta^* - \hat{\theta} \]

\[ e = f^* - \frac{h}{s+h}[y] \]

\[ \tau = \omega t \]

Jacobian of the average system:

\[
J_{av} = \begin{bmatrix}
-\frac{kaf''}{2\omega} & 0 \\
0 & -\frac{h}{\omega}
\end{bmatrix}
\]
Stability Analysis by Averaging

\[
f(\theta) = f^{*} + \frac{f''}{2} (\theta - \theta^*)^2
\]

\[
\tau = \omega t
\]

\[
\tilde{\theta} = \theta^* - \hat{\theta}
\]

\[
e = f^* - \frac{h}{s + h} [y]
\]

**Theorem.** For sufficiently large \( w \) there exists a unique exponentially stable periodic solution of period \( 2p/w \) and it satisfies

\[
\left| \tilde{\theta}_{2\pi/\omega}(t) + e_{2\pi/\omega}(t) - \frac{a^2 f''}{4} \right| \leq O\left( \frac{1}{\omega} \right), \quad \forall t \geq 0
\]

Speed of convergence proportional to \( 1/w, a^2, k, f'' \)
Source Seeking
for Autonomous Vehicles
Without Position Measurement
Source Seeking - Introduction

• Motivation
  – Control a vehicle to locate the source of measurable signal with unknown spatial distribution, **without position measurement**
  – Think: *pursuit-evasion* while blindfolded and left to rely on the nose (with one “nostril”)

• Related work (only partial autonomy)
  – Porat and Nehorai - vehicle has *position information*
  – Ogren, Fiorelli and Leonard - “group” gradient estim. w/ communication
  – Justh and Krishnaprasad - w/ distance information
  – Klein and Morgansen - tracking of slow ground vehicle w/ fast UAV
  – Marshal, Broucke and Francis - cyclic pursuit problem
Introductory Example:
Fully Actuated Point Mass
Model - fully actuated point mass

Dynamics

\[ \dot{x} = v_x \]
\[ \dot{y} = v_y \]

Inputs

\[ v_x, v_y \]
Simulation Results

Point Mass

Vehicle Trajectory
Simulation Results

Circular Pattern of Vehicle Movement

![Graph showing the circular pattern of vehicle movement.](image)
Force-Actuated Point Mass

VEHICLE

\[ u_y \]

\[ \frac{1}{s} \]

\[ v_y \]

\[ \frac{1}{s} \]

\[ y \]

J = f(x, y)

EXTREMUM SEEKING LOOP

\[ u_x \]

\[ \frac{1}{s} \]

\[ v_x \]

\[ \frac{1}{s} \]

\[ x \]

\[ w_x \]

\[ -\alpha \omega^2 \sin \omega t \]

\[ k_x \frac{s - z_x}{s - p_x} \]

\[ C_x \]

\[ \sin(\omega t) \]

\[ J - f* - e \]

\[ \frac{s}{s + h} \]

\[ w_y \]

\[ \alpha \omega^2 \cos \omega t \]

\[ k_y \frac{s - z_y}{s - p_y} \]

\[ C_y \]

\[ -\cos(\omega t) \]
Nonholonomic “Unicycle”

- Kinematically constrained, underactuated
- Control only steering or only speed, not both
Unicycle Model

System is linearly uncontrollable (from inputs $v, \omega_o$) and unobservable (from the output $f(x, y)$ at its peak).

Sensor Dynamics

\[
\begin{align*}
\dot{x}_s &= v \cos \theta_o - r \dot{\theta}_o \sin \theta_o \\
\dot{y}_s &= v \sin \theta_o + r \dot{\theta}_o \cos \theta_o \\
\dot{\theta}_o &= \omega_o = \frac{d}{dt} \theta_o
\end{align*}
\]

Inputs

$v, \omega_o$
Tuning Speed Only; Heading Const.

\[ \omega = k \omega_o \]
Simulation Results

Tuning of Forward Velocity

Vehicle Trajectory
Simulation Results

Triangular Pattern of the Vehicle Center Movement
Simulation Results - Moving Target
Stability Proof by Averaging

\[ J = f(x_{s,c}, y_{s,c}) \]

\[ \tilde{x} = x_c - x^* - a \sin(\omega t) \cos(\omega_o t) \]

\[ \tilde{y} = y_c - y^* - a \sin(\omega t) \sin(\omega_o t) \]

\[ e = \frac{h}{s + h} [J] - f^* \]

\[ \xi = \frac{s}{s + h} [J] = J - f^* - e \]

\[ \tau = \omega t \]

where

\[ \xi = -q_x \left( \tilde{x} + a \sin \tau \cos \left( \frac{\tau}{k} \right) \right)^2 \]

\[ - q_y \left( \tilde{y} + a \sin \tau \sin \left( \frac{\tau}{k} \right) \right)^2 - e \]
Theorem:

For sufficiently large $w$ there exists a unique **exponentially stable periodic solution** of period $2\pi/w$ and it satisfies

$$\left\| \begin{array}{c} 
\frac{2\pi}{\omega} \\
\tilde{x} \\
\frac{2\pi}{\omega} \\
\tilde{y} \\
\frac{2\pi}{e \omega} + \frac{a^2}{4} (q_x + q_y) 
\end{array} \right\| \leq O(1/\omega), \quad \forall \ t \geq 0$$

Speed of convergence proportional to $1/\omega, a^2, c, q_x, q_y$
Alternative Design:
Tuning of *Heading Rate Only*
Tuning the Angular Velocity

Applying a change of notation

$$\theta = \theta_o$$
The Algorithm

\[ \dot{\theta}(t) = a\omega \cos(\omega t) + c \frac{s}{s + h} [J(t)] \sin(\omega t) \]

Linear combination of cosine and sine but non-constant coefficients!
Simulation Results

Tuning Angular Velocity

Vehicle Trajectory
Inspired by…
Track a Diffusive Source
Experimental Results
Level Sets

- Methods using *multiple AUVs*:
  - Kalantar & Zimmer
  - N. Leonard, Fiorelli, Ogren
  - Bertozzi
  - Burian, Singh
  - Bennett, J. Leonard
Level Sets
Level Sets
Level Sets
Level Sets
Level Sets
Navigation Through a “Minefield”
Hardest Problem: Stationary Source

Characterization of *Attractors* (stability proof)
Stability w/ Stationary Source

Unicycle Kinematics & Sensor Position Map

$\mathcal{R}_s$

Signal Field Nonlinear Map $f(x_s, y_s)$

$J$

$\dot{\theta}$

$\omega \cos(\omega t)$

$\sin(\omega t)$

$\xi$

$\frac{s}{s + \bar{h}}$
Stability w/ Stationary Source

\[ \dot{r}_c = V_c e^{j\theta} \]

\[ \dot{\theta} = a\omega \cos(\omega t) + \sin(\omega t) \left\{ c \frac{s}{s + h} [J] - d \left( \frac{s}{s + h} [J] \right)^2 \right\} \]
Theorem. The vehicle center locally exponentially converges to one of the two (almost periodic) attractors of the form

\[ x_c^{\text{attr}}(t) = x^* - \left( \rho + \tilde{r}_\pm^{2\pi/\omega}(t) \right) \cos \left( \frac{V_c}{\rho} t + \alpha_\pm^{2\pi/\omega}(t) + \gamma \right) \]

\[ y_c^{\text{attr}}(t) = y^* \pm \left( \rho + \tilde{r}_\pm^{2\pi/\omega}(t) \right) \sin \left( \frac{V_c}{\rho} t + \alpha_\pm^{2\pi/\omega}(t) + \gamma \right) \]

where \( \rho = \sqrt{\frac{V_c J_0(a)}{2cRq_r J_1(a)}} \)

\( J_0, J_1 = \text{Bessel functions} \)
Residual “Hovering” Motion
\[ \rho = \sqrt{\frac{V_c J_0(a)}{2cRq_r J_1(a)}} \]
Effect of Small “Nonlinear Damping”
Unstable Trajectories

- unstable (repulsive) solutions
- head off to infinity
- measure zero
The Optimal Heading Manifold

\[ \theta^* = \text{arg} \left( r^* - r_c \right) \]
\[ \tilde{\theta} = \theta - \theta^* - a \sin(\omega t) \]
\[ \tilde{r}_c = \left| r^* - r_c \right| \]
The “Average System”

\[ \dot{r}_c^\text{ave} = -\frac{V_c J_0(a)}{\omega} \cos \tilde{\theta}^\text{ave} \]

\[ \dot{\theta}^\text{ave} = \frac{1}{\omega} \sin \tilde{\theta}^\text{ave} \left\{ \frac{V_c J_0(a)}{\tilde{r}_c^\text{ave}} - 2q_r R J_1(a) \tilde{r}_c^\text{ave} \left( c + 2dq_r \left( R^2 + \tilde{r}_c^\text{ave}^2 \right) \right) \right\} \]

\[ + \frac{1}{\omega} 2dq_r R^2 \tilde{r}_c^\text{ave}^2 J_1(2a) \sin \left( 2\tilde{\theta}^\text{ave} \right) \]
The “Average Dynamics”
Extensions to 3D
\[ \dot{y}_c = V_c \cos(\alpha) \sin(\theta) \]
\[ \dot{x}_c = V_c \cos(\alpha) \cos(\theta) \]
\[ \dot{z}_c = V_c \sin(\alpha) \]
\[ \dot{\theta} = \Omega_2 \]
\[ \dot{\alpha} = \Omega_1 \]
Yaw and Pitch Actuated
Simulation Results
**Theorem.** The vehicle center locally exponentially converges to one of the two (almost periodic) attractors of the form

\[
\begin{align*}
x_{c, \text{attr}}(t) &= x^* - \left( \rho + \tilde{r}_{\pm}^{2\pi/\omega}(t) \right) \cos \left( \frac{V_c}{\rho} t + \alpha_{\pm}(t) + \gamma \right) \cos \left( \beta_{\pm}^{2\pi/\omega}(t) \right) \\
y_{c, \text{attr}}(t) &= y^* \pm \left( \rho + \tilde{r}_{\pm}^{2\pi/\omega}(t) \right) \sin \left( \frac{V_c}{\rho} t + \alpha_{\pm}(t) + \gamma \right) \cos \left( \beta_{\pm}^{2\pi/\omega}(t) \right) \\
z_{c, \text{attr}}(t) &= z^* \pm \left( \rho + \tilde{r}_{\pm}^{2\pi/\omega}(t) \right) \sin \left( \beta_{\pm}^{2\pi/\omega}(t) \right)
\end{align*}
\]

where \( \rho = \sqrt{\frac{V_c J_0(\sqrt{2}\alpha)}{\sqrt{2}c_\theta q_r R_1 J_1(\sqrt{2}\alpha)}} \)

\( J_0, J_1 = \text{Bessel functions} \)
“Orbits” for different values of $c$
Vehicle with Const Speed & Const Pitch Up Velocity, Sensor Off the Vehicle

\[ r_c AB \perp r_f QR \]

\[ \dot{y}_c = V_c \cos(\alpha) \sin(\theta) \]
\[ \dot{x}_c = V_c \cos(\alpha) \cos(\theta) \]
\[ \dot{z}_c = V_c \sin(\alpha) \]

\[ x_s = x_c + R_1 \cos(\alpha) \cos(\theta) + R_2 (-\cos(\phi) \sin(\alpha) \cos(\theta) + \sin(\phi) \sin(\theta)) \]
\[ y_s = y_c + R_1 \cos(\alpha) \sin(\theta) + R_2 (-\cos(\phi) \sin(\alpha) \sin(\theta) - \sin(\phi) \cos(\theta)) \]
\[ z_s = z_c + R_1 \sin(\alpha) + R_2 \cos(\phi) \cos(\alpha) \]
Roll Actuated

\[ \dot{\alpha} = \frac{v_2}{r_1} \cos \phi \]

\[ \dot{\theta} = -\frac{v_2}{r_1 \cos \alpha} \sin \phi \]
Simulation Results

Pollutant source moving.
Our vehicle follows it.
3D Boundary Tracing: Yaw+Pitch Actuation

Torpedo Tracing a Level Set

- Vehicle init. pos.
- Vehicle trajectory
- Source pos.
- 2D projections
3D Boundary Tracing: Roll Actuation
Source Seeking with Fish Locomotion Models
Fish Locomotion

Eva Kanso - Jerry Marsden model

Scott Kelly model
Three Links - Two Inputs

move forward

\[ \theta_1 = -a \cos(\omega t) \]
\[ \theta_2 = a \sin(\omega t) \]

turn in a circle

\[ \theta_1 = -a \cos(\omega t) + \beta \]
\[ \theta_2 = a \sin(\omega t) - \beta \]
Tuning of the Fish “Body Bending”

Control:

\[
\begin{align*}
\dot{\beta} &= -c\xi \cos(\omega t) \\
\xi &= \frac{s}{s + h} [J]
\end{align*}
\]
Simulations: Three Link Fish
Joukowsky Foil Fish w/ Vortex Model
One input, with vortex interaction

move forward

\[ \zeta_y = a \sin(\omega t) \]

turn in a circle

\[ \zeta_y = a \sin(\omega t) + \beta \]
Tuning of the Fish “Body Bending”

Control:

\[
\begin{align*}
\dot{\beta} &= c\xi \sin(\omega t) \\
\xi &= \left(\frac{s}{s + h}\right)^2 [J]
\end{align*}
\]
Simulations: Joukowski Foil Fish
E. Coli motility has two phases, run and tumble

- During run phase all flagella spin counter clockwise and bacterium moves forward
- During tumble phase some flagella spin clockwise and bacterium changes orientation

Emulate as Stochastic Source Seeking

$$\theta_{k+1} = \theta_k + w_k + \gamma w_{k-1} \frac{z-1}{z+h} \left[ J(x(\theta_k), y(\theta_k)) \right]$$
**E. Coli** Perform Ext. Seeking

30 sec single bacterium experiment (Berg, Harvard, 2000)
Summary
**Summary**

- **Extremum seeking**
  - a form of *optimization* - continuous in the input
  - a form of *adaptive* control - but not model based

- **Source seeking for nonholonomic vehicles - in lieu of simultaneously solving**
  - motion planning/trajectory generation
  - state and parameter estimation
  - trajectory tracking
  - (subject to an optimality cost)
Thank You