Compensation of Long Input Delays for Unstable Nonlinear and PDE Systems

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Smith predictor

Otto J. M. Smith (1917–)

(Professor at UC Berkeley, 1947–1988)

“A controller to overcome dead time,” *ISA*, vol. 6, pp. 28–33, 1959.

Applications of Delay Systems

- chemical process control
- combustion engines
- rolling mills
- control over communication networks/Internet and MPEG video transmission
- telesurgery
- machine tool “chatter”
- road traffic systems

Basic Observations

- thousands of papers and dozens of books since the 1940's (Tsypkin 1946)
- "golden era" = 1970s and early 1980s
  (Kalman, Sontag, Morse, Mitter, Artstein, Khargonekar, Tannenbaum)
- another "burst" of research activity after the introduction of LMIs (1990s)
- many basic problems still unsolved, still very active area of research
- delay systems are infinite dimensional
- the state is not a vector but a function (or a vector of functions);
  characteristic equation not a polynomial, involves exponentials
- stability analysis requires Krasovskii functionals, rather than Lyapunov functions

Benchmark Delay Systems (Scalar)

$$\dot{X}(t) \;=\; X(t-D)+U(t)$$ (**easy**, cancel or dominate by high gain)

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Outline

- Smith Predictor as a form of infinite-dimensional ‘backstepping’
- Robustness consequences
- Observer design with sensor delay
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$$U(t) = K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]$$

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Artstein's **reduction** approach (1982) and Manitius and Olbrot's **finite spectrum assignment** (1978).

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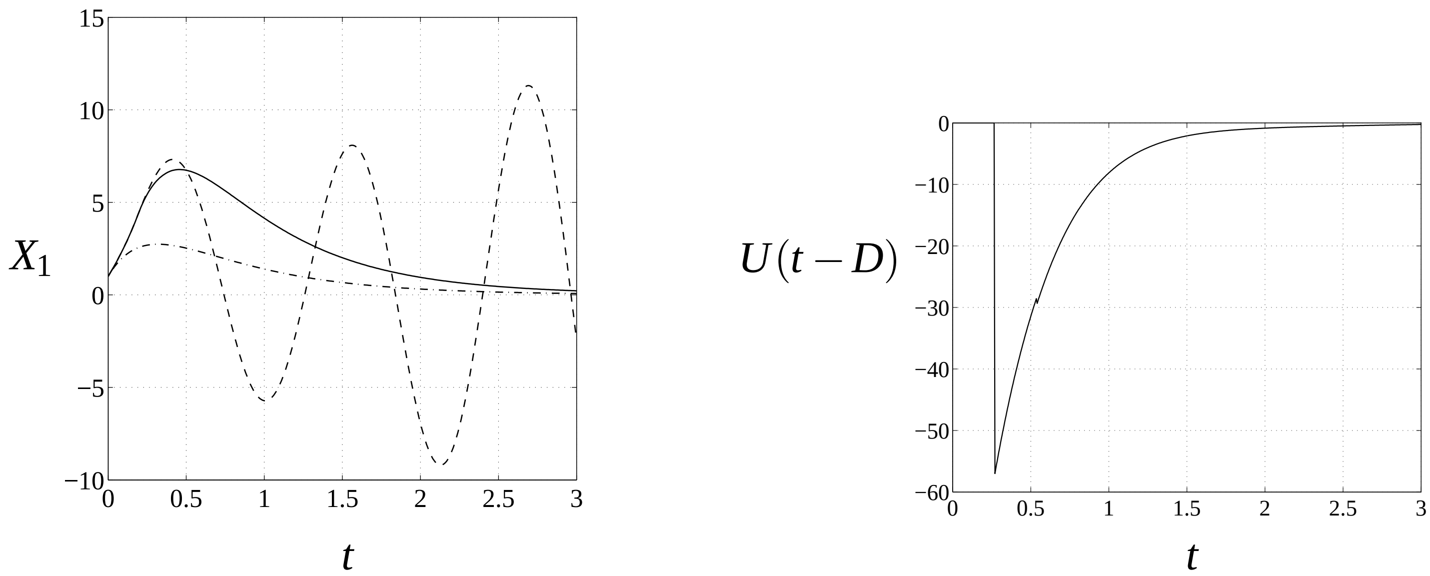
Extension of Smith Predictor to unstable systems.

Example. (Andrey Smyshlyaev)

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & -2 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Open-loop unstable (eigenvalues are 2 and $-1.5 \pm 1.4j$).



Left: stable nominal LQR controller ($Q = I$, $R = 1$) in the absence of delay (dash-dotted); unstable with nominal LQR controller in the presence of delay (dashed); stable with the backstepping controller in the presence of the delay (solid).

Right: delayed control input.

A transport PDE representation of $\dot{X}(t) = AX(t) + BU(t-D)$:

$$\begin{aligned}
\dot{X}(t) &= AX(t) + Bu(0,t) \\
u_t(x,t) &= u_x(x,t) \\
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Note that $u(x,t) = U(t+x-D)$.

Backstepping transformation and its inverse:

$$\begin{aligned} w(x,t) &= u(x,t) - \int_0^x K e^{A(x-y)} B u(y,t) dy - K e^{Ax} X(t) \\ u(x,t) &= w(x,t) + \int_0^x K e^{(A+BK)(x-y)} B w(y,t) dy + K e^{(A+BK)x} X(t) \end{aligned}$$

Target system

$$\begin{aligned} \dot{X}(t) &= (A+BK)X(t) + Bw(0,t) \\ w_t(x,t) &= w_x(x,t) \\ w(D,t) &= 0 . \end{aligned}$$

Cascade $w \mapsto X$. Each subsystem exponentially stable.

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Lyapunov functional

$$V(t) = X(t)^T P X(t) + 2\frac{\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)} \int_0^D (1+x) w(x,t)^2\, dx$$

Theorem 2 *Exp. stable in the norm* $\left(|X(t)|^2 + \int_0^D u(x,t)^2 dx\right)^{1/2}$.

A Short History of Backstepping

- 1990: nonlinear ODEs
- 2000: parabolic linear PDEs
- 2004: linearized Navier-Stokes PDEs
- 2004: 2nd order hyperbolic PDEs (wave equations and beams)
- 2005: adaptive control for linear parabolic PDEs
- 2005: 1st order hyperbolic PDEs and actuator delays
- 2006–: nonlinear parabolic PDEs

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Is there any benefit to having a Lyapunov function besides proving stability?

Inverse optimality and robustness to actuator lag.

Theorem 3 *There exists c^* such that the feedback system with the controller*

$$U(t) = \frac{c}{s+c}\left\{K\left[e^{AD}X(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]\right\},$$

is exponentially stable in the sense of the norm

$$N(t) = \left(|X(t)|^2 + \int_{t-D}^{t} U(\theta)^2 d\theta + U(t)^2\right)^{1/2}$$

for all $c > c^*$. Furthermore, there exists $c^{**} > c^*$ such that, for any $c \geq c^{**}$, the feedback minimizes the cost functional

$$J = \int_0^\infty \left(Q(t) + \dot{U}(t)^2\right) dt,$$

where $Q(t) \geq \mu N(t)^2$ for some $\mu(c) > 0$, which is such that $\mu(c) \to \infty$ as $c \to \infty$.

With a Lyapunov function, one can even quantify disturbance attenuation

$$\dot{X}(t) = AX(t) + BU(t-D) + Gd(t)$$

Theorem 4 *$\exists c^*$ s.t. $\forall c > c^*$, the feedback system is L_∞-stable, i.e., $\exists$ positive constants $\beta_1, \beta_2, \gamma_1$ s.t.*

$$N(t) \leq \beta_1 \mathrm{e}^{-\beta_2 t} N(0) + \gamma_1 \sup_{\tau \in [0,t]} |d(\tau)| .$$

Furthermore, $\exists c^{**} > c^*$ s.t. $\forall c \geq c^{**}$ the feedback minimizes the cost functional

$$J = \sup_{d \in \mathcal{D}} \lim_{t \to \infty} \left[2cV(t) + \int_0^t \left(Q(\tau) + \dot{U}(t)^2 - c\gamma_2 d(\tau)^2 \right) d\tau \right]$$

for any

$$\gamma_2 \geq \gamma_2^{**} = 8 \frac{\lambda_{\max}(PBB^T P)}{\lambda_{\min}(Q)} ,$$

where $Q(t) \geq \mu N(t)^2$ for some $\mu(c,\gamma_2) > 0$, which is such that $\mu(c,\gamma_2) \to \infty$ as $c \to \infty$, and $\mathcal{D}$ is the set of linear scalar-valued functions of X.

Robustness to Delay Mismatch

The biggest open question in robustness of predictor feedbacks.

$$\begin{aligned} \dot{X} &= AX + BU(t - D_0 - \Delta D) \\ U(t) &= K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right] \end{aligned}$$

ΔD either positive or negative

Theorem 5 $\exists\delta > 0$ *s.t.* $\forall\Delta D \in (-\delta, \delta)$ *the closed-loop system is exp. stable in the sense of the state norm*

$$N_2(t) = \left(|X(t)|^2 + \int_{t-\bar{D}}^{t} U(\theta)^2 d\theta\right)^{1/2},$$

where $\bar{D} = D_0 + \max\{0, \Delta D\}$.

Corollary 1 $\exists\delta > 0$ *s.t.* $\forall D_0 \in [0,\delta)$ *the system*

$$\begin{aligned} \dot{X} &= AX + BU(t)\,, \\ U(t) &= K\left[e^{AD_0}X(t) + \int_{t-D_0}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right] \end{aligned}$$

is exp. stable in the sense of the norm $\left(|X(t)|^2 + \int_{t-D_0}^{t} U(\theta)^2 d\theta\right)^{1/2}$.

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Observers for ODE Systems with **Sensor Delay**

$$
\begin{aligned}
\dot{X}(t) &= AX(t) + BU(t) \\
Y(t) &= CX(t - D),
\end{aligned}
$$

where (A, C) is an observable pair.

The output equation can be represented through the transport PDE as

$$
\begin{aligned}
u_t(x,t) &= u_x(x,t) \\
u(D,t) &= CX(t) \\
Y(t) &= u(0,t).
\end{aligned}
$$

Theorem 6 *The observer*

$$\begin{aligned} \dot{\hat{X}}(t) &= A\hat{X}(t) + BU(t) + \mathrm{e}^{AD}L\left(Y(t) - \hat{u}(0,t)\right) \\ \hat{u}_t(x,t) &= \hat{u}_x(x,t) + C\mathrm{e}^{Ax}L\left(Y(t) - \hat{u}(0,t)\right) \\ \hat{u}(D,t) &= C\hat{X}(t), \end{aligned}$$

where L is chosen such that $A-LC$ is Hurwitz, guarantees that the observer error system is e.s. in the norm

$$\left(|X(t) - \hat{X}(t)|^2 + \int_0^D (u(x,t) - \hat{u}(x,t))^2 dx\right)^{1/2}.$$

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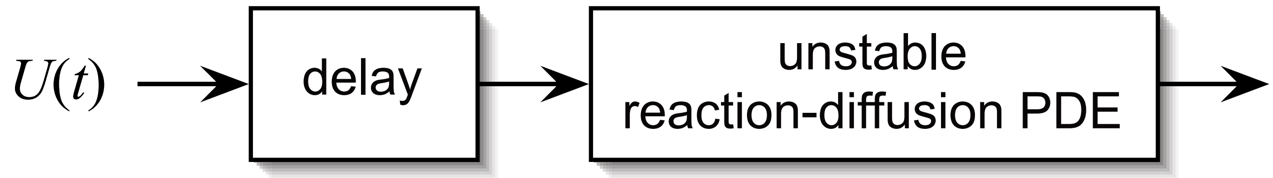
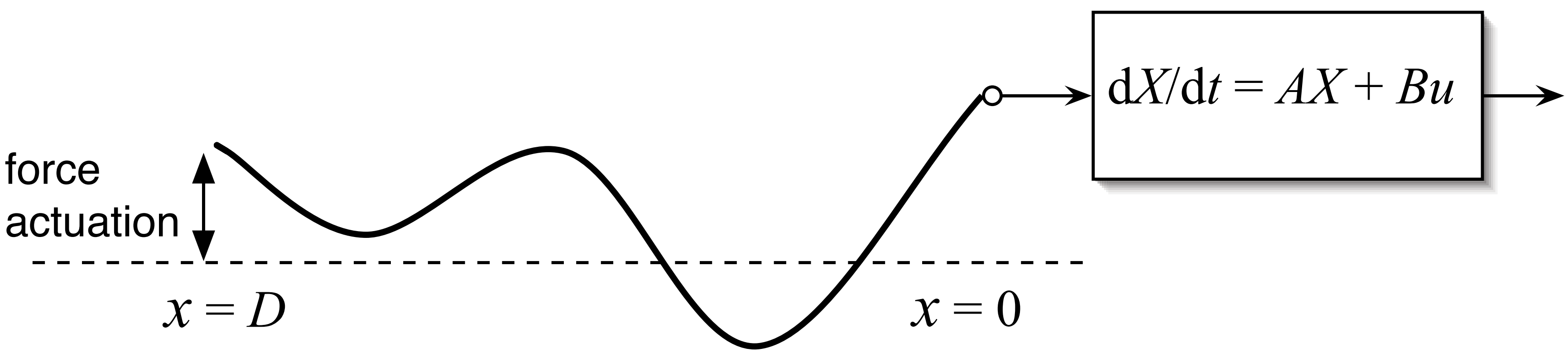
Mostly for **Fun**





Reaction-diffusion PDE plant with input delay:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + \lambda u(x,t) \\ u(0,t) &= 0 \\ u(1,t) &= U(t-D)\,. \end{aligned}$$

Controller

$$U(t) = 2\sum_{n=1}^{\infty}\int_0^1 \sin(\pi n\xi)\lambda\xi \frac{I_1\left(\sqrt{\lambda\left(1-\xi^2\right)}\right)}{\sqrt{\lambda\left(1-\xi^2\right)}}d\xi \qquad \left(I_1(\cdot) = \text{Bessel function}\right)$$

$$\times\left(-\mathrm{e}^{\left(\lambda-\pi^2n^2\right)D}\int_0^1 \sin(\pi ny)u(y,t)dy + \pi n(-1)^n\int_{t-D}^{t}\mathrm{e}^{\left(\lambda-\pi^2n^2\right)(t-\theta)}U(\theta)d\theta\right)$$

Theorem 7 $\exists\rho(D,\lambda)>0$ *s.t.*

$$\Upsilon(t) \le \rho(D,\lambda)\mathrm{e}^{cD}\Upsilon(0)\mathrm{e}^{-\min\{2,c\}t}\,, \qquad \forall t\ge 0\,,$$

where

$$\Upsilon(t) = \int_0^1 u^2(x,t)dx + \int_{t-D}^{t}\left(U^2(\theta)+\dot{U}^2(\theta)\right)d\theta\,.$$

PDE-ODE cascade (actuation through a string):

$$\dot{X}(t) = AX(t) + Bu(0,t)$$
$$u_{tt}(x,t) = u_{xx}(x,t)$$
$$u_x(0,t) = 0$$
$$u(D,t) = U(t)$$

Controller

$$U(t) = K\Sigma(D,c)X(t) + \int_0^D \varphi(D-y)u(y,t)dy + \int_0^D \psi(D-y)u_t(y,t)dy,$$

where $c > 0$ and

$$\begin{aligned}
\Sigma(D,c) &= M(D) + c\int_0^D M(y)Ady \\
\varphi(x) &= \int_0^x KM(\xi)ABd\xi + cK(I+M(x))B \\
\psi(x) &= \int_0^x KM(\xi)Bd\xi + c\int_0^x \int_0^\eta KM(\xi)ABd\xi d\eta - c \\
M(x) &= \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & A^2 \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}.
\end{aligned}$$

Theorem 8 *$\exists G, g > 0$ s.t.*

$$\Gamma(t) \leq G e^{-gt}\Gamma(0)\,, \qquad \forall t \geq 0$$

where

$$\Gamma(t) = |X(t)|^2 + \int_0^D u_x(x,t)^2 dx + \int_0^D u_t(x,t)^2 dx\,.$$

Furthermore, the closed-loop spectrum is

$$\mathrm{eig}\{A+BK\} \cup \left\{ -\frac{1}{2}\ln\left|\frac{1+c}{1-c}\right| + j\frac{\pi}{D}\begin{cases} n+\frac{1}{2}, & 0 \leq c < 1 \\ n\,, & c > 1 \end{cases} \right\}$$

where $n \in \mathbb{Z}$.

Bonus: damping feedback for wave equation with Dirichlet actuation:

$$U(t) = -c\int_0^D u_t(y,t)dy\,.$$

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Delay-Adaptive Control

$$
\begin{aligned}
\dot{X}(t) &= AX(t) + Bu(0,t) \\
Du_t(x,t) &= u_x(x,t) \\
u(1,t) &= U(t),
\end{aligned}
$$

Control

$$
U(t) = K\left[e^{A\hat{D}(t)} X(t) + \hat{D}(t) \int_0^1 e^{A\hat{D}(t)(1-y)} Bu(y,t)dy \right]
$$

Update law

$$
\frac{d}{dt}\hat{D}(t) = -\gamma \frac{\int_0^1 (1+x)w(x,t)Ke^{A\hat{D}(t)x}dx \,(AX(t) + Bu(0,t))}{1 + X(t)^T PX(t) + b\int_0^1 (1+x)w(x,t)^2 dx}
$$

$$
w(x,t) = u(x,t) - \hat{D}(t) \int_0^x Ke^{A\hat{D}(t)(x-y)} Bu(y,t)dy - Ke^{A\hat{D}(t)x} X(t).
$$

Theorem 9 *There exists $\gamma^* > 0$ such that for any $\gamma \in (0,\gamma^*)$ there exist positive constants R and ρ (independent of the initial conditions) such that for all initial conditions satisfying $(X_0, u_0, \hat{D}_0) \in \mathbb{R}^n \times L_2(0,1) \times [0,\bar{D}]$, the following holds:*

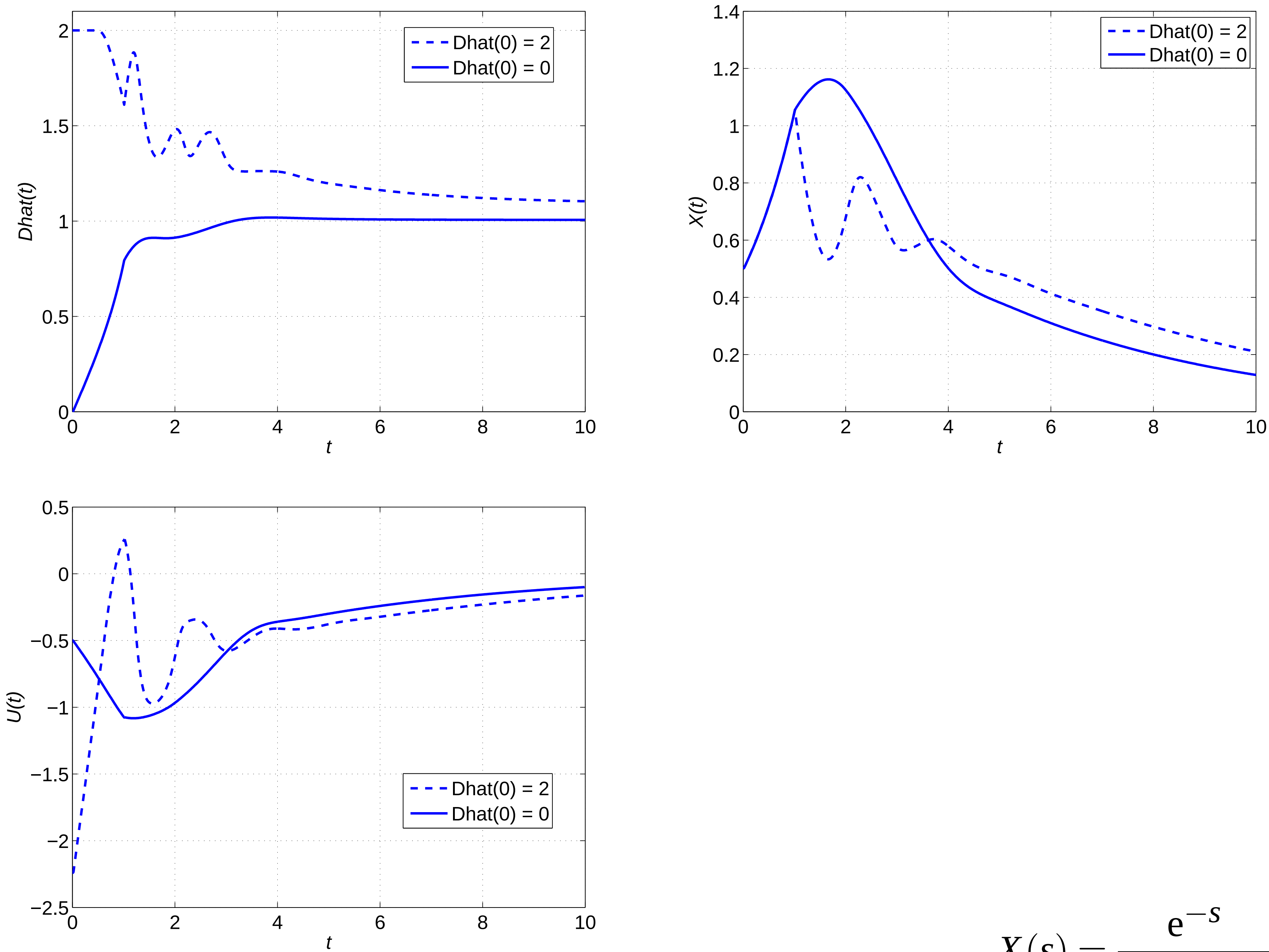
$$\Upsilon(t) \leq R\left(e^{\rho\Upsilon(0)} - 1\right), \quad \forall t \geq 0,$$

where

$$\Upsilon(t) = |X(t)|^2 + \int_0^1 u(x,t)^2 dx + \left(D - \hat{D}(t)\right)^2.$$

Furthermore,

$$\lim_{t\to\infty} X(t) = 0, \quad \lim_{t\to\infty} U(t) = 0.$$



$$X(s) = \frac{e^{-s}}{(s - 0.75)} U(s)$$

0–1 sec The delay precludes any influence of the control on the plant, so $X(t)$ shows an exponential open-loop growth.

1–3 sec The plant starts responding to the control and its evolution changes qualitatively, resulting also in a qualitative change of the control signal.

3–4 sec When the estimation of $\hat{D}(t)$ ends at about 3 seconds, the controller structure becomes linear. However, due to the delay, the plant state $X(t)$ continues to evolve based on the inputs from 1 second earlier, so, a non-monotonic transient continues until about 4 seconds.

4 sec and onwards The (X,U) system is linear and the delay is sufficiently well compensated, so the response of $X(t)$ and $U(t)$ shows a monotonically decaying exponential trend of a first order system.

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Time-Varying Delay

$$\dot{X}(t) = AX(t) + BU(\phi(t))$$

Predictor feedback

$$U(t) = K\left[e^{A\left(\phi^{-1}(t)-t\right)}X(t) + \int_{\phi(t)}^{t} e^{A\left(\phi^{-1}(t)-\phi^{-1}(\theta)\right)}B\frac{U(\theta)}{\phi'\left(\phi^{-1}(\theta)\right)}d\theta\right]$$

Transport PDE representation

$$u(x,t) = U\left(\phi\left(t + x\left(\phi^{-1}(t) - t\right)\right)\right)$$

Time-varying backstepping transformation

$$w(x,t) = u(x,t) - Ke^{Ax\left(\phi^{-1}(t)-t\right)}X(t) - K\int_0^x e^{A(x-y)\left(\phi^{-1}(t)-t\right)}Bu(y,t)\left(\phi^{-1}(t) - t\right)dy$$

Target system

$$
\begin{aligned}
\dot{X}(t) &= (A+BK)X(t)+Bw(0,t)\,,\\
w_t(x,t) &= \pi(x,t)w_x(x,t)\,,\\
w(1,t) &= 0\,,
\end{aligned}
$$

where the variable speed of propagation is

$$
\pi(x,t) = \frac{1+x\left(\frac{d\left(\phi^{-1}(t)\right)}{dt}-1\right)}{\phi^{-1}(t)-t}
$$

Theorem 10 *Let the delay function $\delta(t) = t-\phi(t)$ be strictly positive and uniformly bounded from above. Let the delay rate function $\delta'(t)$ be strictly smaller than 1 and uniformly bounded from below. There exist positive constants G and g (the latter one being independent of ϕ) such that*

$$
|X(t)|^2+\int_{\phi(t)}^{t}U^2(\theta)d\theta \le G\mathrm{e}^{-gt}\left(|X_0|^2+\int_{\phi(0)}^{0}U^2(\theta)d\theta\right), \quad \textit{for all } t\ge 0\,.
$$

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Nonlinear Systems

- Teel (1998) proved local robustness to sufficiently small delays of asymptotically stabilizing feedback laws for nonlinear systems (by nonlinear ISS small gain theorem and equivalence to the Razumikhin theorem). No control design or re-design.

- 90% of the results on nonlinear systems with delays are for state delays, not actuator delays.

- Leading contributors: Jankovic, Mazenc, Pepe, Karafyllis.

Consider the system

$$\dot{Z}(t) = f(Z(t), U(t - D))$$

and suppose that a function $\kappa(Z)$ is known such that

$$\dot{Z} = f(Z, \kappa(Z))$$

is globally asymptotically stable at the origin.

Predictor-based controller (predictor given implicitly):

$$\begin{aligned} U(t) &= \kappa(P(t)) \\ P(t) &= \int_{t-D}^{t} f(P(\theta), U(\theta))d\theta + Z(t) \end{aligned}$$

Theorem 11 *If $\dot{Z} = f(Z, U)$ is forward complete, then $\exists \beta \in \mathcal{KL}$ such that*

$$|Z(t)| + \|U\|_{L_\infty[t-D,t]} \leq \beta\left(|Z(0)| + \|U_0\|_{L_\infty[-D,0]}, t\right)$$

for all $(Z_0, U_0) \in \mathbb{R}^n \times L_\infty[-D, 0]$ and for all t ≥ 0.

Example of a system in *strict-feedforward* form:

$$\begin{aligned}
\dot{X}_1(t) &= X_2(t) + X_3^2(t) \\
\dot{X}_2(t) &= X_3(t) + X_3(t)U(t-D) \\
\dot{X}_3(t) &= U(t-D)
\end{aligned}$$

Controller

$$\begin{aligned}
U(t) = & -P_1(t) - 3P_2(t) - 3P_3(t) - \frac{3}{8}P_2^2(t) \\
& + \frac{3}{4}P_3(t)\left(-P_1(t) - 2P_2(t) + \frac{1}{2}P_3(t) + \frac{P_2(t)P_3(t)}{2}\right. \\
& \left. + \frac{5}{8}P_3^2(t) - \frac{1}{4}P_3^3(t) - \frac{3}{8}\left(P_2(t) - \frac{P_3^2(t)}{2}\right)^2\right)
\end{aligned}$$

where the D-second-ahead predictor of $(X_1(t), X_2(t), X_3(t))$ is given explicitly by

$$\begin{aligned} P_1(t) &= X_1(t) + DX_2(t) + \frac{1}{2}D^2X_3(t) + DX_3^2(t) + 3X_3(t)\int_{t-D}^{t}(t-\theta)U(\theta)d\theta \\ &\quad + \frac{1}{2}\int_{t-D}^{t}(t-\theta)^2U(\theta)d\theta + \frac{3}{2}\int_{t-D}^{t}\left(\int_{t-D}^{\theta}U(\sigma)d\sigma\right)^2 d\theta \\ P_2(t) &= X_2(t) + DX_3(t) + X_3(t)\int_{t-D}^{t}U(\theta)d\theta + \int_{t-D}^{t}(t-\theta)U(\theta)d\theta \\ &\quad + \frac{1}{2}\left(\int_{t-D}^{t}U(\theta)d\theta\right)^2 \\ P_3(t) &= X_3(t) + \int_{t-D}^{t}U(\theta)d\theta \end{aligned}$$

Employs nonlinear Volterra operators on the actuator delay state.

A representative example outside of the *forward complete* class

$$\frac{dZ(t)}{dt} = Z(t)^2 + U(t-D)$$

Nominal feedback (for $D = 0$):

$$U(t) = -Z(t)^2 - cZ(t)\,, \quad c > 0.$$

Theorem 12 *For any given $D > 0$, $\exists$ initial conditions $Z(0) + \sup_{\theta\in[-D,0]} \int_{-D}^{\theta} U(\sigma)d\sigma < \frac{1}{D}$, i.e., not causing finite escape before $t = D$ in open loop, for which the solution escapes to infinity before*

$$t = \frac{3}{2}D\,.$$

Nonlinear predictor feedback:

$$\begin{aligned} U(t) &= -P(t)^2 - cP(t) \\ P(t) &= \int_{t-D}^{t} P(\theta)^2 d\theta + Z(t) + \int_{t-D}^{t} U(\theta) d\theta \end{aligned}$$

Theorem 13 *If*

$$Z(0) + \sup_{\theta \in [-D,0]} \int_{-D}^{\theta} U(\sigma) d\sigma < \frac{1}{D},$$

then the following holds:

$$Z(t)^2 + \int_{t-D}^{t} U(\theta)^2 d\theta \le \rho\left(Z(0)^2 + \int_{-D}^{0} U(\theta)^2 d\theta\right) e^{-t/4}, \quad \forall t \ge 0,$$

where

$$\begin{aligned} \rho(r) &= 8\gamma(r) + 16\gamma^2(r) \\ \gamma(r) &= 4(1+D)r + 16(1+D)^3 \frac{r^2}{(1-D\sqrt{r})^4}. \end{aligned}$$

Any Moral to the Story?

Numerous open problems in delay and other PDE systems,
even in the linear case.