Extremum Seeking for Limit Cycle Minimization

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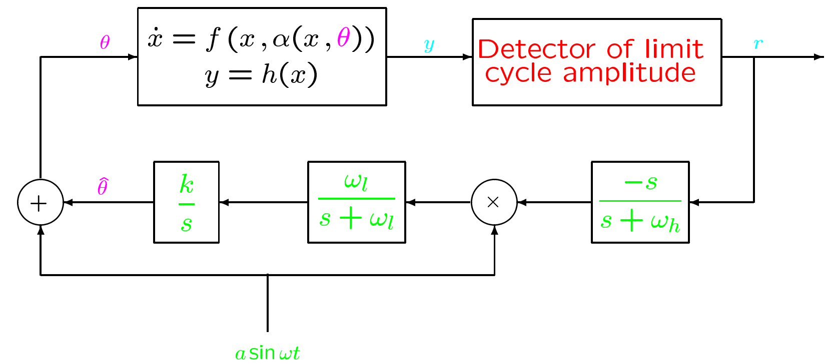
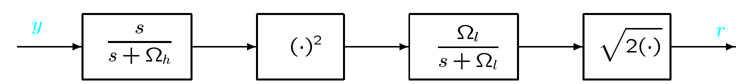
OUTLINE

- An extremum seeking scheme for limit cycle minimization
- A Van der Pol system
- Analysis

An extremum seeking for limit cycle minimization



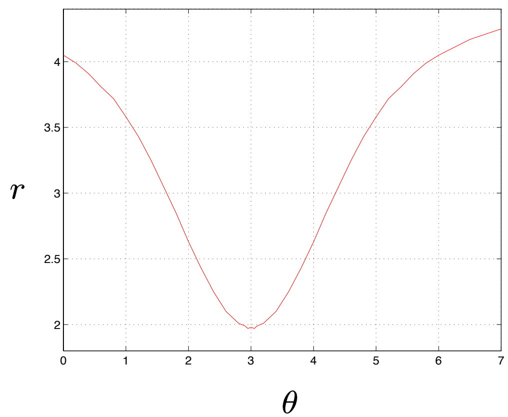
Detector of limit cycle amplitude



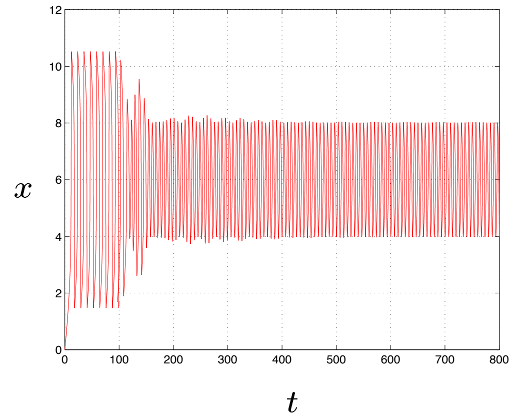
Van der Pol example

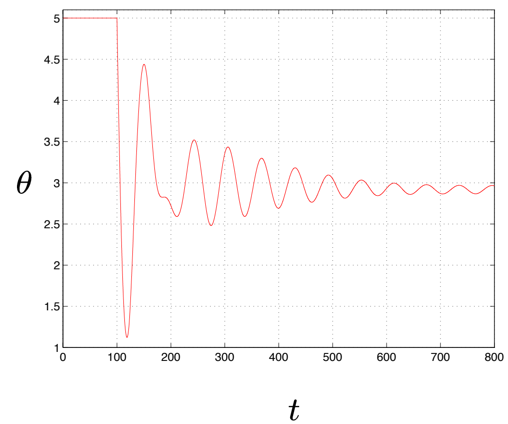
$$\ddot{x} + [(x - x_0)^2 - 1 - (\theta - \theta^*)^2]\dot{x} + x - x_0 = 0$$

Characteristic of the limit cycle "amplitude" r with respect to θ



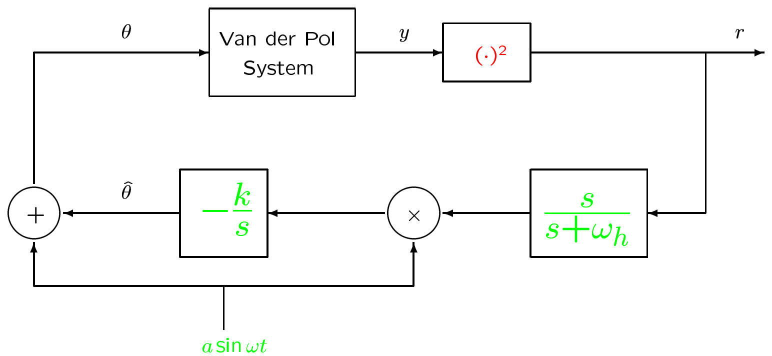
Time responses of θ and x





Analysis

Simplified scheme



Let $\tilde{\theta} = \theta - \theta^*$ **and** $y = x - x_0$

$$\ddot{y} - \epsilon \left(1 + (\tilde{\theta} + a \sin \omega t)^2 - y^2\right) \dot{y} + \mu^2 y = 0$$

To represent the system in polar coordinates, let

$$y = r \sin \phi \quad , \qquad \dot{y} = \mu\, r \cos \phi$$

System in polar coordinates

$$\frac{dr}{d\phi} = \frac{\epsilon}{\mu} \frac{r \cos^2 \phi \left[1 + \left(\tilde{\theta} + a \sin \omega t\right)^2 - r^2 \sin^2 \phi\right]}{1 - \triangle}$$

$$\frac{d\tilde{\theta}}{d\phi} = -\frac{k}{\mu} \frac{a(r^2 \sin^2 \phi - \eta) \sin \omega t}{1 - \triangle}$$

$$\frac{d\eta}{d\phi} = \frac{\omega_h}{\mu} \frac{r^2 \sin^2 \phi - \eta}{1 - \triangle}$$

$$\frac{dt}{d\phi} = \frac{1}{\mu} \frac{1}{1 - \triangle}$$

where

$$\triangle \triangleq \frac{\epsilon}{\mu} \cos \phi \sin \phi \left[1 + \left(\tilde{\theta} + a \sin \omega t\right)^2 - r^2 \sin^2 \phi\right]$$

Average w.r.t. ϕ for $\frac{1}{\mu}$ small

$$\begin{aligned}
\frac{dr^a}{d\phi} &= \frac{\epsilon}{\mu} r^a \left[\frac{1 + \left(\tilde{\theta}^a + a \sin \omega t^a\right)^2}{2} - \frac{(r^a)^2}{8} \right] \\
\frac{d\tilde{\theta}^a}{d\phi} &= -\frac{k}{\mu} a \sin \omega t^a \left(\frac{(r^a)^2}{2} - \eta^a \right) \\
\frac{d\eta^a}{d\phi} &= \frac{\omega_h}{\mu} \left(\frac{(r^a)^2}{2} - \eta^a \right) \\
\frac{dt^a}{d\phi} &= \frac{1}{\mu}
\end{aligned}$$

System in the sigular perturbation form

$$\begin{aligned}
\omega \frac{dr^a}{d\phi_\tau} &= \epsilon r^a \left[\frac{1 + \left(\tilde{\theta}^a + a \sin \phi_\tau\right)^2}{2} - \frac{(r^a)^2}{8} \right] \\
\frac{d\tilde{\theta}^a}{d\phi_\tau} &= -\frac{k}{\omega} a \sin \phi_\tau \left(\frac{(r^a)^2}{2} - \eta^a \right) \\
\frac{d\eta^a}{d\phi_\tau} &= \frac{\omega_h}{\omega} \left(\frac{(r^a)^2}{2} - \eta^a \right)
\end{aligned}$$

where

$$\phi_\tau = \frac{\omega \phi}{\mu}$$

Quasi-steady state

$$(r^a)^2 = 4 \left[1 + (\tilde{\theta}^a + a \sin \phi_\tau)^2 \right]$$

Reduced model

$$\begin{aligned}
\frac{d\tilde{\theta}_r^a}{d\phi_\tau} &= -\frac{k}{\omega} a \sin \phi_\tau \left[2 + 2(\tilde{\theta}_r^a + a \sin \phi_\tau)^2 - \eta_r^a \right] \\
\frac{d\eta_r^a}{d\phi_\tau} &= \frac{\omega_h}{\omega} \left[2 + 2(\tilde{\theta}_r^a + a \sin \phi_\tau)^2 - \eta_r^a \right]
\end{aligned}$$

Average system (w.r.t. ϕ_τ)

$$\begin{aligned}
\frac{d\tilde{\theta}_r^{aa}}{d\phi_\tau} &= -2 \frac{k a^2}{\omega} \tilde{\theta}_r^{aa} \\
\frac{d\tilde{\eta}_r^{aa}}{d\phi_\tau} &= \frac{\omega_h}{\omega} \left[-\tilde{\eta}_r^{aa} + 2 \left(\tilde{\theta}_r^{aa} \right)^2 \right]
\end{aligned}$$

Asymptotic solutions

Let

$$\delta = \frac{\max\{k\,,\omega_h\}}{\omega} \quad \text{— small}$$

Then

$$\begin{aligned} \tilde{\theta}(t) \quad &\longrightarrow \quad O\left(\delta + \omega + \frac{1}{\mu}\right) \\ \sqrt{y(t)^2 + \frac{\dot{y}(t)^2}{\mu^2}} \quad &\longrightarrow \quad 2 + O\left(a + \delta + \omega + \frac{1}{\mu}\right) \end{aligned}$$

(on an $O(1)$ time interval)