Backstepping Boundary Controller and Observer Designs for the Slender Timoshenko Beam

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Literature on Boundary Control of Timoshenko Beams:

- Kim and Renardy (1987); stabilization with classical "boundary damper" feedback which relates spatial and temporal derivatives at the beam tip.

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- Taylor and Yau (2003); controllability of a beam with *spatially varying parameters* using force actuation at the tip and torque at the base.

- Zhang, Dawson, de Queiroz, and Vedagarbha (1997); Timoshenko beam with mass/inertial dynamics at the free end and design a Lyapunov-based adaptive boundary damping feedback, which they also demonstrate experimentally.

Our Objective

Design controllers with actuation only at the base and sensing only at the tip.

An Introductory Example: Wave Equation

$$\varepsilon u_{tt} = (1 + d\partial_t)\, u_{xx}$$

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ε = 1/stiffness

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$$\varepsilon u_{tt} = (1 + d\partial_t)\, u_{xx}$$

d = coefficient of Kelvin-Voigt damping (allowed to be arbitrarily small)

An Introductory Example: Wave Equation

$$\begin{aligned} \varepsilon u_{tt} &= (1 + d\partial_t)\, u_{xx} \\ u_x(0) &= 0 \qquad \text{(free end)} \end{aligned}$$

An Introductory Example: Wave Equation

$$
\begin{aligned}
\varepsilon u_{tt} &= (1 + d\partial_t) u_{xx} \\
u_x(0) &= 0 \\
u(0) &= \text{measured} \\
u(1) &= \text{controlled}
\end{aligned}
$$

The Target System

$$
\begin{aligned}
\varepsilon w_{tt} &= (1+d\partial_t)(w_{xx}-cw) \\
w_x(0) &= 0 \\
w(1) &= 0,
\end{aligned}
$$

where $c > 0$ is a design gain.

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Proposition All the eigenvalues are in the open left-half-plane, have the damping ratios of at least

$$
\frac{\pi d}{4\sqrt{\varepsilon}}\sqrt{1 + \frac{4}{\pi^2}c}
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Proposition All the eigenvalues are in the open left-half-plane, have the damping ratios of at least

$$\frac{\pi d}{4\sqrt{\varepsilon}}\sqrt{1 + \frac{4}{\pi^2}c}$$

and all of their real parts are no larger than

$$-\min\left\{\frac{1}{d}, \frac{\pi^2 d}{8\varepsilon}\left(1 + \frac{4}{\pi^2}c\right)\right\}.$$

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$$-\min\left\{\frac{1}{d}, \frac{\pi^2 d}{8\varepsilon}\left(1+\frac{4}{\pi^2}c\right)\right\}.$$

At most $\frac{4\sqrt{\varepsilon}}{\pi d}\sqrt{1-\frac{d^2}{4\varepsilon}c}-1$ of the eigenvalues are complex, whereas the rest are real.

Controller Design

Invertible spatially-causal/lower-triangular/Volterra state transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy\,.$$

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Boundary feedback law

$$u(1) = \int_0^1 k(1,y)u(y)\,dy\,.$$

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Kernel/gain function $k(x,y)$ needs to satisfy the hyperbolic PDE

$$\begin{aligned} k_{xx} &= k_{yy} + ck \\ k_y(x,0) &= 0 \\ k(x,x) &= -\frac{c}{2}x \end{aligned}$$

on the triangular domain $\{0 \le y \le x \le 1\}$.

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on the triangular domain $\{0 \le y \le x \le 1\}$. Explicit solution to this PDE:

$$k(x,y) = -cx\frac{I_1\left(\sqrt{c\left(x^2-y^2\right)}\right)}{\sqrt{c\left(x^2-y^2\right)}}\,,$$

where I_1 is the modified Bessel function of the first kind/first order.

Observer Design

$$\begin{aligned}
\varepsilon\hat{u}_{tt} &= (1+d\partial_t)\left[\hat{u}_{xx}+\frac{\tilde{c}(1-x)}{x(2-x)}I_2\left(\sqrt{\tilde{c}x(2-x)}\right)(u(0)-\hat{u}(0))\right] \\
\hat{u}_x(0) &= -\frac{\tilde{c}}{2}(u(0)-\hat{u}(0)) \\
\hat{u}(1) &= u(1)
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$$

where $\tilde{c}$ is a positive design parameter.

Timoshenko Beam Model

$$
\begin{aligned}
\varepsilon u_{tt} &= (1 + d\partial_t)(u_{xx} - \alpha_x) \\
\mu\varepsilon\alpha_{tt} &= (1 + d\partial_t)(\varepsilon\alpha_{xx} + a(u_x - \alpha))
\end{aligned}
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$\alpha(x,t)$ = angle of rotation due to bending

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μ small $\Rightarrow$ "slender beam"

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μ small $\Rightarrow$ "slender beam"

$\mu = 0 \Rightarrow$ singular perturbation $\Rightarrow$ "shear beam" model

Shear Beam Model

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Free end BCs:

$$\begin{aligned} u_x(0) &= \alpha(0) \\ \alpha_x(0) &= 0 \end{aligned}$$

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Free end BCs:

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Controlled at the end $x = 1$ through the boundary conditions $u(1,t)$ and $\alpha(1,t)$.

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$$
\begin{aligned}
\alpha(x) &= \frac{\cosh(bx)}{\cosh(b)}\left[\alpha(1) - b\sinh(b)u(0) + b^2\int_0^1 \cosh(b(1-y))u(y)dy\right] \\
&\quad + b\sinh(bx)u(0) - b^2\int_0^x \cosh(b(x-y))u(y)dy
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\varepsilon u_{tt} &= (1+d\partial_t)\left\{u_{xx} + b^2 u + b^3 \int_0^x \sinh(b(x-y))u(y)dy - b^2\cosh(bx)u(0)\right. \\
&\quad \left. -\frac{b\sinh(bx)}{\cosh(b)}\left[\alpha(1) - b\sinh(b)u(0) + b^2\int_0^1 \cosh(b(1-y))u(y)dy\right]\right\} \\
u_x(0) &= \frac{1}{\cosh(b)}\left[\alpha(1) - b\sinh(b)u(0) + b^2\int_0^1 \cosh(b(1-y))u(y)dy\right]
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\end{aligned}$$

Control

$$\alpha(1)=b\sinh(b)u(0)-b^2\int_0^1\cosh(b(1-y))u(y)dy$$

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$$u(1)=\int_0^1 k(1,y)\hat{u}(y)\,dy$$

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$$u(1)=\int_0^1 k(1,y)\hat{u}(y)\,dy$$

Gain kernel PDE

$$\begin{aligned} k_{xx} &= k_{yy}+\left(c+b^2\right)k-b^3\sinh(b(x-y))+b^3\int_y^x k(x,\xi)\sinh(b(\xi-y))d\xi \\ k(x,x) &= -\frac{c+b^2}{2}x-c_0 \\ k_y(x,0) &= b^2\left(-\cosh(bx)+\int_0^x k(x,y)\cosh(by)dy\right) \end{aligned}$$

Observer

$$
\begin{aligned}
\varepsilon \hat{u}_{tt} &= (1+d\partial_t)\Big\{\hat{u}_{xx} + b^2\hat{u} + b^3\int_0^x \sinh(b(x-y))\hat{u}(y)dy \\
&\quad -b^2\cosh(bx)u(0) - b\sinh(bx)\alpha(0) + p_y(x,0)\left(u(0)-\hat{u}(0)\right)\Big\} \\
\hat{u}_x(0) &= \alpha(0) + p(0,0)\left(u(0)-\hat{u}(0)\right) \\
\hat{u}(1) &= u(1)
\end{aligned}
$$

$$
\begin{aligned}
p_{yy} &= p_{xx} + \left(\tilde{c}+b^2\right)p - b^3\sinh(b(x-y)) + b^3\int_y^x p(\xi,y)\sinh(b(x-\xi))d\xi \\
p(x,x) &= \frac{\tilde{c}+b^2}{2}(x-1) \\
p(1,y) &= 0
\end{aligned}
$$

Stability

Lyapunov functions for observer error state and observer state

$$\widetilde{V} = \frac{1}{2}\left[\left(1+\tilde{\delta}d\right)\left(\|\tilde{w}_x\|^2+\tilde{c}\|\tilde{w}\|^2\right)+\varepsilon\|\tilde{w}_t\|^2+2\tilde{\delta}\varepsilon\langle\tilde{w},\tilde{w}_t\rangle\right]$$

$$\widehat{V} = \frac{1}{2}\left[\left(1+\hat{\delta}d\right)\left(\|\hat{w}_x\|^2+c\|\hat{w}\|^2\right)+\varepsilon\|\hat{w}_t\|^2+2\hat{\delta}\varepsilon\langle\hat{w},\hat{w}_t\rangle\right]+c_0\frac{d+\hat{\delta}}{2}\hat{w}(0)^2.$$

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By Poincare's inequality, for sufficiently small $\hat{\delta},\tilde{\delta}>0$ there exist $\tilde{m}_1,\tilde{m}_2,\hat{m}_1,\hat{m}_2>0$ s.t.

$$\tilde{m}_1\widetilde{U}\leq\widetilde{V}\leq\tilde{m}_2\widetilde{U}$$

$$\hat{m}_1\widehat{U}\leq\widehat{V}\leq\hat{m}_2\widehat{U},$$

where

$$\widetilde{U} = \|\tilde{w}_x\|^2+\|\tilde{w}_t\|^2$$

$$\widehat{U} = \|\hat{w}_x\|^2+\|\hat{w}_t\|^2.$$

A long calculation shows that

$$\begin{aligned}
\dot{\tilde{V}} &= -\tilde{\delta}\left(\tilde{c}\|\tilde{w}\|^2 + \|\tilde{w}_x\|^2\right) - \left(\tilde{c}d - \tilde{\delta}\varepsilon\right)\|\tilde{w}_t\|^2 - d\|\tilde{w}_{xt}\|^2 \\
\dot{\hat{V}} &= -\hat{\delta}\left(c\|\hat{w}\|^2 + \|\hat{w}_x\|^2\right) - \left(cd - \hat{\delta}\varepsilon\right)\|\hat{w}_t\|^2 - d\|\hat{w}_{xt}\|^2 - c_0\left(\hat{\delta}\hat{w}(0)^2 + d\hat{w}_t(0)^2\right) \\
&\quad + \Xi,
\end{aligned}$$

where

$$\Xi = -\left(\widehat{\Gamma}_0 + d\dot{\widehat{\Gamma}}_0\right)\left(\hat{w}_t(0) + \hat{\delta}\hat{w}(0)\right) + \left\langle \hat{w}_t + \hat{\delta}\hat{w}, \widehat{\Gamma} + d\widehat{\Gamma}_t \right\rangle$$

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and

$$\widehat{\Gamma}(x) = Q_0(x)\tilde{w}(0) + Q_1(x)\int_0^1 Q_p(1,y)\tilde{w}(y)dy$$

$$\widehat{\Gamma}_0 = +p(0,0)\tilde{w}(0) + \frac{b^2}{\cosh(b)}\int_0^1 Q_p(1,y)\tilde{w}(y)dy$$

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\end{aligned}$$

$$\begin{aligned}
Q_0(x) &= p_y(x,0) - b^2\cosh(bx) - \int_0^x k(x,\eta)\left(p_y(\eta,0) - b^2\cosh(b\eta)\right)d\eta \\
Q_1(x) &= \frac{b^2}{\cosh(b)}Q_2(x) \\
Q_2(x) &= -b\left(\sinh(bx) - \int_0^x k(x,\eta)\sinh(b\eta)d\eta\right) \\
Q_p(x,y) &= b^2\left(\cosh(b(x-y)) - \int_y^x \cosh(b(x-\xi))p(\xi,y)d\xi\right).
\end{aligned}$$

Using the Poincare, Agmon, and Cauchy-Schwartz inequalities, it can be shown that

$$|\Xi| \leq \bar{m}\left(\|\tilde{w}_x\|^2 + \|\tilde{w}_{xt}\|^2 + \|\hat{w}_x\|^2 + \|\hat{w}_{xt}\|^2\right)$$

for sufficiently large $\bar{m}$.

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for sufficiently large $\bar{m}$.

So,

$$\begin{aligned}
\dot{\tilde{V}} &= -\tilde{\delta}\left(\tilde{c}\|\tilde{w}\|^2 + \|\tilde{w}_x\|^2\right) - \left(\tilde{c}d - \tilde{\delta}\varepsilon\right)\|\tilde{w}_t\|^2 - d\|\tilde{w}_{xt}\|^2 \\
\dot{\hat{V}} &\leq -\hat{\delta}\left(c\|\hat{w}\|^2 + \|\hat{w}_x\|^2\right) - \left(cd - \hat{\delta}\varepsilon\right)\|\hat{w}_t\|^2 - d\|\hat{w}_{xt}\|^2 - c_0\left(\hat{\delta}\hat{w}(0)^2 + d\hat{w}_t(0)^2\right) \\
&\quad + \bar{m}\left(\|\tilde{w}_x\|^2 + \|\tilde{w}_{xt}\|^2 + \|\hat{w}_x\|^2 + \|\hat{w}_{xt}\|^2\right).
\end{aligned}$$

Taking a Lyapunov function of the form

$$V = \widehat{V} + \Lambda \widetilde{V} \,,$$

one can show that there exists a sufficiently large positive Λ such that

$$\dot{V} \leq -\lambda V$$

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From the invertibility of the transformations, it follows that

$$\|u_x(t)\|^2 + \|u_t(t)\|^2 + \|\hat{u}_x(t)\|^2 + \|\hat{u}_t(t)\|^2 \\ \leq \\ \bar{M} e^{-t/\bar{M}} \left(\|u_x(0)\|^2 + \|u_t(0)\|^2 + \|\hat{u}_x(0)\|^2 + \|\hat{u}_t(0)\|^2 \right)$$

Extra

The design extends to a beam model destabilized at the tip (AFM).

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In the Works

Design for undamped *shear beam* with Andras Balogh.