Robust Adaptive Control and the **Greeks** who Made it Possible

Miroslav Krstic

joint work with **Iasson Karafyllis**

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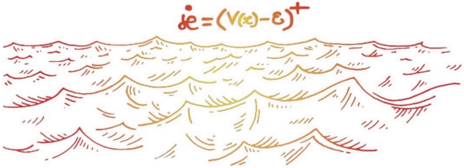
Robust Adaptive Control

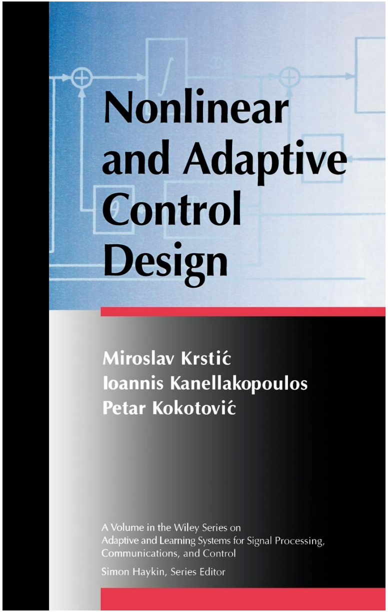
Deadzone-Adapted Disturbance Suppression



Iasson Karafyllis • Miroslav Krstic

Advances in Design and Control

Adaptive Backstepping



- *Nonlinear & Adaptive Control Design*—1995
- adapt. backstepping & strict-fbk systems (1991): Kanellakopoulos, Kokotovic, Morse
- prescribed performance (2008): Bechlioulis & Rovithakis

Robust Adaptive Control:

40+ years since the KEY (unimproved) ideas

Robust Adaptive Control — the Ordeal

- adaptive control stability (MRAC) solved $\approx$ 1980

- **Rohrs-Athans** (1982) unstable *simulations* (disturbances, unmod. dynamics)
- deadzone: Peterson-Narendra-Annaswamy (1982-1989)
  small disturbance, requires PE
- parameter projection must know parameter bound
- **σ-modification**:
  Ioannou (**1983**–1996), + Kokotovic, Tsakalis/Tao/Sun/Datta/Polycarpou, etc.
- last 30 years of **my** editorial observation of adaptive (nonlinear) control:
  σ-modification owns 95% of the "robustification market"

“IOANNOU effect”

Effect of 1996 book by Ioannou & Sun

- Systematizes, comprehensively covers, and advances **linear** (robust) adaptive control to date (800 pages)
- Deals not only with disturbances but also (multiplicative) unmodeled dynamics,

$$y = G_0(s)\,(1 + \Delta_m(s))\,u\,.$$

Example of a theorem in [IS'96], roughly expressed:
Theorem 9.3.3. If

$$\min\left\{1 + \frac{\|\Delta_m\|^2_{\infty\delta_0}}{\delta_0^{2(n^*+1)}}, \sigma + \|H\Delta_m\|^2_{2\delta_0}\right\} \quad \text{suffic. small}$$

then the tracking error $e(t)$ is bounded-in-the-mean, i.e.,

$$\frac{1}{T}\int_t^{t+T} e^2(\tau)d\tau \leq c\left(\frac{1}{T} + \sigma + \|\Delta_m\|^2_{\infty\delta_0} + \|H\Delta_m\|^2_{2\delta_0}\right)$$

Effect of 1996 book by Ioannou & Sun

Results *dishearteningly* **clear and precise** regarding

- the complexity of the robust adaptive problem,
- the technical virtuosity required to keep making progress.

Disturbance-Robustness under σ-Modification

Scalar Example

$$\boxed{\dot{x} = \theta x + u + d}$$

Adaptive controller with leakage:

$$\begin{aligned} u &= -cx - \hat{\theta} x \\ \dot{\hat{\theta}} &= \Gamma x^2 \ -\sigma\hat{\theta} \end{aligned}$$

p-IOS and p-OAG

Lyapunov calculation yields practical input-output-stability (p-IOS) w.r.t. disturbance d:

$$|x(t)| \quad \leq \quad e^{-\sigma t/2}\left(|x_0| + \left|\hat{\theta}_0 - \theta\right|\right) + \frac{\sqrt{\Gamma}}{\sigma}\|d\|_\infty + |\theta|$$

Additionally, practical output asymptotic gain (p-OAG):

$$\boxed{\limsup_{t\to+\infty}|x(t)| \leq \frac{\sqrt{\Gamma}}{\sigma}\limsup_{t\to+\infty}|d(t)| + |\theta|}$$

The p-OAG $\sqrt{\Gamma}/\sigma$ can be reduced (though not to zero) by increasing leakage σ.

But the bias $|\theta|$ is neither known nor reducible.

The culprit for the bias $|\theta|$ is the leakage itself

- The term $\boxed{\Gamma x^2}$ in the update

$$\dot{\hat{\theta}} = \boxed{\Gamma x^2} - \sigma\hat{\theta}$$

  pumps up $\hat{\theta}$, i.e., pumps error $\tilde{\theta} = \theta - \hat{\theta}$ to *negative*/disturbance-attenuating values.

- But in the parameter error system

$$\dot{\tilde{\theta}} = -\sigma\tilde{\theta} - \Gamma x^2 + \sigma\theta$$

  the term $+\sigma\theta$ **counters** the work of Γx^2.

σ-modification's task CONTRADICTS the task of adaptation.

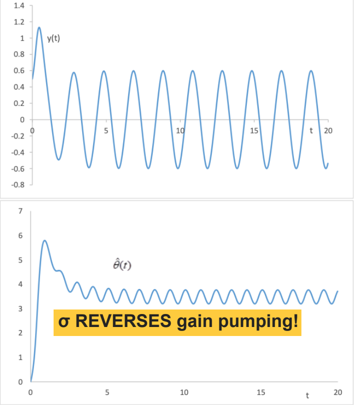


Fig. 1. $(y, \hat{\theta}, u)$ for closed-loop system with **σ-mod**,
with $d(t) = 2\sin(\pi t)$, $\theta = 3$,
$c = 1, \Gamma = 10, \sigma = q = 0.5$,
$y_0 = 0.5$, $\hat{\theta}(0) = 0$.

DADS: the Scalar Case

(DADS = Deadzone-Adapted Disturbance Suppression)

The gist of what you will see:

$$
\begin{aligned}
\dot{x} &= \theta x + u + d \\
u &= -x - \kappa \left(2 + x^2\right) x \\
\dot{\kappa} &= \Gamma \max\left\{0, x^2 - \delta^2\right\}
\end{aligned}
$$

guarantees

$$\boxed{\limsup_{t \to +\infty} |x(t)| \le \delta}$$

for arbitrarily small $\delta > 0$ independent of $\sup_{t \ge 0} |d(t)|$ and $|\theta|$,

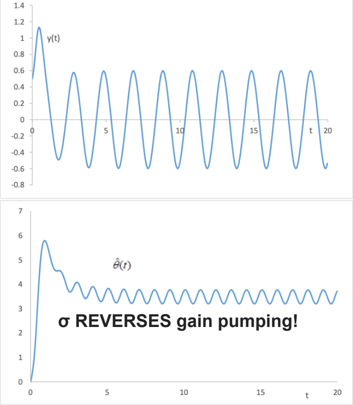


Fig. 1. $(y, \hat{\theta}, u)$ for closed-loop system with σ**-mod**,
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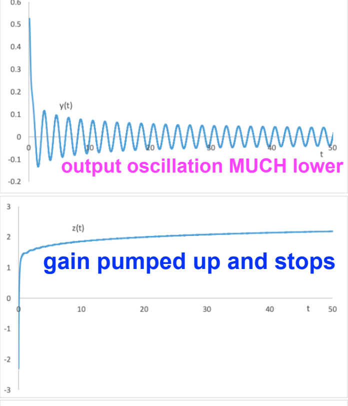


Fig. 2. $(y, \hat{\theta}, u)$ for **DADS** closed-loop system,
with $d = 2\sin(\pi t)$, $\theta = 3$,
$c = 1/2, q = 1/4, \Gamma = 50, \delta = 0.01$,
$y_0 = 0.5$, $z_0 = -\ln(10)$.

TENFOLD improved disturbance suppression under DADS, using the SAME CONTROL EFFORT, relative to σ-modification in Figure 1.

Intuition behind *deadzone*

- update "pumps up" the controller dynamic gain κ to a sufficient size, adapting the gain to the unknown disturbance d,
- deadzone shuts off further pumping up of the gain once the state is in the prespecified residual set.

Unlike σ-mod, **the deadzone's task does not contradict the adaptation's task.**

Innovations behind DADS

In **design**:

- deadzone is not new
- new uses of completing squares/dynamic nonlinear damping, to obliterate ("practically," ultimately) the effects of disturbance and parametric uncertainty

In **analysis**:

- A new comparison lemma (I call it "**Karafyllis lemma**")

Notation for deadzone function $(\cdot)^+$

deadzone acting on
$s = x^2 - \delta^2$

$$\boxed{s^+ := \max\{0, s\}} = \mathrm{ReLU}(s)$$

ReLU — notation common in neural networks, representing the “rectifier linear unit”

Control Design Idea

Recall the adaptive controller

plant: $\dot{x} = \theta x + u + d$

controller: $\boxed{u = -x - \kappa \left(2 + x^2\right) x}$

gain update: $\dot{\kappa} = \Gamma \left(x^2 - \delta^2\right)^+ , \qquad \kappa(0) > 1$

What drives this **Control Law Design?**

Lyapunov function (independent of gain κ)

$$V(x) = \frac{x^2}{2}$$

Control to employ dynamic nonlinear damping and ensure

$$\boxed{\dot{V} \leq -2cV + \frac{d^2 + \left((|\theta| - \kappa)^+\right)^2}{4q\kappa}}$$

This form of inequality ensures

- ISS w.r.t. d
- gain $\kappa(t)$ *ultimately dominates* the unknown $|\theta|$; makes $x(t) \longrightarrow 0$ when $d(t) \longrightarrow 0$

Stability/Convergence Properties

Basic results from Lyapunov inequality

Practical IOS

$$\boxed{|x(t)| \leq e^{-ct}\,|x_0| + \frac{1}{2\sqrt{cq}}\|d\|_\infty + \frac{1}{2\sqrt{cq}}\left(|\theta| - 1\right)^+}$$

(Practical) Asymptotic Gain to Disturbance

$$\boxed{\limsup_{t \longrightarrow +\infty} |x(t)| \leq \frac{1}{2\sqrt{cq}}\left(\limsup_{t \longrightarrow +\infty} |d(t)| + (|\theta| - \kappa_\infty)^+\right)}$$

Main Result

Theorem

The DADS closed-loop system satisfies

p-IOS	*assignable gain to d*, *with* *bias dependent on $\|\theta\|$*

All of these also achieved with σ**-mod**.

No Parameter Drift

No Parameter Drift

This is the most novel and complex part of our analysis approach.

Lemma (*Comparison lemma* on deadzone in "exterminator-pest" feedback)

Consider the differential inequalities

$$\begin{aligned} 0 \leq \dot{\kappa} &\leq (V - \varepsilon)^+ , & \kappa_0 > 1 \\ \dot{V} &\leq -V + \frac{\mu}{\kappa} , & V_0 \geq 0 \end{aligned}$$

Then,

$$\kappa_0 \leq \kappa(t) \leq \max\left\{\kappa_0, \frac{\mu}{\varepsilon}\right\} + \left(V_0 - \varepsilon + \frac{\mu}{\kappa_0}\right)^+ , \qquad \forall t \geq 0 \, .$$

Karafyllis

Pest (V) grows then decays. Exterminator (κ) grows but bounded.

Explicit bound on parameter update

Practical uniform bounded-input bounded state (p-UBIBS) property:

$$\kappa_0 \leq \kappa(t) \leq \kappa_\infty \leq \max\left\{\kappa_0, \frac{\mu}{4\delta^2}\right\} + \frac{1}{2}\left(x_0^2 - \delta^2 + \frac{\mu}{4\kappa_0}\right)^+, \qquad \forall t \geq 0$$

where

$$\mu = \|d\|_\infty^2 + \left((|\theta| - \kappa_0)^+\right)^2 .$$

The bound on $\kappa(t)$ is

- increasing in $\|d\|_\infty$ and $|x_0|$
- non-decreasing in $|\theta|$

and

- increasing in $1/\delta$

(The tighter the asymptotic regulation δ desired, the higher $\kappa(t)$ needs to be pumped.)

Main Result

Theorem

The DADS closed-loop system satisfies

p-UBIBS	*no parameter drift*
p-IOS	*assignable gain to d*, *with* *bias dependent on $|\theta|$*

STILL,

All of these also achieved with **σ-mod**.

Zero *Asymptotic Gain* to Disturbance

Practical Regulation for Arbitrary Disturbance

By the Karafyllis lemma - and - Barbalat's lemma for $\dot{\kappa}(t) = \Gamma\left(x^2(t) - \delta^2\right)^+$, we obtain the zero practical asymptotic gain (zero p-OAG) property

$$\boxed{\limsup_{t\to+\infty} |x(t)| \leq \delta}$$

Perfect Regulation under Asymptotically Vanishing Disturbance

When $\lim_{t \to +\infty} d(t) = 0$ (the disturbance-vanishing case),

and when either $|\theta| \le 1$ or $\kappa_\infty \ge |\theta|$,

σ prevents convergence of x(t) to 0

perfect regulation is achieved,

$$\boxed{\lim_{t \to +\infty} x(t) = 0}$$

To increase the *chance* of $\kappa_\infty \ge |\theta|$, increase Γ.

Perfect Regulation under Asymptotically Vanishing Disturbance

When $\lim_{t \to +\infty} d(t) = 0$ (the disturbance-vanishing case),

and when either $|\theta| \leq 1$ or $\kappa_\infty \geq |\theta|$, (similar to switching-σ)

perfect regulation is achieved,

$$\boxed{\lim_{t \to +\infty} x(t) = 0}$$

To increase the *chance* of $\kappa_\infty \geq |\theta|$, increase Γ.

Summary:
MAIN RESULT

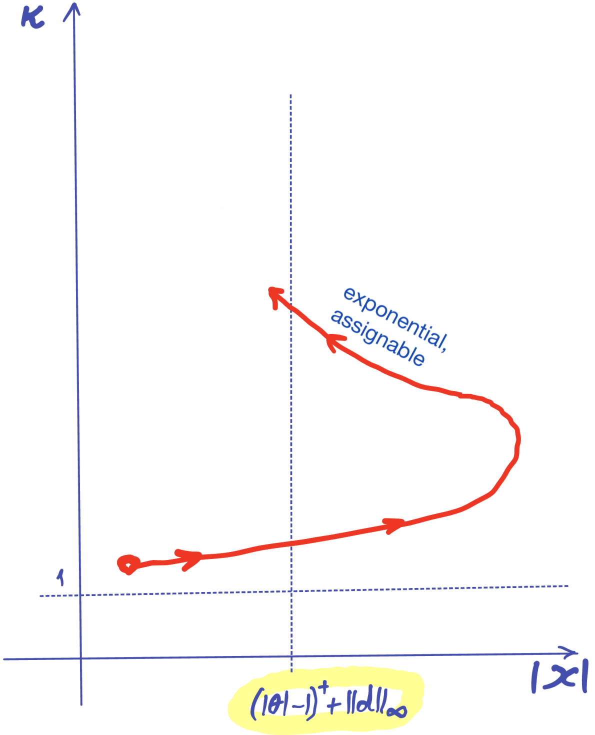
Main Result

Theorem

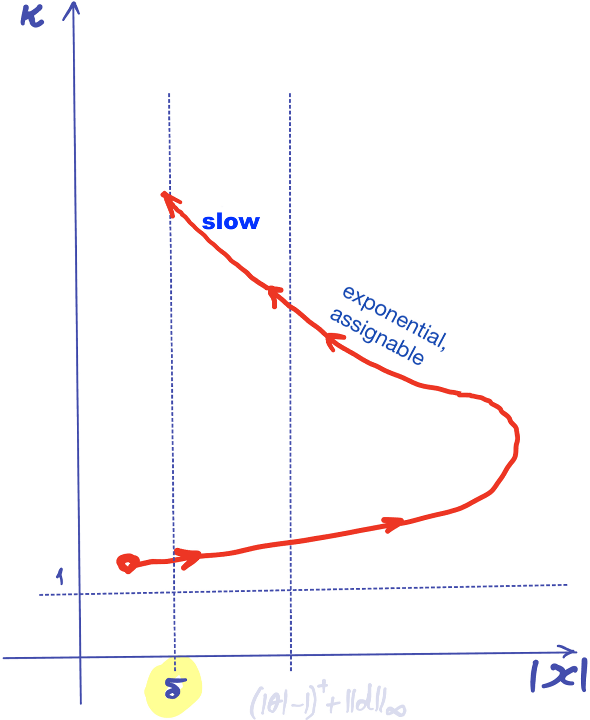
The DADS closed-loop system satisfies

p-UBIBS	*no parameter drift*
p-IOS	*assignable gain to d, with bias dependent on $\|\theta\|$*
zero p-OAG	*residual error indep. of (d, θ) and assignably small*
$x(t) \to 0$	*when $d(t) \to 0$ and κ_∞ large*
IOS w/ small $\|\theta\|$	*IOS gain & asymptotic gain to d are assignable*

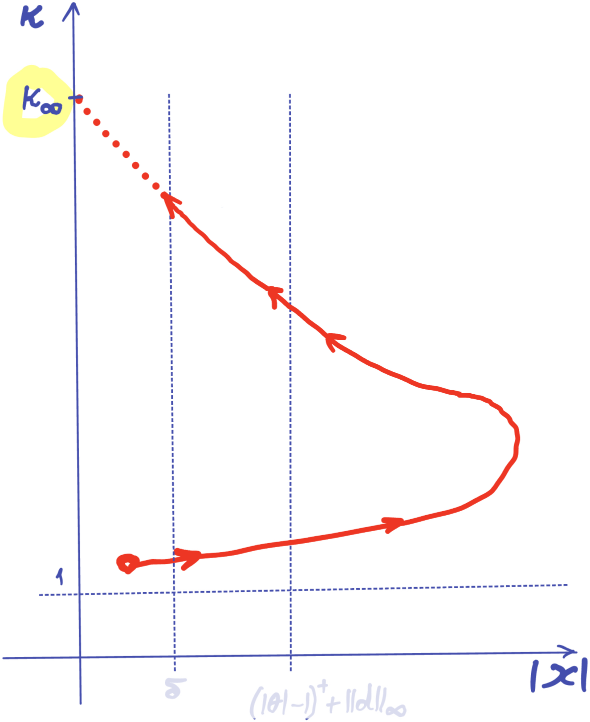
The properties in BLUE are NOT guaranteed with σ-mod.



κ
1
exponential,
assignable
$(|\theta|-1)^+ + \|d\|_\infty$
$|x|$



κ
slow
exponential,
assignable
1
δ
$(|\theta|-1)^+ + \|d\|_\infty$
$|x|$



κ
κ_∞
1
δ
$(|\theta|-1)^+ + \|d\|_\infty$
$|x|$

BACKSTEPPING
for Strict-Feedback Systems

DADS Backstepping for Strict-Feedback Systems ★

Theorem

Consider the strict-feedback system

UNMATCHED parametric uncertainty

$$\boxed{\dot{x}_i = x_{i+1} + \varphi_i'(x_1, \ldots, x_i)\theta + \alpha_i'(x_1 \ldots, x_i)d} \qquad i = 1, \ldots, n, \qquad x_{i+1} = u$$

Explicit backstepping design guarantees, globally, the p-OAG, p-IOS, p-UBIBS properties

$$\limsup_{t\to+\infty} |x_1(t)| \leq \varepsilon$$

$$|x_1(t)|^2 \leq 2e^{-2c_0 t} V_n(x_0, z_0) + \frac{\|d\|_\infty^2 + \left(\left(\|\theta\|_\infty - 1 - e^{z_0}\right)^+\right)^2}{4c_0\beta_0\left(1 + e^{z_0}\right)}$$

$$|x(t)| \leq B\left(x_0, z_0, \|d\|_\infty, \|\theta\|_\infty\right)$$

$$z_0 \leq z(t) \leq \lim_{s\to+\infty} z(s) \leq B\left(x_0, z_0, \|d\|_\infty, \|\theta\|_\infty\right)$$

Explicit DADS Backstepping

$$\zeta_i = x_i - k_{i-1}(\bar{x}_{i-1}, z)\,, \qquad i = 1, \ldots, n$$

$$V_i(\bar{x}_i, z) = \frac{1}{2}\sum_{j=1}^{i} \zeta_i^2$$

$$\pi_i'(\bar{x}_i, z) = \varphi_i' - \sum_{j=1}^{i-1} \frac{\partial k_{i-1}}{\partial x_j}\varphi_j' =: p_i'(\bar{x}_i, z)\bar{\zeta}_i\,, \qquad \rho_i'(\bar{x}_i, z) = \alpha_i' - \sum_{j=1}^{i-1} \frac{\partial k_{i-1}}{\partial x_j}\alpha_j'$$

$$k_i(\bar{x}_i, z) = -\left[c_i + r_i(\bar{x}_i, z)\right]\zeta_i - \zeta_{i-1} + \sum_{j=1}^{i-1} \frac{\partial k_{i-1}}{\partial x_j} x_{j+1}\,, \qquad c_i > \sum_{j=i}^{n} b_j + \sum_{j=2}^{i-1} \gamma_j + n g_i$$

$$r_i(\bar{x}_i, z) = (1 + \exp(z))\left(\beta_i|\rho_i|^2 + \beta_i|\pi_i|^2 + \frac{1}{4b_i}|p_i|^2\right)$$

dynamic nonlinear damping

$$+\frac{\Gamma^2}{4}\exp(-2z)\left[\sum_{j=2}^{i-1} \frac{1}{4\gamma_j}\zeta_j^2 \left(\frac{\partial k_{j-1}}{\partial z}\right)^2 + \left(\frac{\partial k_{i-1}}{\partial z}\right)^2 \sum_{j=1}^{i} \frac{1}{4g_j}\zeta_j^2\right]$$

domination of adaptation

Adaptive controller: $u = k_n(w, z)$

$$\dot{z} = \Gamma \exp(-z)\left(V_n - \frac{\varepsilon^2}{2}\right)^+$$

Aircraft Wing Rock

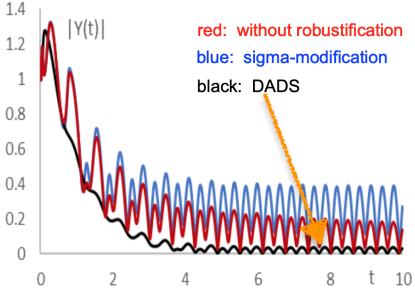
Limit cycle at high angle-of-attack

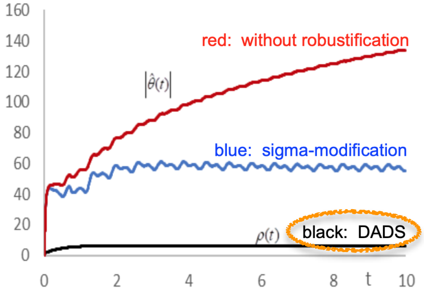
$$
\begin{array}{lrcl}
\text{roll angle} & \dot{x}_1 & = & x_2 \\
\text{roll rate} & \dot{x}_2 & = & \varphi^{\mathrm{T}}(x_1, x_2)\theta + x_3 + d_2 \\
\text{aileron angle} & \dot{x}_3 & = & u + d_3
\end{array}
$$

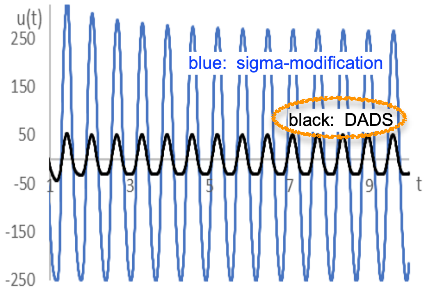
φ(roll-angle, roll-rate) = aerodynamic forces vector

disturbance vector $d(t)$ = taken sinusoidal for simulations

Aircraft Wing Rock







But what about robustness to NOISE?

$$\begin{array}{lrcl} \text{plant:} & \dot{x} & = & \theta x + u \\ \text{output:} & y & = & x + n(t) \\ \text{controller:} & u & = & -y - \kappa\left(2 + y^2\right) y \\ \text{gain update:} & \dot{\kappa} & = & \Gamma\left(y^2 - \delta^2\right)^+ \end{array}$$

$$\boxed{\limsup_{t \to +\infty} |x(t)| \leq \delta + \limsup_{t \to +\infty} |n(t)|}$$

$$\kappa_0 \leq \kappa(t) \leq \kappa_\infty \leq \max\left\{\kappa_0, \frac{\mu}{4\delta^2}\right\} + \frac{1}{2}\left(\|n\|_\infty^2 - \delta^2 + \frac{\mu}{4\kappa_0}\right)^+$$

$$\mu = \left\|\frac{\mathrm{d}n}{\mathrm{d}t}\right\|_\infty^2 + |\theta| \|n\|_\infty^2 + \left(\left(|\theta| - \kappa_0\right)^+\right)^2$$

Future Work

- tracking — treats reference as a disturbance (brute force), due to state reference trajectory being affected by unknown parameters
- output feedback — adaptive observer error likely to manifest itself as a disturbance and be tractable
- **unmodeled dynamics**, ISS/small-gain

Recap

Adaptive control can achieve arbitrarily good regulation

- for arbitrarily large disturbances
- without PE

No "fundamental limitation," **NO "CURSE OF UNCERTAINTY"!**