Extremum Seeking for Discrete-Time Systems

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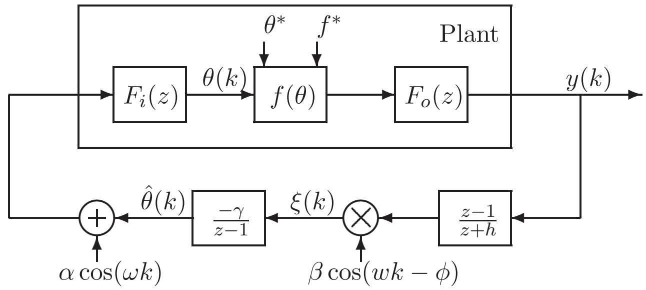
Introduction

- The plant model and compensator structure are the same as in the continuous-time extremum seeking.
- We do not have the frequency property of continuous time systems.
- Stability and ultimate bounds on error signals are established using two-time scale averaging theory (Bai, Fu & Sastry, 1988)

Discrete-Time Extremum Seeking Control



- $F_i(z)$ and $F_o(z)$ are assumed to be exponentially stable.
- Without loss of generality, the static nonlinear block $f(\theta)$ is assumed to have a minimum at $\theta = \theta^*$, and to be of the form
$$f(\theta) = f^* + (\theta - \theta^*)^2.$$
- The extremum seeking scheme consists of a modulation signal $\alpha\cos(\omega k)$, a demodulation signal $\beta\cos(\omega k - \phi)$, a high-pass filter $\frac{z-1}{z+h}$ $(0 < h < 1)$, and an integrator $\frac{-\gamma}{z-1}$.

System Description

- Governing equations:
$$y(k) = F_o(z)\Big[f^* + (\theta(k) - \theta^*)^2\Big],$$
$$\theta(k) = F_i(z)\Big[\alpha\cos(\omega k) - \frac{\gamma}{z-1}[\xi(k)]\Big],$$
$$\xi(k) = \beta\cos(\omega k - \phi)\frac{z-1}{z+h}[y(k)].$$
- For the convenience of analysis, the following terms are defined:
$$\theta_0(k) = F_i(z)[\alpha\cos(\omega k)],$$
$$\tilde{\theta}(k) = \theta^* - \theta(k) + \theta_0(k),$$
$$\tilde{y}(k) = y(k) - F_o(z)[f^*],$$

  where $\tilde{\theta}(k)$ is the tracking error and $\tilde{y}(k)$ is the output error.

Closed-Loop System

- Governing equation:

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \epsilon\Big(L(z)[\tilde{\theta}] + \Phi_1(k) + \Phi_2(k)\Big) + \delta(k),$$

where

$$\begin{aligned}
\epsilon &= \gamma\beta \\
L(z) &= -\frac{\alpha}{2}F_i(z)\Big(\mathrm{e}^{j\phi}M(z,\mathrm{e}^{j\omega}) + \mathrm{e}^{-j\phi}M(z,\mathrm{e}^{-j\omega})\Big), \ \text{(linear time invariant)} \\
\Phi_1(k) &= \alpha F_i(z)\Big[s(2\omega k)\mathbf{Im}\Big\{M(z,\mathrm{e}^{j\omega})[\tilde{\theta}]\Big\} - c(2\omega k)\mathbf{Re}\Big\{M(z,\mathrm{e}^{j\omega})[\tilde{\theta}]\Big\}\Big], \ \text{(linear time varying)} \\
\Phi_2(k) &= F_i(z)\Big[c(\omega k)\frac{z-1}{z+h}F_o(z)[\tilde{\theta}^2]\Big], \ \text{(nonlinear time varying)} \\
\delta(k) &= \epsilon F_i(z)\left[c(\omega k)\frac{z-1}{z+h}F_o(z)[f^* + \theta_0^2] + \alpha\varepsilon^{-k}\right], \ \text{(function of time)}
\end{aligned}$$

$$M(z,\mathrm{e}^{j\omega}) = F_i(\mathrm{e}^{j\omega})\frac{\mathrm{e}^{j\omega}z-1}{\mathrm{e}^{j\omega}z+h}F_o(\mathrm{e}^{j\omega}z), \ s(2\omega k) \triangleq \sin(2\omega k - \phi), \ c(2\omega k) \triangleq \cos(2\omega k - \phi).$$

Stability Analysis Outline

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \epsilon\Big(L(z)[\tilde{\theta}] + \Phi_1(k) + \Phi_2(k)\Big) + \delta(k),$$

$\delta(k)$ in the closed loop dynamics satisfies the following:

Lemma 1 *$\delta(k)$ exponentially converges to an $O(\epsilon\alpha^2)$ neighborhood of zero:*

$$|\delta(k)| \le \varepsilon^{-k} + \kappa_1\epsilon\alpha^2,$$

where κ_1 is a constant.

The bound on $\delta(k)$ depends only on modulation signal magnitude α and is independent of ϵ.

- This permits a two step stability analysis of the closed loop system
- The first step is to analyze the homogeneous system without $\delta(k)$. The second step is to derive bounds on system state and output using the bound on $\delta(k)$.

Stability and Performance Results

- Sufficient condition under which the homogeneous $\tilde{\theta}$-error system is locally exponentially stable at the origin:

  Theorem 1 *If* $F_i(1)\mathbf{Re}\Big\{e^{j\phi}F_i(e^{j\omega})\frac{e^{j\omega}-1}{e^{j\omega}+h}F_o(e^{j\omega})\Big\} > 0$*, then there exists a positive constant* ϵ^* *such that the state-space realization of the* $\tilde{\theta}$*-error system is locally exponentially stable at the origin for all* $0 < \epsilon (= \gamma\beta) \leq \epsilon^*$.

- Convergence of $\tilde{\theta}$ in the overall system:

  Theorem 2 *Suppose that the conditions of Theorem 1 are satisfied. Then, for sufficiently small* α*, there exists* ϵ_1^**,* $0 < \epsilon_1^* \leq \epsilon^*$*, such that* $\tilde{\theta}$ *in the original system locally exponentially converges to an* $O(\alpha^2)$ *neighborhood of zero for all* $0 < \epsilon \leq \epsilon_1^*$.

- Convergence of the output error $\tilde{y}(k)$:

  Corollary 1 *Under the conditions of Theorem 2, the output error* $\tilde{y}(k)$ *locally exponentially converges to an* $O(\alpha^2)$ *neighborhood of zero.*

Stability Analysis for $\tilde{\theta}$-error System

- The homogeneous part of the closed-loop system

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \epsilon\Big(L(z)[\tilde{\theta}] + \Phi_1(k) + \Phi_2(k)\Big),$$

  is periodic in k. This motivates the use of averaging to prove stability.

- Choose minimal state space realizations of $L(z)$, $\Phi_1(k)$, and the linear part of $\Phi_2(k)$ as (A_1, B_1, C_1, D_1), $(A_2(k), B_2(k), C_2(k), D_2(k))$, and $(A_3(k), B_3(k), C_3(k), D_3(k))$, respectively.
- A_1, $A_2(k)$, and $A_3(k)$ are exponentially stable (the last two from Lyapunov analysis).
- The $\tilde{\theta}$-error system can now be expressed in the following state space form

$$x'(k+1) = A(k)x'(k) + h(k, \tilde{\theta}(k))$$

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + \epsilon f'(k, \tilde{\theta}(k), x'(k)),$$

  where $A(k) = \begin{bmatrix} A_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2(k) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_3(k) \end{bmatrix}$, $h(k, \tilde{\theta}(k)) = \Big[B_1^T\tilde{\theta} \mid B_2^T(k)\tilde{\theta} \mid B_3^T(k)\tilde{\theta}^2\Big]^T$, and $f'(k, \tilde{\theta}(k), x'(k)) = D_1\tilde{\theta} + D_2\tilde{\theta} + D_3\tilde{\theta}^2 + \Big[C_1 \mid C_2(k) \mid C_3(k)\Big]x'(k)$.

Coordinate Transformation for Averaging

The change of variables

$$x(k) = x'(k) - w(k, \tilde{\theta}),$$

where $w(k, \tilde{\theta}) = \sum_{i=0}^{k-1} \Psi(k, i+1) h(i, \tilde{\theta})$, and $\Psi(k, i) = \prod_{l=i}^{k-1} A(i + k - 1 - l)$, transforms the error system to the following two time scale system:

$$x(k+1) = A(k)x(k) + \epsilon g(k, \tilde{\theta}, x)$$
$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + \epsilon f(k, \tilde{\theta}, x),$$

where

$$g(k, \tilde{\theta}, x) = -\left(\int_0^1 \frac{\partial w}{\partial \tilde{\theta}}(k+1, s\tilde{\theta}(k+1) + (1-s)\tilde{\theta}(k)) ds \right) \cdot f'(k, \tilde{\theta}, x + w(k, \tilde{\theta}))$$
$$f(k, \tilde{\theta}, x) = f'(k, \tilde{\theta}(k), x + w(k, \tilde{\theta})).$$

Averaging

- The averaged system is defined as

$$\tilde{\theta}_{av}(k+1) = \tilde{\theta}_{av}(k) + \epsilon f_{av}(\tilde{\theta}_{av}(k)),$$

  where f_{av} is calculated by the averaging operator $\mathbf{AVG}\{\cdot\}$ (Bai, Fu & Sastry, 1988) defined by

$$f_{av}(\tilde{\theta}) = \mathbf{AVG}\Big\{ f(k, \tilde{\theta}, 0) \Big\} = \lim_{T \to \infty} \frac{1}{T} \sum_{k=s+1}^{s+T} f(k, \tilde{\theta}, 0).$$

- The resulting averaged system is obtained as

$$\tilde{\theta}_{av}(k+1) = (1 - \kappa_2 \epsilon \alpha) \tilde{\theta}_{av}(k),$$

  where $\kappa_2 = \frac{1}{2} F_i(1) \mathbf{Re}\Big\{ \mathrm{e}^{j\phi} F_i(\mathrm{e}^{j\omega}) \frac{\mathrm{e}^{j\omega}-1}{\mathrm{e}^{j\omega}+h} F_o(\mathrm{e}^{j\omega}) \Big\} = \frac{1}{2} F_i(1) \Big| F_i(\mathrm{e}^{j\omega}) \frac{\mathrm{e}^{j\omega}-1}{\mathrm{e}^{j\omega}+h} F_o(\mathrm{e}^{j\omega}) \Big| \cos(\psi_M + \phi)$ and $\psi_M = \angle\Big(F_i(\mathrm{e}^{j\omega}) \frac{\mathrm{e}^{j\omega}-1}{\mathrm{e}^{j\omega}+h} F_o(\mathrm{e}^{j\omega}) \Big)$. This yields the sufficient conditions for stability.

Simulation Study

- Plant parameters $\theta^* = 3$, $f^* = 2$ and plant dynamics:

$$F_i(z) = \frac{z + 0.4}{(z + 0.5)(z + 0.6)} \text{ and } F_o(z) = \frac{z - 0.2}{z + 0.6}.$$

- Extremum seeking design parameters: $h = 0.9$, $\alpha = 0.05$, $\beta = 0.05$, and $\phi = 0$.

- Simulation is conducted for $\omega = \frac{\pi}{1.1}$ and $\omega = \frac{\pi}{1.5}$, giving $\left|M(\mathrm{e}^{j\frac{\pi}{1.1}})\right| = 4.57$, $\measuredangle\Big(M(\mathrm{e}^{j\frac{\pi}{1.1}})\Big) = -0.75$ rad, $\left|M(\mathrm{e}^{j\frac{\pi}{1.5}})\right| = 2.68$, $\measuredangle\Big(M(\mathrm{e}^{j\frac{\pi}{1.5}})\Big) = 0.93$ rad, and $F_i(1) = 0.58$, where $M(\mathrm{e}^{j\omega}) = F_i(\mathrm{e}^{j\omega})\frac{\mathrm{e}^{j\omega}-1}{\mathrm{e}^{j\omega}+h}F_o(\mathrm{e}^{j\omega})$.

- Since $\cos\left(\measuredangle\Big(M(\mathrm{e}^{j\frac{\pi}{1.1}})\Big)\right) > 0$, $\cos\left(\measuredangle\Big(M(\mathrm{e}^{j\frac{\pi}{1.5}})\Big)\right) > 0$, and $F_i(1) > 0$, the sufficient condition of Theorem 1 is satisfied for both $\omega = \frac{\pi}{1.1}$ and $\frac{\pi}{1.5}$.

Simulation results

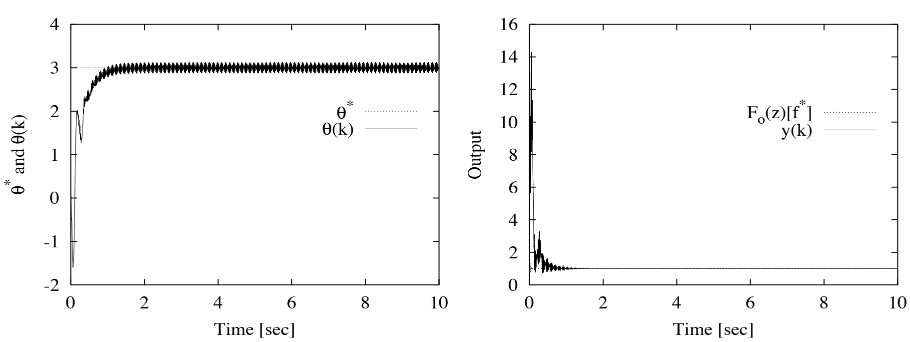


Figure 1: Responses for $\omega = \frac{\pi}{1.1}$ rad/sample and $\gamma = 0.6$

Simulation results

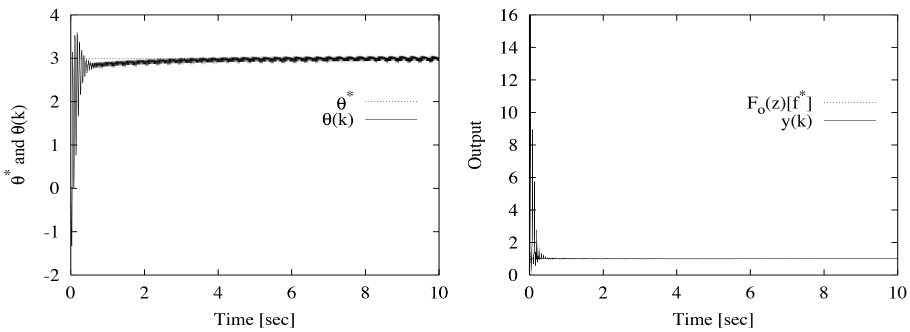


Figure 2: Responses for $\omega = \frac{\pi}{1.5}$ rad/sample and $\gamma = 2.1$

Summary

- The use of two-time scale averaging theory (Bai, Fu & Sastry, 1988), yields a very mild sufficient condition under which the system output exponentially converges to an $O(\alpha^2)$ neighborhood of the extremum value.
- The sufficient condition is related to *positive realness* of linear parts of the plant but only at the *modulation frequency* ω.

Unsolved Problems

- Method to improve and analyze the transient performance.
- Practical design guidelines for selecting modulation signal frequency ω, phase shift of demodulation signal ϕ, and compensator parameters.
- Tracking of time-varying f^* and θ^*.

Notation

- A transfer function in front of a bracketed time function, such as $G(z)[u(k)]$, means a time-domain signal obtained as an output of $G(z)$ driven by $u(k)$.
- ε^{-k} denotes exponentially decaying terms.