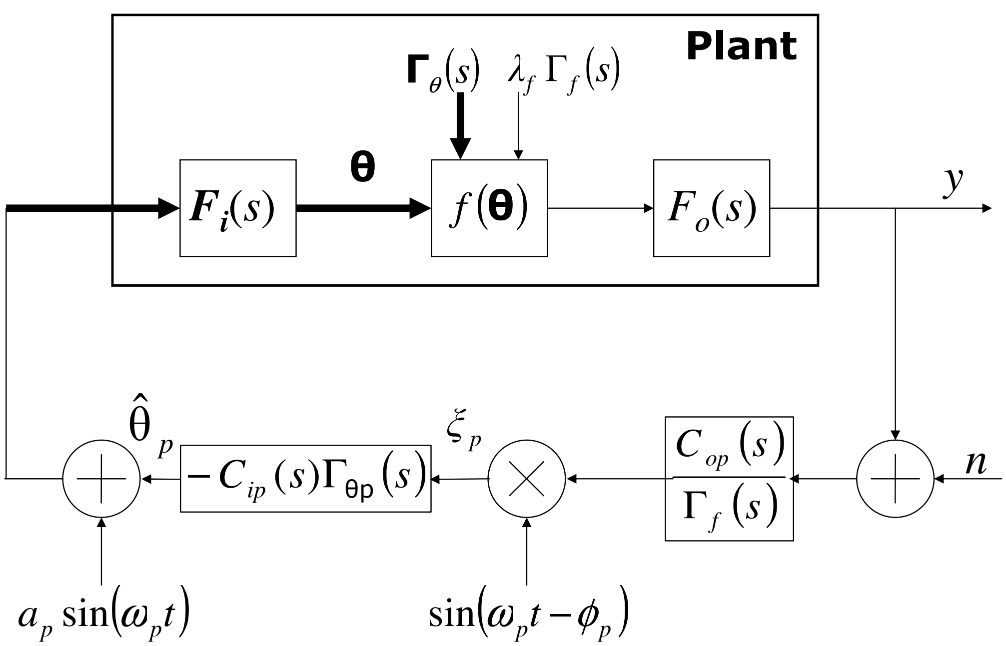
Multiparameter Extremum Seeking

General Multiparameter Scheme



Assumptions

1. $f(\theta) = f^*(t) + \left(\theta - \theta^*(t)\right)^T \mathbf{P}\left(\theta - \theta^*(t)\right)$ and $\mathbf{P}_{l\times l} > \mathbf{0}$

2. $\omega_p + \omega_q \neq \omega_r$ for any $p, q, r = 1,2,\ldots,l.$

Other assumptions as for the single parameter case for each parameter tracking loop.

Governing Equations for the p^th loop

$$
\begin{aligned}
y &= F_o(s)\left[f^* + \left(\theta - \theta^*\right)^T \mathbf{P}\left(\theta - \theta^*\right)\right] \\
\theta_p &= F_{ip}(s)\left[a_p \sin \omega_p t - C_{ip}(s)\Gamma_{\theta p}(s)\left[\xi_p\right]\right] \\
\xi_p &= \sin\left(\omega_p t - \phi_p\right)\frac{C_{op}(s)}{\Gamma_f(s)}\left[y + n\right]
\end{aligned}
$$

Definitions:

$$
\begin{aligned}
\theta_{0p} &= F_{ip}(s)\left[a_p \sin \omega_p t\right] \\
\tilde{\theta}_p &= \theta^*_p - \theta_p + \theta_{0p} \\
\tilde{y} &= y - F_o(s)\left[f^*\right] \\
&= F_o(s)\left[\left(\theta - \theta^*\right)^T \mathbf{P}\left(\theta - \theta^*\right)\right]
\end{aligned}
$$

Multiparameter Stability Test...

Output error $\tilde{y} = y - F_o(s)\left[f^*(t)\right]$ achieves local exponential convergence to an $O\left(\Delta^2 + l\sum_{p=1}^{l} a_p^2\right)$ neighbourhood of the origin where $\Delta = 1/\omega_1 + 1/M$ provided $n = 0$ and:

1. Perturbation frequencies $\omega_1 < \omega_3 < \cdots < \omega_l$ are rational, sufficiently large, and $\pm j\omega_p$ is not a zero of $F_{ip}(s)$.
2. Zeros of $\Gamma_f(s)$ that are not asymptotically stable are also zeros of $C_{op}(s)$ for all $p = 1, \ldots, l$.
3. Poles of $\Gamma_{\theta p}(s)$ that are not asymptotically stable are not zeros of $C_{ip}(s)$, for any $p = 1, \ldots, l$.

...Multiparameter Stability Test

4. $C_{op}(s)$ are asymptotically stable for all $p = 1, \ldots, l$

and $\dfrac{1}{\det(\mathrm{I}_l + \mathbf{X}(s))}$ is asymptotically stable, where

$$\mathbf{X}(\mathbf{s}) = \begin{pmatrix} X_{11}(s) & X_{12}(s) & \cdots & X_{1l}(s) \\ X_{21}(s) & X_{22}(s) & \cdots & X_{2l}(s) \\ \vdots & \ddots & \ddots & \vdots \\ X_{l1}(s) & X_{l2}(s) & \cdots & X_{ll}(s) \end{pmatrix}, \text{ and } X_{pq}(s) = P_{pq} a_p L_p(s)$$

$$L_p(s) = \frac{1}{2}\mathrm{Re}\left\{e^{j\phi_p} F_{ip}(j\omega_p)\right\} H_{ip}(s)$$

and $H_{ip}(s) = C_{ip}(s)\Gamma_{\theta p}(s) F_{ip}(s)$.

Multiparameter Design

Challenges:

- Need for asymptotic stabilization of $\frac{1}{\det\left(\mathrm{I}_1 + \mathbf{X}(\mathbf{s})\right)}$
- Available methods of decentralized control inapplicable

Theorem: Multiparameter Design

If $\dfrac{X_{pp}(s)}{1+X_{pp}(s)}$ are asymptotically stable and

$$\left|P_{pq}\right| < \frac{P_{pp}}{\left\|\dfrac{X_{pp}}{1+X_{pp}}\right\|_{H_\infty} \sqrt{\mathbf{I!}-\mathbf{1}}}$$

for each $q \neq p$, then $\dfrac{1}{\det\left(\mathbf{I}+\mathbf{X}(s)\right)}$ is asymptotically stable.

Proposition: Diagonal Domination Design

Let ρ_l^* denote the unique solution in the interval $(0,1]$

of the polynomial equation $\operatorname{per}(\Sigma(\rho)) = 2,\ \Sigma(\rho) = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$.

If $\dfrac{X_{pp}(s)}{1+X_{pp}(s)}$ are asymptotically stable and $\left\| \dfrac{X_{pq}}{1+X_{pp}} \right\|_{H_\infty} < \rho_l^*$ for

all $p \neq q$, then $\dfrac{1}{\det(\mathrm{I} + \mathbf{X}(s))}$ is asymptotically stable.

Note : $\dfrac{1}{\rho_l^*} \leq \sqrt{l! - 1}$

The Permanent of a Matrix

$\operatorname{per}(\mathbf{A}) = \sum_{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}$, where the sum runs over all $n!$ permutations σ of $\{1,\ldots,n\}$, and $\sigma(i)$ is the i^{th} element of the permutation σ.

Key Steps in Proof

Diagonal domination:

$$\det(\mathrm{I}_l + \mathbf{X}(s)) = \det\begin{pmatrix} 1 & \frac{X_{12}(s)}{1+X_{11}(s)} & \cdots & \frac{X_{1l}(s)}{1+X_{11}(s)} \\ \frac{X_{21}(s)}{1+X_{22}(s)} & 1 & \cdots & \frac{X_{2l}(s)}{1+X_{22}(s)} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{X_{l1}(s)}{1+X_{ll}(s)} & \frac{X_{l2}(s)}{1+X_{ll}(s)} & \cdots & 1 \end{pmatrix} \prod_{p=1}^{l}(1+X_{pp}(s))$$

$$= \det(\mathbf{Y}(s))\prod_{p=1}^{l}(1+X_{pp}(s)) = (1+W(s))\prod_{p=1}^{l}(1+X_{pp}(s)).$$

Bounding of off diagonal contribution:

$$W(s) = \sum_{\sigma} \operatorname{sgn}\sigma \prod_{i=1}^{l} y_{i,\sigma(i)}(s)$$

$$\|W\|_{H_\infty} \le \sum_{\sigma}\left\|\prod_{i=1}^{l} y_{i,\sigma(i)}(s)\right\|_{H_\infty} \le \sum_{\sigma}\prod_{i=1}^{l}\left\|y_{i,\sigma(i)}(s)\right\|_{H_\infty} < \operatorname{per}\Sigma(\rho_1^*) - 1 = 1\,.$$

11

Theorem: Multiparameter Design

Let separate forcing frequencies $\omega_1 < \omega_3 < \cdots < \omega_l$ be used for each of the parameter tracking loops.

If $\dfrac{X_{pp}(s)}{1+X_{pp}(s)}$ are asymptotically stable and $|P_{pq}| < \dfrac{\rho_l^*}{\left\|\dfrac{X_{pp}}{1+X_{pp}}\right\|_{H_\infty}} P_{pp}$

for each $q \ne p$, then $\dfrac{1}{\det(\mathbf{I}+\mathbf{X}(s))}$ is asymptotically stable.

Proof idea: $\dfrac{X_{pq}(s)}{1+X_{pp}(s)} = \dfrac{P_{pq}}{P_{pp}}\dfrac{X_{pp}(s)}{1+X_{pp}(s)}$

MPES Examples

Example 1: $F_{ij}(s)=1,\ F_o(s)=\dfrac{1}{s+2},$

$$f(\theta)=f^*+\frac{1}{2}\sum_{j=1}^{4}\sum_{m=1}^{4}\left(\theta_j-\theta_j^*\right)\left(\theta_m-\theta_m^*\right),$$

$$f^*=0.5u(t-5),\ \theta_j^*=u(t-j)$$

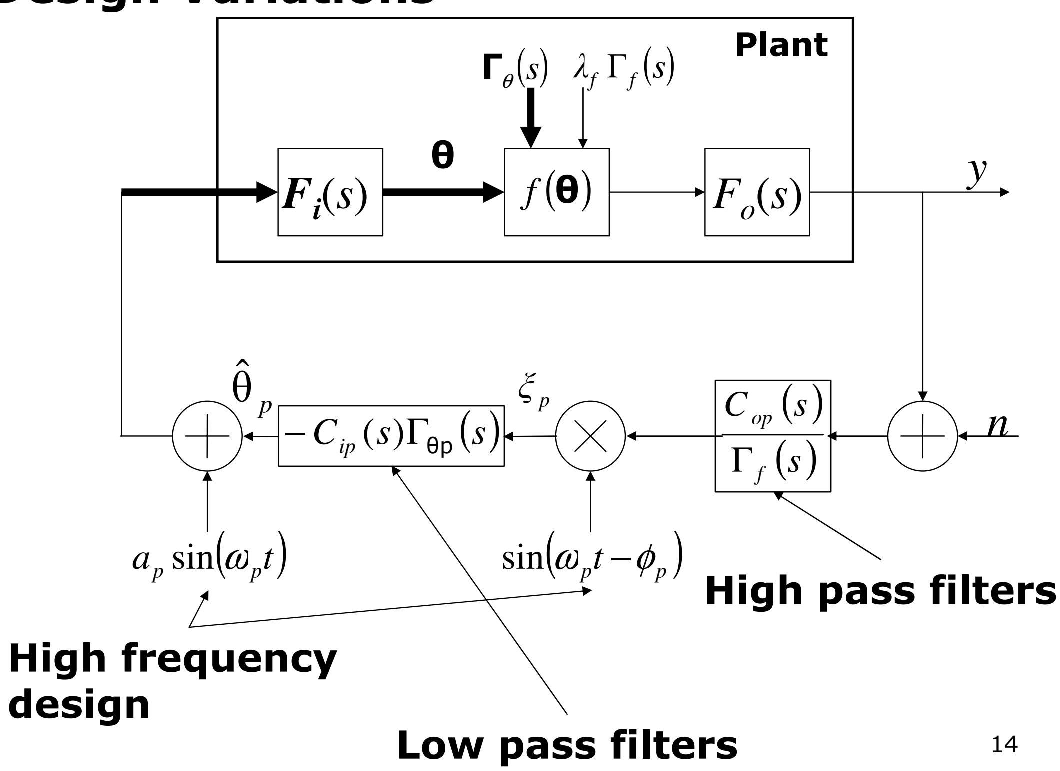
Example 2: $F_{ij}(s)=1,\ F_o(s)=\dfrac{1}{s+2},$

$$f(\theta)=f^*+\frac{1}{2}\sum_{j=1}^{4}\sum_{m=1}^{4}\left(\theta_j-\theta_j^*\right)\left(\theta_m-\theta_m^*\right)$$

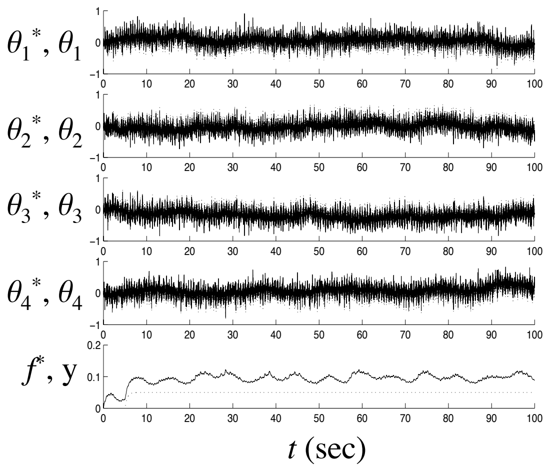
$$f^*=0.5u(t-30),\ \theta_1^*=0.01e^{0.01t},\theta_2^*=0.2t,\theta_3^*=0.1\sin t,$$

$$\theta_4^*=0.1u(t-20)$$

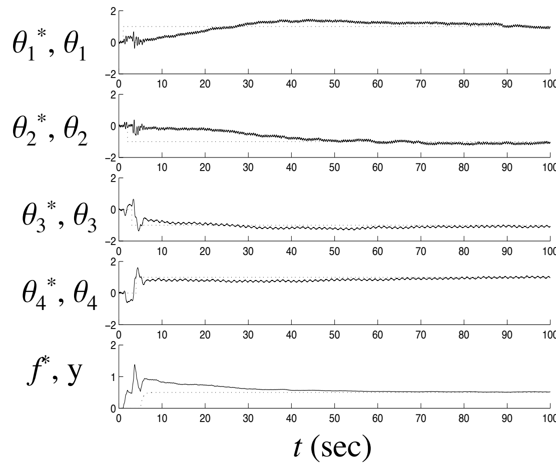
Design Variations



Example 1: Simulation Results

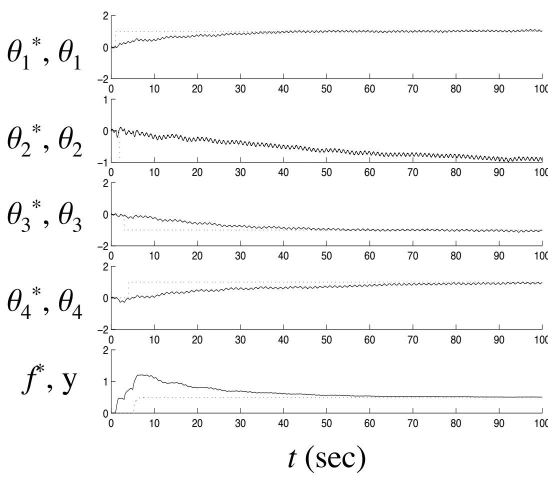


General design

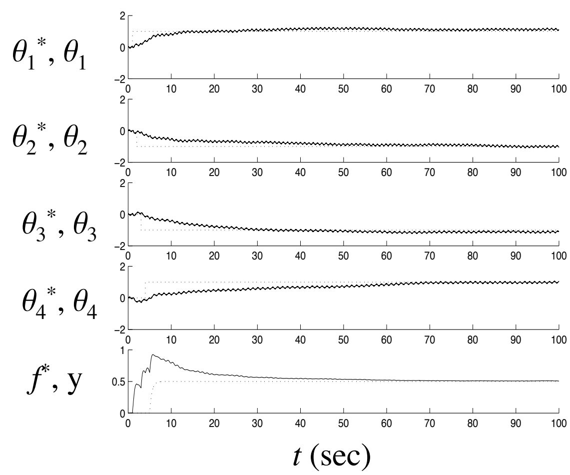


High frequency design

Example 1: Simulation Results

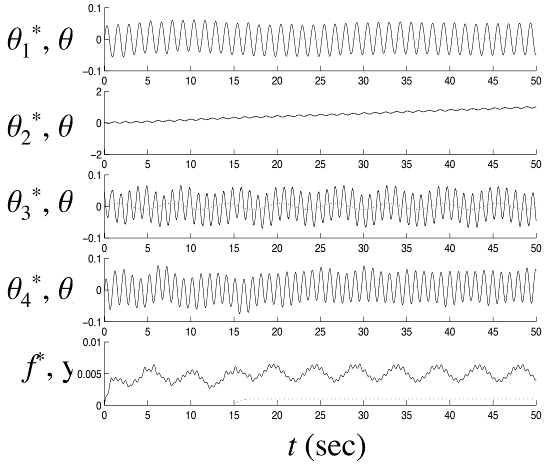


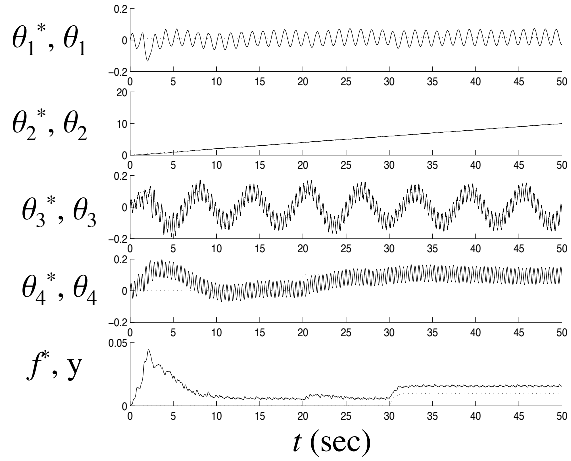
Band-pass filter design



Low-pass filter design

Example 2: Simulation Results

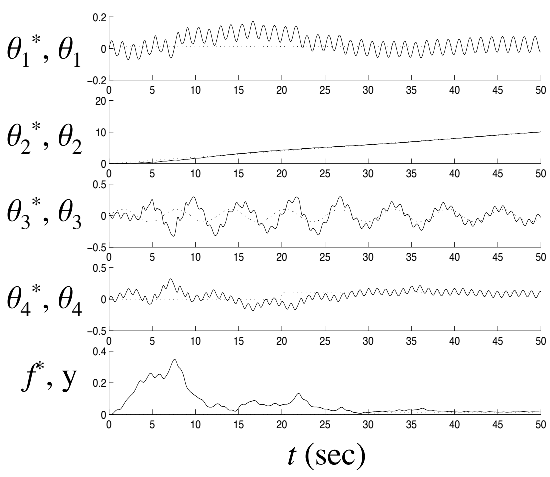


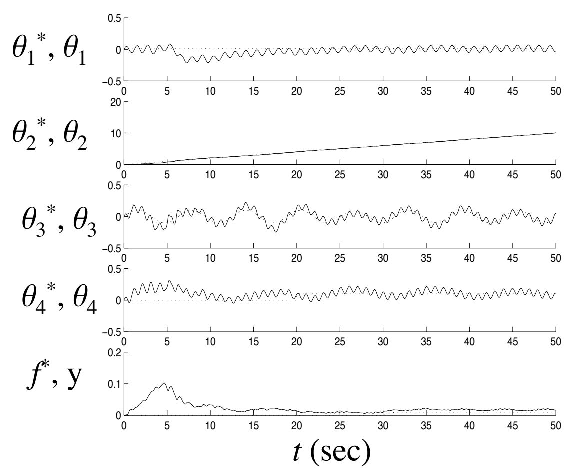


General design

High frequency design

Example 2: Simulation Results





Band-pass filter design

Low-pass filter design